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Existence and Uniqueness of Solution of a Neutral Functional Differential Equation

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ABSTRACT

Up to now, many things were said about differential equations without time delay, the so called ordinary differential equations or partial differential equations, and their solutions. The fixed point theorems have been used to show the existence and uniqueness of solution of initial value problem of these equations. Since time delay occurs naturally in just about every interaction of the real world, here in this paper we see some differential equations with time delay, the so called functional differential equations or delay differential equations. In general, we used Banach-Cacciopoli Theorem and Schauder's fixed point theorem to show the existence and uniqueness of solution of a neutral functional differential equation.

KEYWORDS: Neutral, Functional differential equation, existence, uniqueness

1. INTRODUCTION

In several applications, one assumes the system under consideration is governed by a principle of causality. That is, the future state of the system is independent of the past states and is determined solely by the present. If it is also assumed that the system is governed by an equation involving the state and rate of change of the state, then one is considering either ordinary or partial differential equations. However, under closer analysis it becomes evident that the principle of causality is often only a first approximation to the true situation and a more realistic model would include some of the past states of the system. Furthermore, in some problems it is meaningless not to have dependence on the past. This has been recognized for some time, but the theory for such systems has been extensively developed only recently. In reality until the time of Volterra [1] most of the outcome obtained during the previous 200 years were concerned with special properties for very special equations. There were some very interesting developments concerning the closure of the set of exponential solutions of linear equations and the expansion of solutions in terms of these special solutions. On the other hand, there seemed to be little concern about a qualitative theory in the same spirit as for ordinary differential equations.

A functional differential equation is a differential equation with deviating argument. That is, a functional differential equation is an equation that contains some function and some of its derivatives to different argument values [2]. Functional differential equations find use in mathematical models that assume a specified behavior or phenomenon depends on the present as well as the past state of a system [3]. Functional differential equations of retarded type depend on the past and present values of the function with delays. Also, functional differential equations of neutral type depend on past and present values of the function, similarly to retarded differential equations, except it also depends on derivatives with delays. In other words, retarded differential equations do not involve the given function's derivative with delays while neutral differential equations do [4]. This implies that, past events explicitly influence future results. For this reason, functional differential equations are used to in many applications rather than ordinary differential equations (ODE), in which future behavior only implicitly depends on the past.

In many years there have been a lot of papers written on the various aspects for the theory of neutral functional differential equations (NFDE). The existence problem for neutral functional differential equations was considered by Wright [5] and Bellman & Cooke [6] in the case of constant delays. While, El'sgol'ts [7], Kamenskii [8, 9] and Driver [10] studied in the case of variable delays. Meng Fan and Ke Wang [11] also considered the existence of periodic solutions of neutral functional differential equations. The results showed that for convex neutral functional differential equations of D-operator type with finite (or infinite) delay and hyper neutral functional differential equations with finite delay, there is a periodic solution if and only if there is a bounded solution. This result was proved by Massera, Chow and Makay are generalized.

Moreover, Lianglong Wang, Zhicheng Wang and Xingfu Zou [12] Periodic neutral functional differential equations are considered. In this study, sufficient conditions for existence, uniqueness and global attractivity of periodic solutions are established by combining the theory of monotone semiflows generated by neutral functional differential equations and Krasnosel'skii's fixed point theorem. This result is applied to a concrete neutral functional differential equation that can model single-species growth, the spread of epidemics, and the dynamics of capital stocks in a periodic environment.

In addition to this, the existence and regularity of mild solutions for a class of abstract neutral functional differential equations with infinite delay was studied by Xianlong Fu [13]. This study used fraction power theory and α- norm to discuss the problem so that the obtained results can be applied to equations with terms involving spatial derivatives. A stability result for the autonomous case is also established. Results further concluded with an example that illustrates the applications of the results obtained. However, most of recent study as in Feng Jiang and Yi Shen [14] studied the existence and uniqueness of mild solutions to neutral stochastic partial functional differential equations under some Carathéodory-type conditions on the coefficients by means of the successive approximation. In particular, they generalized and improved the results that appeared in Govindan and Bao &Hou.

To our knowledge, most existing results on the existence of periodic solutions of functional differential equations are for the retarded type, and these existence results are usually obtained by the technique of bifurcation, fixed point theorems or by degree theory. In general, it is more difficult to study the uniqueness and uniqueness of the solutions neutral functional differential equations. This is, therefore we are motivated to undertake this study for fulfilling this entire gap.

2. PRELIMINARY

In this section we discussed some basic definition and theorems which is useful to show the existence and uniqueness of neutral functional differential equations (NFDE). Fixed point theorems can be considered in metric spaces where distance is used. Here we will use it in Banach spaces where norm is used since it is applied to many areas of current interest in analysis.

Definition 2.1: Let X be a non empty set and $d: \mathrm{X} \times \mathrm{X} \longrightarrow \overline{\mathbb{R}^+}$ a function. Then d is called metric on X if the following properties hold.

i. $d(x, y) = 0$ if and only if $x = y$ for some $x, y \in \mathrm{X}$;
ii. $d(x, y) = d(y, x)$ for all $x, y \in \mathrm{X}$;
iii. $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in \mathrm{X}$.

The value of metric d at (x, y) is distance between x and y and the ordered pair (X, d) is called metric space.

Definition 2.2: A norm on a linear space X is a functional whose value at x is denoted by $\|x\|$ with the following properties.

1. $\|x_1 + x_2\| \le \|x_1\| + \|x_2\|$, for all $x_1, x_2 \in \mathrm{X}$;
2. $\|\alpha x\| = |\alpha|\|x\|$, for all $x \in \mathrm{X}$ and scalar α;
3. $\|x\| \ge 0$ and $\|x\| = 0$ if $x = 0$, for all $x \in \mathrm{X}$.

$(\mathrm{X}, \|.\|)$ is called normed linear space. Then every normed space $(\mathrm{X}, \|.\|)$ is a metric space (X, d) with induced metric $d(x, y) = \|x - y\|$.

Definition 2.3: A sequence $\{x_n\}$ is said to be a Cauchy sequence if for each $\varepsilon > 0$ there exists a positive integer N such that $|x_n - x_m| < \varepsilon$ for all $n \ge m \ge N$.

Definition 2.4: A normed linear space X is complete if every Cauchy sequence in X converges in X.

Definition 2.5: A complete normed linear space is called Banach space.

Definition 2.6: A subset A of a normed linear space B is said to be compact if and only if every sequence $\{x_n\}$ in A has a convergent subsequence with limit in A.

Definition 2.7: Let $\{f_m\}$ be a sequence of real valued functions in a subset D of $\mathbb{R}^n$. Let $x \in D$. The sequence is equi-continuous at x if for all $\varepsilon > 0$ there exists $\delta > 0$, independent of m such that $|f_m(y) - f_m(x)| < \varepsilon$ for all $y \in D$ with $|y - x| < \delta$.

Definition 2.8: Let $U \subseteq \mathbb{R}^n$ be open. A function $f: U \longrightarrow \mathbb{R}$ is said to be

i. Uniformly Lipschitz continuous if and only if there exists $L \in \mathbb{R}$ such that $\|f(x) - f(y)\| \le L\|x - y\|$, for all $x, y \in U$. Here L is called Lipschitz constant.
ii. Lipschitz continuous if and only if for all $x \in U$ there exists a neighborhood V of x such that the restriction of f to $U \cap V, f_{/(U \cap V)}$ is uniformly Lipschitz continuous.

Definition 2.9: Let S and B be Banach spaces. A transformation $T: V \subseteq S \longrightarrow B$ such that $\|T_x - T_y\| \le L\|x - y\|$ for some $L \in \mathbb{R}^+$ is said to be:

a) A contraction map on V if $0 < 1 < 1$;
b) Non expansive map on V if $L = 1$.

Definition 2.10: A point $x \in U$ is said to be a fixed point of a transformation $T: U \longrightarrow U$ if
$T_x = x$.

Definition 2.11: Let D is a subset of a Banach space X and $T: D \longrightarrow X$ be a map. Given $\varepsilon > 0$, a point $x \in D$ with $\|x - T_x\| < \varepsilon$ is called an ε –fixed point.

Remark 2.1: A convergent sequence is bounded. A Cauchy sequence is convergent and hence it is bounded. So it has a bounded subsequence.

Theorem 2.1 (Arzera-Ascole Theorem): Every bounded and equi-continuous sequence $\{f_m\}$ of real valued functions on a compact subset S of $\mathbb{R}^n$ has a subsequence which converges uniformly on S.

Theorem 2.2 (Darbo's Theorem): If $U(\sigma, t)$ is asymptotically smooth, point dissipative and positive orbits of bounded sets are bounded, then there exists a connected global attractor. Finally there is an ω −periodic trajectory.

Theorem 2.3 (Gronwall's Inequality): If $n(t)$ and $\alpha(t)$ are real valued continuous functions on $[a,b], \alpha(t)$ is nondecreasing and $\beta(t) \geq 0$ is integrable on $[a,b]$ with $u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds,$ for $a \leq t \leq b,$

Then

$u(t) \leq \alpha(t)e^{\int_a^t \beta(r)dr}$, for $a \leq t \leq b.$

Theorem 2.4 (Banach-Cacciopoli Theorem): Let $T: V \longrightarrow V$ is a contraction, where V is a closed subset of a Banach space X, then T has a unique fixed point in V.

Theorem 2.5: Let D be a closed subset of a normed linear space X and $F: D \longrightarrow X$ is a compact continuous map. Then F has a fixed point if and only if it has an ε −fixed point.

Theorem 2.6 (Schauder's Fixed Point Theorem): Let C be a closed, convex subset of a normed linear space X. Then every compact continuous map $F: C \longrightarrow C$ has at least one fixed point.

3. EXISTENCE AND UNIQUENESS OF SOLUTION

A functional equation is an equation involving an unknown function for different argument values. A functional differential equation is an equation that involves a functional equation together with derivatives. So combining the notions of differential and functional equation we obtain the notion of functional differential equation (FDE).

In general, when a situation (model) does not depend on its past history, it consists of the so called ordinary differential equation (ODE) or partial differential equation (PDE).These equation admit the principle of causality which says that,"The future state of the system is independent of the past and solely determined by the present". Models incorporating past history include functional differential equation (FDE) or delay differential equation (DDE).

When the past dependence is through the state variable and the derivatives of the state variables we call it neutral functional differential equation (NFDE). Therefore neutral functional differential equation is a dynamical system where the rate of change $\dot{x}(t)$ of the state $x(.)$ at time t depends not only on the present value $x(t)$, but also on past values $x(.)$ where there is a derivative of the state variable x as an argument value.

3.1. Axioms for the phase space

Let $\hat{B}$ be a linear real vector space of function mapping $(-\infty, 0]$ into $\mathbb{R}^n$ with elements denoted by $\hat{\varphi}, \hat{\psi}, \ldots,$ where $\hat{\varphi} = \hat{\psi}$ means $\hat{\varphi}(t) = \hat{\psi}(t)$ for $t \leq 0$. Assume that a seminorm $|.|_{\hat{B}}$ is given in $\hat{B}$ so that $B = \frac{\hat{B}}{|.|}$ is a Banach space with the induced norm $|\varphi|_B = |\hat{\varphi}|_{\hat{B}}$ for $\varphi \in B, \hat{\varphi} \in \hat{B}$ if $\hat{\varphi} \in \varphi$.

For $\hat{x}: (-\infty, \sigma) \longrightarrow \mathbb{R}^n, t \in (-\infty, \sigma)$, we define $\hat{x}_t: (-\infty, \sigma] \longrightarrow \mathbb{R}^n$ by $\hat{x}_t(s) = \hat{x}(t+s)$ for $s \leq 0$. For $\alpha \geq 0, t_0 \in \mathbb{R}$ and $\hat{\varphi} \in \hat{B}$, let $\mathcal{F}_{\alpha,t_0}(\hat{\varphi})$ be the set of all functions $\hat{x}: (-\infty, t_0 + \alpha] \longrightarrow \mathbb{R}^n$ with $\hat{x}_{t_0} = \hat{\varphi}$ and $\hat{x}$ being continuous on $[t_0, \ t_0 + \alpha]$ (on $[t_0, \infty)$ in case $\alpha = \infty$). Furthermore we put $\mathcal{F}_{\alpha,t_0} = U_{\hat{\varphi} \in \hat{B}} \mathcal{F}_{\alpha,t_0}(\hat{\varphi})$. In case $t_0 = 0$ we simply write $\mathcal{F}_\alpha(\hat{\varphi})$ and $\mathcal{F}_\alpha$. Thus we have the following axioms for the phase space is

(A1) $\hat{x}_t \in \hat{B}$ for $\hat{x}_t \in \mathcal{F}_\alpha$ and $t \in [0, \alpha]$.

For $\beta \geq 0$ and $\hat{\varphi} \in \hat{B}$, let $\hat{\varphi}^\beta$ denoted the restriction of $\hat{\varphi}$ to $(-\infty, -\beta)$. For $\varphi \in B$, define

$|\varphi|_\beta = inf\left\{|\hat{\psi}|_{\hat{\beta}}\right\};\ \hat{\psi} \in \hat{B}$ and $\hat{\psi}^\beta = \hat{\varphi}^\beta$ for some $\hat{\varphi} \in \varphi$.

$B^\beta = B_{|.|_\beta}$ is the space of all equivalent classes $\{\varphi\}_\beta = \left\{\psi \in B : |\varphi - \psi|_\beta = 0\right\}$ for $\varphi \in B$ with respect to the seminorm $|.|_\beta$. In B^β, we define the norm $|.|_\beta$ naturally induced by the seminorm $|.|_\beta$.

For $\beta \geq 0$ and $\hat{\varphi} \in \hat{B}$, define

$$(\hat{S}_\beta \hat{\varphi})(\theta) = \begin{cases} \hat{\varphi}(\beta + \theta), \theta \in (-\infty, -\beta), \\ \hat{\varphi}(0), \qquad \theta \in [-\beta, 0]. \end{cases}$$

(A2) If $|\hat{\varphi} - \hat{\psi}|_{\hat{\beta}} = 0$, then $|\hat{S}_\beta \hat{\varphi} - \hat{S}_\beta \hat{\psi}|_{\hat{\beta}} = 0$.

This axiom justifies the definition of S_β given by $S_\beta \varphi = \psi$ if $\hat{S}_\beta \hat{\varphi} \in \psi$ for $\hat{\varphi} \in \varphi$ and if $\varphi = \psi$ in B, then $|S_\beta \varphi - S_\beta \psi|_\beta = 0$ for $\beta \geq 0$.

(A3) there exists a positive constant K such that for any $\varphi \in B$, $|\varphi(0)| \leq K|\varphi|_B$.

(A4) there exist a continuous function $K_1(s)$ and d locally bounded function $M_1(s)$ such that

i. $|\tau^\beta \varphi|_\beta \leq M_1(\beta)|\varphi|_B$ for $\beta \geq 0, \varphi \in B$,

ii. If $x \in \mathcal{F}_{\alpha,t_0}$, then for $t \in [t_0, \ t_0 + \alpha]$, we have $|x_t|_B \leq K_1(t - t_0) \sup_{t_0 \leq s \leq t} |x(s)| + M_1(t - t_0)|x_{t0}|_B$.

(A5) if $x \in \mathcal{F}_\alpha, \alpha > 0$, then x_t is continuous in $t \in [0, \ \alpha]$.

Definition 3.1: Suppose that Ω is an open set in $\mathbb{R} \times B$, $G: \Omega \longrightarrow \mathbb{R}^n$ is continuous, $G(t, \varphi)$ has a continuous Frέchet derivative $G_\varphi(t, \varphi)$ with respect to φ on Ω and

$G_\varphi(t, \varphi)\psi = A(t, \varphi)\psi(0) + L(t, \varphi, \psi)$, for $(t, \varphi) \in \Omega, \psi \in$ B. (3.1)

If $A(t, \varphi)$ is an $n \times n$ matrix such that $det\, A(t, \varphi) \neq 0$ and $A(t, \varphi), A^{-1}(t, \varphi)$ are continuous, and if $L(t, \varphi, \psi)$ is linear with respect to ψ and satisfies:

(H₁) there are an $\alpha_0 > 0$ and a continuous map $r(t, \varphi, \alpha): \Omega \times [0, \alpha_0] \longrightarrow \mathbb{R}^+, r(t, \varphi, 0) = 0$, such that for $\psi \in B$ satisfying $|\psi|_\alpha = 0$,

$$L(t, \varphi, \psi) < r(t, \varphi, \alpha)|\psi|_B \qquad (3.2)$$

Then we say G is generalized atomic at zero on Ω.

Definition 3.2: Suppose $\Omega \subseteq \mathbb{R} \times B$ is open, $f: \Omega \longrightarrow \mathbb{R}^n, G: \Omega \longrightarrow \mathbb{R}^n$ are given continuous functions with G atomic at zero. Then we say;

$$\frac{d}{dt} G(t, x_t) = f(t, x_t) \qquad (3.3)$$

is a neutral functional differential equation with infinity delay (NFDE). The function G will be called the difference operator for the NFDE.

By a solution of equation (3.3) we mean an $x \in \mathcal{F}_{A,\sigma}$ for some $A > 0$ and $-\infty < \sigma < \infty$ such that

i. $(t, x_t) \in \Omega$ for $t \in [\sigma, \sigma + A]$,

ii. $G(t, x_t)$ is continuously differentiable and satisfies (3.3) on $[\sigma, \sigma + A]$.

If, in addition, $x_\sigma = \varphi$, then we say x is a solution of (3.3) through (σ, φ) and we denote it by $x(t, \sigma, \varphi)$.

Theorem 3.1 (Existence of solution): Suppose Ω is an open set of $\mathbb{R} \times B$. Then for any $(\sigma, \varphi) \in \Omega$ there exist a solution of NFDE (G, f, Ω) through (σ, φ).

Proof: For $\alpha > 0, \beta > 0$, define

$A(\alpha, \beta) = \{z \in C((-\infty, \alpha], \mathbb{R}^n): z(s) = 0, s \leq 0, |z(t)| \leq \beta, t \in [0, \alpha]\}$.

Obviously, $A(\alpha, \beta)$ is a bounded, closed, convex subset of $BC((-\infty, \alpha], \mathbb{R}^n)$ (the space of bounded and continuous functions with the supnorm $\|.\|$).

We define two operators on $A(\alpha, \beta)$:

$$S: \begin{cases} S_z(t) = 0, \quad -\infty < t \leq 0, \\ A(\sigma + t, \hat{\varphi}_t)(S_z)(t) = -L(\sigma + t, \hat{\varphi}_t, z_t) - g(\sigma + t, \hat{\varphi}_t, z_t) + G(\sigma, \varphi) - G(\sigma + t, \hat{\varphi}_t), \quad 0 \leq t \leq \alpha \end{cases} \tag{3.4}$$

$$U: \begin{cases} U_z(t) = 0, \quad -\infty < t \leq 0, \\ A(\sigma + t, \hat{\varphi}_t)(U_z)(t) = \int_0^t f(\sigma + s, \hat{\varphi}_s + z_s) ds, \quad 0 \leq t \leq \alpha \end{cases} \tag{3.5}$$

Under equation (3.1), where $\hat{\varphi} \in \mathcal{F}_\alpha(\varphi)$ with $\hat{\varphi}(t) = \varphi(0)$ for $t > 0$, and

$$g(\sigma, \varphi, \psi) = G(\sigma, \varphi + \psi) - G(\sigma, \psi) - G_\varphi(\sigma, \varphi)\psi \tag{3.6}$$

(i) By equation (2.2) and the continuity of f, G, G_φ, there are $\beta_0 > 0$ and a positive function $\alpha_1(\beta)$ defined for $0 < \beta < \beta_0$ such that for $0 < \beta < \beta_0$ and $0 < \alpha < \alpha_1(\beta)$, $S + U$ maps $A(\alpha, \beta)$ into itself.

(ii) S is a contraction on $A(\alpha, \beta)$ for suitable α, β.

By the continuity of G_φ, for any $\varepsilon > 0$, there are $\beta(\varepsilon) \in (0, \beta_0), \alpha(\varepsilon) \in (0, \alpha_1(\beta(\varepsilon)))$ such that for $y, z \in A(\alpha(\varepsilon), \beta(\varepsilon)), t \in [0, \alpha(\varepsilon)]$,

$$|g(\sigma + t, \hat{\varphi}_t, z_t) - g(\sigma + t, \hat{\varphi}_t, y_t)| \leq \varepsilon |z_t - y_t|_B$$

Therefore, for $0 < \beta < \beta(\varepsilon)$, $0 < \alpha < \alpha(\varepsilon)$, we have

$$\begin{aligned} \|S_z - S_y\| &\leq \sup_{0 \leq t \leq \alpha} \{|A^{-1}(\sigma + t, \hat{\varphi}_t)|[|L(\sigma + t, \hat{\varphi}_t, y_t - z_t)| + |g(\sigma + t, \hat{\varphi}_t, z_t) - g(\sigma + t, \hat{\varphi}_t, z_t)|]\} \\ &\leq \sup_{0 \leq t \leq \alpha} |A^{-1}(\sigma + t, \hat{\varphi}_t)|[r(\sigma + t, \hat{\varphi}_t, t) + \varepsilon] |z_t - y_t|_B \\ &\leq \sup_{0 \leq t \leq \alpha} |A^{-1}(\sigma + t, \hat{\varphi}_t)| K_1(t)[r(\sigma + t, \hat{\varphi}_t, t) + \varepsilon] \|z - y\| \end{aligned}$$

Therefore, for a constant $k \in (0,1)$, there are $0 < \beta_2 < \beta_0$ and a function $\alpha_2(\beta)$defined on $[0, \beta_2], \alpha_2(\beta) < \alpha_1(\beta)$ such that for $0 < \beta < \beta_2, 0 < \alpha < \alpha_2(\beta), z, y \in A(\alpha, \beta)$, we have

$$\|S_z - S_y\| \leq k\|z - y\|$$

iii. U is completely continuous on $A(\alpha, \beta)$ for $0 < \beta < \beta_2, 0 < \alpha < \alpha_2(\beta)$.

For any $B \subseteq A(\alpha, \beta), z \in B, 0 \leq t, \tau \leq \alpha$,

$$|U_z(t) - U_z(\tau)| \leq \left| A^{-1}(\sigma + t, \hat{\varphi}_t) \int_0^t f(\sigma + s, \hat{\varphi}_s + z_s) ds - A^{-1}(\sigma + t, \hat{\varphi}_t) \int_0^\tau f(\sigma + s, \hat{\varphi}_s + z_s) ds \right|$$

$$\leq \left| A^{-1}(\sigma + t, \hat{\varphi}_t) \int_\tau^t f(\sigma + s, \hat{\varphi}_s + z_s) ds \right| + |A^{-1}(\sigma + t, \hat{\varphi}_t) - A^{-1}(\sigma + t, \hat{\varphi}_t)| \times \left| \int_0^\tau f(\sigma + s, \hat{\varphi}_s + z_s) ds \right|$$

$$\leq N|t - \tau| + N|A^{-1}(\sigma + t, \hat{\varphi}_t) - A^{-1}(\sigma + t, \hat{\varphi}_t)|$$

Where N is a positive constant (by the continuity of A^{-1} and f^{-1}, for sufficiently small α, β, we can find such an N). So, UB is uniformly bounded and equicontinuous, and hence UB is precompact by Ascoli's theorem. This implies U is completely continuous on $A(\alpha, \beta)$.

Obviously, by (ii), (iii), $S + U$ is an α-contraction on $A(\alpha, \beta)$. By Darbo's theorem 2.2, $S + U$ has a fixed point. Since;

$$A(\sigma + t, \hat{\varphi}_t)(S_z + U_z)(t) = \int_0^t f(\sigma + s, \hat{\varphi}_s + z_s) ds + G(\sigma, \varphi) + G(\sigma + t, \hat{\varphi}_t + z_t) + A(\sigma + t, \hat{\varphi}_t) Z(t)$$

The integral equation

$$\begin{cases} G(\sigma+t,\hat{\varphi}_t+\ z_t)-G(\sigma,\varphi)=\int_0^t f\,(\sigma+s,\hat{\varphi}_s+\ z_s)ds \\ z_0=0 \end{cases}$$

Has a continuous solution. This implies there exist a solution of (3.3) through (σ,φ).

Theorem 3.2 (Uniqueness of solution): If there is a constant $L>0$ such that $|f(t,\varphi)-f(t,\psi)|\le L|\varphi-\psi|_B$ for $(t,\varphi),(t,\psi)\in\Omega$, then for any $(\sigma,\varphi)\in\Omega$, there is a unique solution of (3.3) through (σ,φ).

Proof: It is sufficient to prove $S+U$ has a unique fixed point in $A(\sigma,\beta)$. Suppose there are $z_1,z_2\in A(\alpha,\beta)$ such that $z_i=(S+U)_{z_i}(i=1,2)$. Then by (ii), we have

$$\sup_{0\le s\le t}\left|S_{z_1}(s)-S_{z_2}(s)\right|\le k\sup_{0\le s\le t}|z_1(s)-z_2(s)|$$

And

$$\left|U_{z_1}(t)-U_{z_2}(t)\right|\le|A^{-1}(\sigma+t,\hat{\varphi}_t)|\left|\int_0^t[f\,(\sigma+s,\hat{\varphi}_s+\ z_{1s})-f(\sigma+s,\hat{\varphi}_s+\ z_{2s})ds\right|$$

$$\le|A^{-1}(\sigma+t,\hat{\varphi}_t)|L\int_0^t|z_{1s}-z_{2s}|_B ds$$

$$\le|A^{-1}(\sigma+t,\hat{\varphi}_t)|L\int_0^t K_1(s)\sup_{0\le\theta\le s}|z_1(\theta)-z_2(\theta)|ds$$

$$\le LL_1\int_0^t\sup_{0\le\theta\le s}|z_1(\theta)-z_2(\theta)|ds$$

Where, $L_1=\sup_{0\le t\le\alpha}|A^{-1}(\sigma+t,\hat{\varphi}_t)|\sup_{0\le t\le\alpha}K_1(t)$.

Putting $m(t)=\sup_{0\le s\le t}|z_1(s)-z_2(s)|$, we have

$$m(t)\le km(t)+LL_1\int_0^t m(s)\,ds$$

And

$$m(t)\le\frac{LL_1}{1-k}\int_0^t m(s)\,ds$$

By Gronwall's inequality, $m(t)=0$ for $0\le t\le\alpha$.

Therefore, $z_1=z_2$.

Hence, equation (3.3) has a unique solution.

4. CONCLUSION

In this study, we discussed the existence and uniqueness of solutions of neutral functional differential equations. Some preliminary definition and theorem are discussed which are used to show the existence and uniqueness solution of neutral functional differential equation. The Banach-Cacciopoli Theorem and Schauder's fixed point theorems have been used to show the existence and uniqueness of solution of initial value problem of neutral functional differential equations.

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