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New Results on k-prime Labeling

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Abstract. In this paper, we introduce new results on k-prime labeling. First, we discuss the k-prime labeling of cycles C_n for some values of and n. Also we give the k-prime labeling of combs $P_n \odot K_1$ and some case of the crowns $C_n \odot K_1$. Second, we show that all wheels W_{2n+1} are not k-prime for every positive integers k and n while W_{2n} $(n \geq 2)$ is not k-prime for every even positive integers k. Finally, we give the k-prime labeling of the helm H_n $(n \geq 3)$ for $2 \leq k \leq 11$ and we show that if $k+2(n-1)$ and $k+2n$ are twin primes where $n \geq 3$ and $k \geq 1$, then H_n is k-prime.

Introduction

The notion of prime labeling originated with Entringer and was introduced in a paper by Tout et al. [12]. A graph G with vertex set $V(G)$ is said to have a prime labeling if there exist a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for each edge $xy \in E(G)$, $f(x)$ and $f(y)$ are relatively prime. A graph that admits a prime labeling is called a *prime graph.* Around 1980 Entringer conjectured, that all trees are prime. Paths, stars, caterpillar, complete binary trees, spiders have prime labeling (see [4]). In 1999, Seoud and Youssef [10] conjectured that all unicyclic graphs are prime. In 2011, Vaidya and Prajapati [13] gave a variation of the definition of prime labeling. They call a graph G of order n is k-prime for some positive integer k if its vertices can labeled bijectively by the labels $k, k+1, \ldots, k+n-1$ such that adjacent vertices receive relatively prime labels. For known results on the prime labeling and its variations see [1-14]. The reference [4] surveyed the known results to all variations of graph labelings appearing in this paper. In this paper we give new results on k-prime labeling. All graphs in this chapter are simple, finite, and undirected.

2. k-prime labeling of cycles and related graphs

In this section we investigate the -prime labeling of cycles, combs and crowns for some positive integer values k. We well denoted to the vertices of the path or the cycle by

$u_1, u_2, \ldots, u_n$ and the pendant edges of the comb or the crown by $v_1, v_2, \ldots, v_n$. However, we have the following necessary condition for the graph to be k-prime.

Lemma 2.1 If G is a k-prime graph of order n with a k-prime labeling function f and let $E_2 = \{k \le t \le k+n-1 : t \text{ is even}\}$, then $\beta(G) \ge |E_2|$, where

$$|E_2| = \begin{cases} \left\lfloor \dfrac{n}{2} \right\rfloor & \text{if } k \text{ is odd} \\ \left\lceil \dfrac{n}{2} \right\rceil & \text{if } k \text{ is even} \end{cases}$$

Proof Since we have $|E_2|$ even vertex labels, then we must have $\beta(G) \ge |E_2|$. ■

Note that if n is even in the above lemma, then we have $|E_2| = \frac{n}{2}$ for every value of k. The following result shows that if C_n is k-prime then either n is even or k is odd.

Corollary 2.2 C_{2n+1} is not k-prime for all even positive integer k .

Proof Comes directly from Lemma 3.1.1, since $\beta(C_{2n+1}) = n < |E_2| = n+1$. ■

From the above result we may deal with the k-prime labeling of cycles when n is odd and k is odd or when k is even and for every k.

Theorem 2.3 C_{2^n+1} is k-prime if and only if k is odd.

Proof Necessity comes from Corollary 3.1.2. For sufficiency, label the vertices of the cycle consecutively by the labels $k, k+1, \ldots, k+2^n$ where k is the label of u_1 and $k+2^n$ is the label of u_{2^n+1}. Since $(k, k+2^n) = (k, 2^n) = 1$, as k is odd. Thus C_{2^n+1} is k-prime.■

Theorem 2.4 C_4 is k-prime if and only if $k \not\equiv 0 (\text{mod} 3)$.

Proof Necessity, let $k \equiv 0(\text{mod}3)$, the set of vertex labels is: $\{k, k+1, k+2, k+3\}$ so , we put the labels k and k+3 on non adjacent vertices of C_4. We check the other two nonadjacent vertices: We note one of the labels of the set $\{k, k+3\}$ is odd and one is even and as k, k+1 are two consecutive integers. So, none of the remaining vertices cannot assign the even label from the set $\{k+1, k+2\}$. Hence C_4 is not k-prime. For sufficiency, let

$k \not\equiv 0(\text{mod}3)$. Define $f: V(C_4) \longrightarrow \{k, k+1, k+2, k+3\}$ as follow:

$f(u_i) = k - 1 + i, \quad 1 \le i \le 4$

We have to show that the labeling works:

Since, $(k, k+3) = (k, 3) = 1$ since $k \not\equiv 0 \pmod 3$. ■

Theorem 2.5 The comb $P_n \odot K_1$ is k-prime for every positive integer k.

Proof Define $f : V(P_n \odot K_1) \to \{k, k+1, \ldots, k+2n-1\}$ as in the following two cases according to parity of k.

Case 1. k is even:

$$f(u_i) = k + 2i - 1, \qquad 1 \le i \le n$$
$$f(v_j) = k + 2(j-1), \quad 1 \le j \le n.$$

Case 2. k is odd:

$$f(u_i) = k + 2(i-1), \quad 1 \le i \le n$$
$$f(v_j) = k + 2j - 1, \quad 1 \le j \le n.$$

■

Theorem 2.6 $C_n \odot K_1$ is 2^i-prime for all $n \ge 3$ and $i \ge 1$.

Proof Put $k = 2^i$. Define $f : V(C_n \odot K_1) \to \{k, k+1, \ldots, k+2n-1\}$ as follows:

$$f(u_1) = k$$
$$f(u_i) = k + 2i - 3, \qquad 2 \le i \le n$$
$$f(v_1) = k + 2n - 1$$
$$f(v_j) = k + 2(j-1), \quad 2 \le j \le n.$$

■

The following result enlarge the class of crowns that have k-prime labeling.

Theorem 2.7 If p is an odd prime, then $C_n \odot K_1$ is p^i-prime for all $i \ge 1$ and $n \not\equiv 1 (\mathrm{mod}\, p)$.

Proof Let $f : V(C_n \odot K_1) \longrightarrow \{p^i, \ldots, 2n + p^i - 1\}$. We have two cases:

Case 1: $n \not\equiv 1 (\mathrm{mod}\, p)$

Define f as follow:

$$f(u_i) = 2i + (p^i - 2), \qquad 1 \le i \le n$$

$f(v_j) = 2j + (p^i - 1), \quad 1 \le j \le n$.

Case 2: $n \equiv 1(\mathrm{mod}\, p)$

Define f as follow:

$$f(u_i) = 2i + (p^i - 2), \quad 1 \le i \le n-2$$

$$f(u_n) = 2n + (p^i - 4)$$

$$f(u_{n-1}) = 2n + (p^i - 2)$$

$$f(v_j) = 2j + (p^i - 1), \quad 1 \le j \le n-2$$

$$f(v_n) = 2n + (p^i - 3).$$

$f(v_{n-1}) = 2n + (p^i - 1)$. ■

We investigate the k-prime labeling of wheels W_n for some value of n and for fixed value integer k. In the wheels $W_n = C_n + K_1$, the vertex corresponding to K_1 is called the *apex vertex* and is denoted by u_0, while the vertices corresponding to cycle C_n are called the *rim vertices* and are denoted by $u_1, u_2, \ldots, u_n$ where u_i is adjacent to u_0 for each $1 \le i \le n$. However we have the following results.

Lemma 2.8 W_{2n+1} is not k-prime for all $k \ge 1$ and $n \ge 1$.

Proof Comes straightforward from Lemma 2.1, since the order of the wheel W_{2n+1} is even and $\beta(W_{2n+1}) = n < |E_2| = \dfrac{2n+2}{2} = n+1$ for any k, where E_2 is the set of even vertex labels, then W_{2n+1} is not k-prime.■

Lemma 2.9 W_{2n} is not k-prime for all $n \ge 2$ and for every even positive integer k.

Proof Comes straightforward from Lemma 2.1, since k is even and $\beta(W_{2n}) = n < |E_2| = \left\lceil \dfrac{2n+1}{2} \right\rceil = n+1$, where E_2 is the set of even vertex labels, W_{2n} is not k-prime.■

Theorem 2.10 W_n is k-prime if and only if $k \equiv 1(\mathrm{mod}\, 6)$.

Proof Necessity, we have two cases

Case 1: $k \equiv 0, 2 \text{ or } 4 (mod\ 6)$

By Lemma 2.2, W_n is k-prime.

Case 2: $k \equiv 3 \text{ or } 5 (mod\ 6)$

Suppose W_n is k-prime, then we must label two independent rim vertices of the wheel by the even labels $k+1$ and $k+3$. Then we can not find a vertex of the wheel to put the vertex label k in case $k \equiv 3(\bmod 6)$ or to put the vertex label $k+5$ in case $k \equiv 5(\bmod 6)$, a contradiction and hence the wheel is not k-prime in this case.

For sufficiency, let $k \equiv 1(\bmod 6)$, we define a function $f: V\left(W_n\right) \longrightarrow \{k, k+1, \dots, k+4\}$ as follow:

$$f(u_0) = k$$

$$f(u_i) = k + i \ , \qquad 1 \le i \le 4$$

We have to show that the labeling works:

For $1 \le i \le 4$,

- $\left(f(u_0), f(u_i)\right) = (k, k+i) = (k, i) = (6t+1, i) = 1.$
- $\left(f(u_1), f(u_4)\right) = (k+1, k+4) = (k+1, 3) = 1.$

  as $k \equiv 1(\bmod 6)$. ■

3. k-prime labeling of helms

In this section we investigate the k-prime labeling of helms for some values of k. The helm H_n is the graph obtained from the wheel $W_n = C_n + K_1$ ($n \ge 3$) by attaching a pendant edge at every vertex of the n-cycle. We shall denote to the centre vertex of the helm by u_0, the vertices of the n-cycle by $u_1, u_2, \dots, u_n$ and the pendant edges by $v_1, v_2, \dots, v_n$ where u_i is adjacent to v_i for each $1 \le i \le n$. Although the wheel W_{2n+1} is not $2k$-prime for all positive integer k, we did not find yet positive integers $k \ge 2$ and $n \ge 3$ for which a helm H_n is not k-prime and we conjecture that the helm H_n is k-prime for every positive integer k. Seoud and Youssef [10] showed that H_n is prime for all $n \ge 3$. However, we have the following results.

Theorem 3.1 H_n is 2-prime for all $n \ge 3$.

Proof Let $f: V(H_n) \to \{2, 3, \dots, 2n+2\}$ be a function. We have two cases:

<u>**Case 1**</u>: $n \not\equiv 1(mod 3)$

Define f as follows:

$$f(u_0) = 2$$

$$f(u_i) = 2i + 1 \quad , 1 \le i \le n$$

$$f(v_j) = 2j + 2 \quad , 1 \le j \le n$$

<u>**Case 2**</u> : $n \equiv 1(mod 3)$

Define f as follows:

$$f(u_0) = 2$$

$$f(u_i) = 2i + 1 \ ,1 \leq i \leq n - 2$$

$$f(u_n) = 2n - 1$$

$$f(u_{n-1}) = 2n + 1$$

$$f(v_i) = 2i + 2 \quad ,1 \leq i \leq n$$

Cleary f is injective function in both cases. It can be easily verified that f is a 2-prime labeling of H_n . ■

Theorem 3.2 H_n is 3-prime for all $n \geq 3$.

Proof Let $f\colon \mathrm{V(H_n)} \rightarrow \{3,4,\dots,2n+3\}$ be a function. We have two cases:

Case 1 : $n \equiv 0 (mod 3)$

In this case define as follow:

$$f(u_0) = 4$$

$$f(u_i) = 2i + 1 \ \ ,1 \leq i \leq n$$

$$f(v_j) = 2j + 2 \ \ ,2 \leq j \leq n - 1$$

$$f(v_1) = 2n + 2 \quad , \quad f(v_n) = 2n + 3$$

We have to show that the labeling works:

- $\big(f(u_i), f(u_0)\big) = (2i + 1,4) = 1 \, for \, each \ 1 \leq i \leq n$
- $(f(u_i), f(v_i) = (2i + 1,2i + 2) = (2i + 1,1) = 1 \, ,2 \leq i \leq n - 1$
- $(f(u_1), f(v_1) = (3,2n + 2) = \big(3, (n + 1)\big) = 1$
- $\big(f\big(u_n\big), f(v_n)\big) = (2n + 1,2n + 3) = 1$

Case 2 : $n \not\equiv 0 (mod 3)$

Define f as follows:

$$f(u_0) = 4$$

$$f(u_i) = 2i + 1 \ ,1 \leq i \leq n$$

$$f(v_j) = 2j + 2 \ ,2 \leq j \leq 2$$

$$f(v_1) = 2n + 3$$

Again, this labeling is works since moreover:

- $\big(f(v_1), f(u_1)\big) = (3,2n + 3) = (3,2n) = (3, n) = 1$

as $n \not\equiv 0 (mod 3)$. ■

Theorem 3.3 H_n is 4-prime for all $n \geq 3$.

Proof We have two cases :

<u>**Case 1**</u>: $n \not\equiv 1(mod5)$

Define a function $f : V(H_n) \to \{4, 5, \ldots, 2n + 4\}$ as follows:

$$f(u_0) = 4$$

$$f(u_i) = 2i + 3 \ ,1 \leq i \leq n$$

$f(v_j) = 2j + 4 \quad ,1 \leq j \leq n.$

Clearly f is injective function. We show that all adjacent vertices receive relatively prime labels:

- $(f(u_0), f(u_i) = (4,2i + 3) = 1$
- $(f(u_i), f(v_i)) = (2i + 3,2i + 4) = 1$
- $(f(u_i), f(u_{i+1})) = (2i + 3,2i + 5) = (2i + 3,2) = 1 \ ,1 \leq i \leq n - 1$
- $(f(u_1), f(u_n)) = (5,2n + 3) = (5,2n - 2) = (5, n - 1) = 1$

<u>**Case 2**</u>: $n \equiv 1(mod5)$

Define a function $f : V(H_n) \to \{4, 5, \ldots, 2n + 4\}$ as follows:

$$f(u_0) = 4$$

$$f(u_i) = 2i + 3 \ ,1 \leq i \leq n - 2$$

$$f(u_{n-1}) = 2n + 3$$

$$f(u_n) = 2n + 1$$

$f(v_i) = 2i + 4 \ , \ 1 \leq i \leq n.$

Clearly f is injective function. Again, it is straightforward to verify that all adjacent vertices receive relatively prime labels. ■

Theorem 3.4 H_n is 5-prime for all $n \geq 3$.

Proof We have three cases :

<u>**Case 1:**</u> $n \equiv 0(mod5)$

Define $f: \mathrm{V}(\mathrm{H_n}) \to \{5,6, \ldots ,2n + 5\}$ as follow:

$$f(u_0) = 8$$

$$f(u_i) = 2i + 5 \ , \qquad 1 \leq i \leq n - 2$$

$$f(u_n) = 2n + 3$$

$$f(u_{n-1}) = 2n + 5$$

$$f(v_j) = 2j + 6 \ ,2 \leq j \leq n - 1$$

$$f(v_1) = 6$$

$$f(v_n) = 5$$

We have to show that the labeling works:

- As $f(u_i)$ is odd for each $1 \leq i \leq n$, then $\big(f(u_i), f(u_0)\big) = (f(u_i), 8) = 1$.
- $(fu_i), f(u_{i+1})) = 1\ , 1 \leq i \leq n-1$.
- $\big(f(u_n), f(v_n)\big) = (2n+3, 5) = (5, n-1) = 1$

<u>**Case 2**</u>: $n \equiv 1(mod5)$

$$f(u_0) = 8$$

$$f(u_i) = 2i + 3\ , 1 \leq i \leq n-2$$

$$f(u_{n-1}) = 2n + 3$$

$$f(u_n) = 2n + 1$$

$$f(v_1) = 2n + 4$$

$$f(v_2) = 6$$

$$f(v_j) = 2j + 4\ , 3 \leq j \leq n-1$$

$$f(v_n) = 2n + 5$$

We have to show that the labeling works:

- As in case 1, $\big(f(u_i), f(u_0)\big) = 1\ , 1 \leq i \leq n$.
- $\big(f(u_1), f(u_n)\big) = (5, 2n+1) = (5, 2n-4) = (5, n-2) = 1$.
- $\big(f(u_1), f(v_1)\big) = (5, 2n+4) = (5, 2n-1) = 1$.

<u>**Case3**</u>: $n \not\equiv 0\ and\ 1(mod5)$

$$f(u_0) = 8$$

$$f(u_i) = 2i + 3\ , 1 \leq i \leq n$$

$$f(v_j) = 2j + 4\ , 3 \leq j \leq n$$

$$f(v_2) = 5$$

$$f(v_1) = 2n + 5$$

We must show that the labeling works:

- $(f(u_1), f(u_n) = (5, 2n+3) = (5, 2n-2) = (5, n-1) = 1, as\ n \not\equiv 1(mod5)$.
- $\big(f(u_1), f(v_1)\big) = (5, 2n+5) = (5, n) = 1\ , as\ n \not\equiv 0(mod5)$. ■

Theorem 3.5 H_n is 6-prime for all $n \geq 3$.

Proof We have two cases:

Case 1: $n \equiv 1(mod7)$

Define $f: \mathrm{V}(\mathrm{H_n}) \to \{6,7,\dots,2n+6\}$ as follow:

$$f(u_0) = 8$$

$$f(u_i) = 2i + 5 \;, \qquad 1 \le i \le n-2$$

$$f(u_n) = 2n + 3$$

$$f(u_{n-1}) = 2n + 5$$

$$f(v_j) = 2j + 6 \;, 2 \le j \le n$$

$$f(v_1) = 6.$$

Case 2: $n \not\equiv 1(mod7)$

$$f(u_0) = 8$$

$$f(u_i) = 2i + 5 \;, 1 \le i \le n$$

$$f(v_1) = 6$$

$$f(v_j) = 2j + 6 \,, 2 \le j \le n \,. \quad \blacksquare$$

Theorem 3.6 H_n is 7-prime for all $n \ge 3$.

Proof We have three cases:

Case 1: $n \equiv 1(mod7)$

Define $f: \mathrm{V}(\mathrm{H_n}) \to \{7,8,\dots,2n+7\}$ as follow:

$$f(u_0) = 8$$

$$f(u_i) = 2i + 5 \;, \qquad 1 \le i \le n-2$$

$$f(u_n) = 2n + 3$$

$$f(u_{n-1}) = 2n + 5$$

$$f(v_j) = 2j + 6 \;, 2 \le j \le n-1$$

$$f(v_1) = 2n + 6$$

$$f(v_n) = 2n + 7$$

Case 2: $n \equiv 0(mod7)$

$$f(u_0) = 8$$

$$f(u_i) = 2i + 5 \;, 1 \le i \le n$$

$$f(v_1) = 2n + 6$$

$$f(v_j) = 2j + 6 \,, 2 \le j \le n-1$$

$$f(v_n) = 2n + 7$$

Case 3: $n \not\equiv 0,\ 1(mod7)$

$$f(u_0) = 8$$

$$f(u_i) = 2i + 5\ ,1 \le i \le n$$

$$f(v_j) = 2j + 6\ ,2 \le j \le n$$

$$f(v_1) = 2n + 7.\ \blacksquare$$

Theorem 3.7 H_n is 8-prime for all $n \ge 3$.

Proof We have two cases:

Case 1: $n \equiv 1(mod3)$

Define $f: \mathrm{V(H_n)} \rightarrow \{8,9, \dots ,2n + 8\}$ *as follows*:

$$f(u_0) = 8$$

$$f(u_i) = 2i + 7\ ,1 \le i \le n - 2$$

$$f(u_n) = 2n + 5$$

$$f(u_{n-1}) = 2n + 7$$

$$f(v_j) = 2j + 8$$

Case 2: $n \not\equiv 1(mod3)$

$$f(u_0) = 8$$

$$f(u_i) = 2i + 7\ ,1 \le i \le n$$

$$f(v_j) = 2j + 8\ \blacksquare$$

Theorem 3.8 H_n is 9-prime for all $n \ge 3$.

Proof We have three cases:

Case 1: $n \equiv 0(mod3)$

a) If $n > 3$:

Define $f: \mathrm{V(H_n)} \rightarrow \{9,10, \dots ,2n + 9\}$ as follow:

$$f(u_0) = 16$$

$$f(u_i) = 2i + 7\ , \quad 1 \le i \le n$$

$$f(v_1) = 2n + 8$$

$$f(v_j) = 2j + 6\ ,2 \le j \le 4$$

$$f(v_j) = 2j + 8\ , \quad 5 \le i \le n - 1$$

$$f(v_n) = 2n + 9$$

This labeling is work since:

- $(9,2n+8)=(9,n+4)=1$.

b) If $n=3$:

Define $f:V(H_3)\to\{9,10,\ldots,15\}$ as follows:

$$f(u_0)=13$$

c) $f(u_1)=9,\quad f(u_2)=11,\quad f(u_3)=14$

$$f(v_1)=10,\quad f(u_2)=12,\quad f(u_3)=15.$$

__Case 2__: $n\equiv 2(mod3)$

The same as in Case 1(a) except for $f(v_1), f(v_n)$.

Define $g:\mathrm{V(H_n)}\to\{9,10,\ldots,2n+9\}$ as follow:

$$g(v)=f(v),\qquad \forall\, v\in V(H_n): v\neq v_1, v_n .$$

$$g(v_1)=2n+9$$

$$g(v_n)=2n+8$$

__Case 3__: $n\equiv 1(mod3)$

$$f(u_0)=16$$

$$f(u_i)=2i+7\ ,1\le i\le n-2$$

$$f(u_{n-1})=2n+7$$

$$f(u_n)=2n+5$$

$$f(v_1)=2n+9$$

$$f(v_j)=2j+6\ ,2\le j\le 4$$

$$f(v_j)=2j+8\ ,5\le j\le n$$

This labeling is work since:

- (9,2n+9)=(9,2n)=(9,n)=1 , as n=1 (mod3).
- (9,2n+5)=(9,2n-4)=(9,n-2)=1
- (2n+5,2n+8)=(2n+5,3)=1. ■

Theorem 3.9 H_n is 10-prime for all $n\geq 3$.

Proof We have two cases:

Case 1: $n \not\equiv 1 (\bmod 11)$

Define $f : V(H_n) \to \{10, 11, ..., 2n+10\}$ as follow:

$$f \; u_0 \; = 16$$

$$f(u_i) = 2i + 9 \quad , 1 \le i \le n$$

$$f(v_j) = 2j + 8 \; , 1 \le j \le 3$$

$$f(v_j) = 2j + 10 \quad , 4 \le j \le n$$

This labeling is work since:

- $(f(u_1), f(u_n)) = (11, 2n+9) = (11, 2n-2) = (11, n-1) = 1$.
- $(f(u_i), f(v_i)) = (2i+9, 2i+8) = 1, \; 1 \le j \le 3$.
- $(f(u_i), f(v_j)) = (2i+9, 2i+10) = 1, \; 4 \le i \le n.$

Case 2: $n \equiv 1 (\bmod 11)$

$f(u_o) = 16$

$$f(u_i) = \begin{cases} 2i+9 \;, & 1 \le i \le n-2 \\ 2n+9 \;, & i = n-1 \\ 2n+7 \;, & i = n \end{cases}$$

$$f(v_j) = \begin{cases} 2j+8 \;, & 1 \le j \le 3 \\ 2j+10 \;, & 4 \le j \le n-2 \\ 2n+10, & j = n-1 \\ 2n+8 \;, & j = n \end{cases}$$

This labeling is work since:

- $(f(u_1), f(u_n)) = (11, 2n+7) = (11, 2n-4) = (11, n-2) = 1$

- $(f(u_i), f(v_i)) = \begin{cases} (2i+9, 2i+8) = 1, & 1 \le i \le 3 \\ (2i+9, 2i+10) = 1, & 4 \le i \le n-2 \\ (2n+9, 2n+10) = 1, & i = n-1 \\ (2n+7, 2n+8) = 1, & i = n \end{cases}$

- $(f(u_n), f(u_{n-1})) = (2n+7, 2n+9) = (2n+7, 2) = 1.$
- $(f(u_{n-2}), f(u_{n-1})) = (2n+5, 2n+9) = (2n+5, 4)) = 1.$ ■

Theorem 3.10 H_n is 11-prime for all $n \geq 3$.

Proof We have three cases :

Case 1: $n \equiv 0(mod11)$

Define $f: V(H_n) \to \{11,12, \dots, 2n+11\}$ as follow:

$$f(u_0) = 16$$

$$f(u_i) = 2i + 9 \ , \quad 1 \leq i \leq n$$

$$f(v_j) = 2j + 8 \, , \quad j = 2,3$$

$$f(v_j) = 2j + 10 \, , \quad 4 \leq j \leq n-1$$

$$f(v_1) = 2n + 10$$

$$f(v_n) = 2n + 11$$

This labeling is work since:

- $(f(u_1), f(u_n)) = (11, 2n+9) = (11, 2n-2) = (11, n-1) = 1$
- $(f(u_1), f(v_1)) = (11, 2n+10) = (11, 2n-1) = 1$

Case 2: $n \equiv 1(mod11)$

$$f(u_0) = 16$$

$$f(u_i) = 2i + 9 \ , \quad 1 \leq i \leq n-1$$

$$f(u_n) = 2n + 11$$

$$f(v_j) = 2j + 8 \, , \quad j = 2,3$$

$$f(v_j) = 2j + 10 \, , \quad 4 \leq j \leq n-1$$

$$f(v_1) = 2n + 10$$

$$f(v_n) = 2n + 9$$

This labeling is work since:

- $(f(u_1), f(u_n)) = (11, 2n+11) = (11, 2n) = (11, n) = 1$
- $(f(u_1), f(v_1)) = (11, 2n+10) = (11, 2n-1) = 1$
- **Case 3**: $n \not\equiv 0 \ ,1(\mathrm{mod}\,11)$

$$f(u_0) = 16$$

$$f(u_i) = 2i + 9 \quad , 1 \le i \le n$$

$$f(v_j) = 2j + 8 \ , \ j = 2,3$$

$$f(v_j) = 2j + 10 \ , \ 4 \le j \le n$$

$$f(v_1) = 2n + 11$$

This labeling is work since:

- $(f(u_1), f(u_n)) = (11, 2n+9) = (11, 2n-2) = (11, n-1) = 1$, as $n \not\equiv 1 (\text{mod} 11)$.
- $(f(u_1), f(v_1)) = (11, 2n+11) = (11, n) = 1$, as $n \not\equiv 0 \ (\text{mod} 11)$.

Theorem 3.11 If $k + 2(n-1)$ and $k + 2n$ are twin primes where $n \ge 3$ and $k \ge 1$, then H_n is k-prime.

Proof Clearly, k must be an odd integer, define

$f : V(H_n) \to \{k, k+1, ..., k+2n\}$ as follows:

$$f(u_i) = k + 2(i-1) \quad , 1 \le i \le n$$

$$f(u_0) = k + 2n$$

$$f(v_j) = k + 2(j-1) + 1 = k + 2j - 1 \quad , 1 \le j \le n$$

$|f(u_i) - f(v_i)| = 1 \Rightarrow (f(u_i), f(v_i)) = 1$. ■

Our previous results about the k-prime labeling of the helms for $2 \le k \le 11$ motivated us to arise the following conjecture.

Conjecture All helms are k-prime for every positive integer k.

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