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Numerical Simulation of Partial Differential Equations Using Finite Difference Methods

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ABSTRACT

This paper explores the use of the software package, Matlab and Excel in the implementation of the finite difference method to solve partial differential equations (PDE's.). It aimed to examine the strength of the forward explicit method and backward implicit method in solving PDE's. A comparison was made between the forward explicit method and the backward implicit method for their stability. The FDM method was used to solve partial differential equations of heat. Numerical examples were also created and analysed to show the strengths of each method. The results shows that the forward explicit method is conditionally stable because the stability it requires a small step size of time 't' compared to space 'x' for stability. The backward implicit method is unconditionally stable because it depends on the local truncation error considerations.

KEYWORDS: Finite difference, Convergences, Truncation error, Partial differential equations, Numerical Simulation.

1. INTRODUCTION

Many years ago, numerical simulation of engineering problems has been successfully implemented (de Campos, 2014). In recent years, the study of fractional derivatives has added a significant improvement in both ordinary and partial differential equations.

The idea of fractional calculus is considered as a situation of widespread activity in the fields of physics, mathematics, and engineering (Baleanu et al., 2010 and Uchaikin, V. V., 2013). The method allows us to solve various problems of applied mechanics. Due to successful solutions of the difficulties discussed, in particular, for problems with physical or geometrical nonlinearities, optimization and time and/or temperature dependence the forward difference method at irregular meshes is customary ample to be aggressive with the finite element approach (Liszka and Orkisz, 1980**).** Many time-dependent partial differential equations are observed combining low-order nonlinear with higher-order linear terms (Owolabi and Patidar, 2016). They in additional discretized the governing models in space with the usage of a fourth-order central finite difference scheme and integrating the resulting ODEs with the exponential time differencing schemes whose formulations were totally Runge-Kutta based and multistep techniques of Adams-type.

A considerable number of physical problems are mathematically modelled by the systems of differential equations (Owolabi, 2017). An active research undertaken in this paper is nothing but finding efficient and accurate methods to numerically simulate partial differential equations. Among many authors that have studied the numerical simulations of such problems can be found in Bhrawy, et al. (2015) and Owolabi and Atangana, (2016).

2. FINITE DIFFERENCE APPROXIMATIONS

The finite difference method involves using discrete approximation like

$$\frac{\partial \phi}{\partial x} \approx \frac{\phi_{i+1} - \phi_i}{\Delta x}$$

Where $\frac{\partial \phi}{\partial x}$ are define on the finite dfference mesh. Approximation to the behaviour of differential equation is obtained by replacing derivatives formulas such as those in the above equation. Observe that finding ϕ_i^m from the finite difference model is a distinct stage from translating the continuous problem to the discrete problem.

Finite difference formulas are initially developed with the dependent variable ϕ as a function of only one independent variable,x i.e $\phi = \phi(x)$ the formula obtained are the use to approximate derivatives with respect to either space or time.

2.1 First order Forward Difference

Consider a Taylor series expansion of $\phi(x)$ about the point x_i

$$\phi(x_i + \delta x) = \phi(x_i) + \delta x \frac{\partial \phi(xi)}{\partial x} \quad + \delta x^2 \frac{\partial \phi^2(x_i)}{\partial x^2}$$
$$+ \delta x^3 \frac{\partial^3 \phi(x_i)}{\partial x^3} + \cdots \qquad (2)$$

Where δx is a change in x relative to x_i . Let $\delta x = \Delta x$ in equation (2)

Consider the value of ϕ at the location of x_{i+1} mesh line.

$$\phi(x_i + \Delta x) = \phi(x_i) + \Delta x \frac{\partial \phi(xi)}{\partial x} \quad + \Delta x^2 \frac{\partial^2 \phi(x_i)}{\partial x^2}$$
$$+ \Delta x^3 \frac{\partial^3 \phi(x_i)}{\partial x^3} + \cdots$$

Solve for $(\frac{\partial \phi}{\partial x}) x_i$

$$\frac{\partial \phi}{\partial x} = \frac{\phi(x_i + \Delta x) - \phi(x_i)}{\Delta x} - \frac{\Delta x}{2} \frac{\partial^2 \phi}{\partial x^2} - \frac{\Delta x^2}{3!} \frac{\partial^3 \phi}{\partial x} + \cdots$$

Observe that the powers of Δx multiplying the partial derivatives on the right hand side have been reduced by one.

Replacing the approximate solution for exact solution,

That is, use $\approx \phi(x_i)$ and $\phi_{i+1} \approx \phi(x_i + \Delta x)$

$$\frac{\partial \phi(x_i)}{\partial x} = \frac{\phi_{i+1} - \phi_i}{\Delta x} - \frac{\Delta x}{2} \frac{\partial^2 \phi}{\partial x^2} - \frac{\Delta x^2}{3!} \frac{\partial^3 \phi}{\partial x^3} + \cdots \qquad (3)$$

The mean value theorem can be used to replace the higher order derivatives.

$$\frac{\Delta x^2}{2} \frac{\partial^2 \phi(x_i)}{\partial x^2} + \frac{\Delta x^3}{3!} \frac{\partial^3 \phi(x_i)}{\partial x^3} + \cdots = \frac{\Delta x^2}{2} \frac{\partial^2 \phi(\xi)}{\partial x^2}$$

Where $x_i \leq \xi \leq x_{n+1}$ known

$$\Rightarrow \frac{\partial \phi(x_i)}{\partial x} \approx \frac{\phi_{i+1} - \phi_i}{\Delta x} + \frac{\Delta x^2}{2} \frac{\partial^2 \phi(\xi)}{\partial x^2}$$

$$or \ \frac{\partial \phi(x_i)}{\partial x} - \frac{\phi_{i+1} - \phi_i}{\Delta x} \approx \frac{\Delta x^2}{2} \frac{\partial^2 \phi(\xi)}{\partial x^2} \qquad (4)$$

The term on the right-hand side (R.H.S) of equation of (4) is the truncation error of finite difference approximation. It is the resulting error from truncating the series in equation (3).

Generally ξ is not known. Moreover since the function $\phi(x,t)$ is also unknown, $\frac{\partial^2 \phi}{\partial x^2}$ cannot be computed. Despite the fact the magnitude of the truncation error cannot be known. The big 'O' notation can be used to express the dependence of the truncation error on the mesh spacing.

Observe that the right hand side of the equation (4) contain mesh Parameter, Δx , using the finite difference simulation parameter is choosing by the individual.

The truncation error is written as

$$\frac{\Delta x^2}{2} \frac{\partial^2 \phi(\xi)}{\partial x^2} = O(\Delta x^2)$$

The above equation implies that the left hand side is a product of an unknown and Δx^2. Although the expression does not give us the exact magnitude of $\frac{\Delta x^2}{2} \frac{\partial^2 \phi(\xi)}{\partial x^2}$, it gives us an idea on how quickly the term approaches zero as Δx is reduced.

Equation (4) can be written, in terms of the big 'O' notation

$$\frac{\partial \phi(x_i)}{\partial x} = \frac{\phi_{i+1} - \phi_i}{\Delta x} + O(\Delta x) \qquad (5)$$

The resulting equation (5) is called the forward difference formula for $\frac{\partial \phi(x_i)}{\partial x}$ because it involves nodes x_i and x_{i+1} . The forward difference approximation has a truncation error that is $O(\Delta x)$. Mostly the part of the truncation error is under control because we can choose the mesh size Δx.while the $\left|\frac{\partial \phi}{\partial x}\right|$ is not under control.

2.2 First order Backward Difference

Another way to find the first order if the Taylor series in the equation 2 is written with

$$\delta x = -\Delta x.$$

by using the discrete mesh variables in place of the unknowns we get

$$\phi_{i-1} = \phi_i - \Delta x \frac{\partial \phi(x_i)}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 \phi(x_i)}{\partial x^2} -$$
$$\frac{\Delta x^3}{3!} \frac{\partial^3 \phi(x_i)}{\partial x^3} + \cdots$$

Solving $\frac{\partial \phi(x_i)}{\partial x}$ to get

$$\frac{\partial \phi(x_i)}{\partial x} = \frac{\phi_{i+1} - \phi_i}{\Delta x} + \frac{\Delta x}{2} \frac{\partial^2 \phi(x_i)}{\partial x^2} - \frac{\Delta x^2}{3!} \frac{\partial^3 \phi(x_i)}{\partial x^3} + \cdots$$

Using the big 'O' notation

$$\frac{\partial \phi(x_i)}{\partial x} = \frac{\phi_i - \phi_{i-1}}{\Delta x} + O(\Delta x) \qquad (6)$$

The resulting formula is called the backward difference because of ϕ at x_i and x_{i-1}. The order of magnitude of the truncation error for the backward difference approximation is the same as that of the forward difference approximation. We can find the first order difference formula for $\frac{\partial \phi(x_i)}{\partial x}$ with smaller truncation error.

2.3 First order central difference

Writing the Taylor series expansions for x_{i+1} and x_{i-1}

$$\phi_{i+1} = \phi_i + \Delta x \frac{\partial \phi(x_i)}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 \phi(x_i)}{\partial x^2} + \frac{\Delta x^3}{3!} \frac{\partial^3 \phi(x_i)}{\partial x^3} + \cdots \qquad (7)$$

$$\phi_{i-1} = \phi_i - \Delta x \frac{\partial \phi(x_i)}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 \phi(x_i)}{\partial x^2} - \frac{\Delta x^3}{3!} \frac{\partial^3 \phi(x_i)}{\partial x^3} + \cdots \qquad (8)$$

Subtracting (8) from (7), we have

$$\phi_{i+1} - \phi_{i-1} = 2 \Delta x \frac{\partial \phi(x_i)}{\partial x} + 2 \frac{\Delta x^3}{3!} \frac{\partial^3 \phi(x_i)}{\partial x^3} + \cdots$$

Solving for $\frac{\partial \phi(x_i)}{\partial x}$ gives

$$\frac{\partial \phi(x_i)}{\partial x} = \frac{\phi_{i+1} - \phi_{i-1}}{2 \Delta x} - \frac{\Delta x^3}{3!} \frac{\partial^3 \phi(x_i)}{\partial x^3} + \cdots$$

Or

$$\frac{\partial \phi(x_i)}{\partial x} = \frac{\phi_{i+1} - \phi_{i-1}}{2 \Delta x} + O(\Delta x^2) \qquad (9)$$

Equation (9) is the central difference approximation *to* $\frac{\partial \phi(x_i)}{\partial x}$ .In order to obtain a very good approximation to the continuous problem, we choose small value for Δx. the truncation error goes to zero much faster than the truncation error in equation (5). Equation (9) involve complication because it does not include the value for ϕ_i . When the central difference approximation is included in an approximation to a differential equation it may cause problem.

2.4 Second Order Central Difference

Finite difference approximations to higher order derivatives can be evaluated with the additional manipulations of the Taylor series expansion about $\phi(x_i)$. Adding equation (7) and equation (8) yields

$$\phi_{i+1} + \phi_{i-1} = 2\phi_i + \delta x^2 \frac{\partial^2\phi(x_i)}{\partial x^2} + 2\frac{\delta x^4}{4!}\frac{\partial^4\phi(x_i)}{\partial x^4} + \cdots$$

Solving for $\frac{\partial^2\phi(x_i)}{\partial x^2}$,

$$\frac{\partial^2\phi(x_i)}{\partial x^2} = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2} + \frac{\delta x^4}{4!}\frac{\partial^4\phi(x_i)}{\partial x^4} + \cdots$$

Or

$$\frac{\partial^2\phi(x_i)}{\partial x^2} = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2} + O(\Delta x^2) \qquad (10)$$

This is called the central approximation to the second derivative, where (9) is the central approximations to the first derivative.

3. APPLICATIONS

Consider $f(x) = sin^2 2\pi x$ with the boundary condition $\phi(0,t) = \phi(1,t) = 0$ for all time.

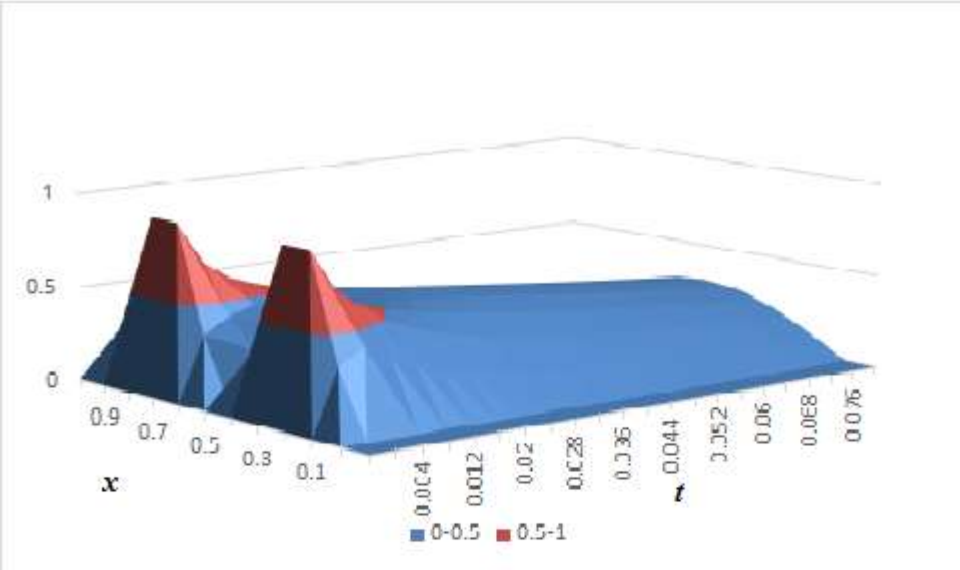

Fig.1 approximate solution of heat equation with step sizes h =0.1, k = 0.004. Method is stable.

With time the initial temperature should diffuse away, resulting to a graph like the one shown in fig 1. In the graph the forward scheme is used with step size h = 0.1 along the rod and k = 0.004 in time. The explicit forward difference method gives the approximate solution. In fig.a, the smooth flow of heat is shown to a near equilibrium after less than one time units. This corresponds to the temperature of the rod $u \to 0$ as $t \to \infty$.

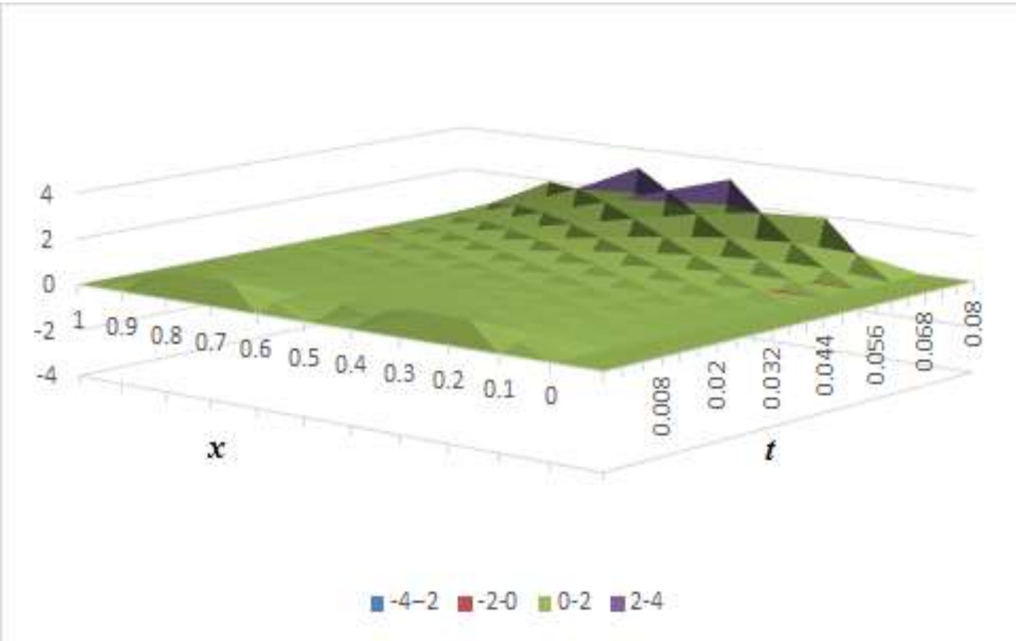

Fig. 2. Approximate solution of heat equation with step sizes h =0.1, k = 0.004. Method is unstable.

In fig. 2 is slightly large time step, at the beginning, the heat bumps start to die down as expected but after additional time steps, small lapses in the estimate get to be amplified by the forward difference method, causing the answer for move far from right equilibrium of zero. This is a sign that the technique is unstable. If the recreation were permitted to continue further, these errors would develop without bound. Therefore, we are forced to keep the time step k somewhat little to ensure convergence.

Applying the backward difference method to heat equation,

$\phi_t = \phi_{xx}$ for all $0 \leq x \leq 1, t \geq 0$

$\phi(x, 0) = sin^2 2\pi x$ for all $0 \leq x \leq 1$

$\phi(0, t) = 0$ for all $t \geq 0$

$\phi(1,0) = 0$ for all $t \geq 0$

Using the step sizes h=k=0.1 we have the approximate solution in figure below

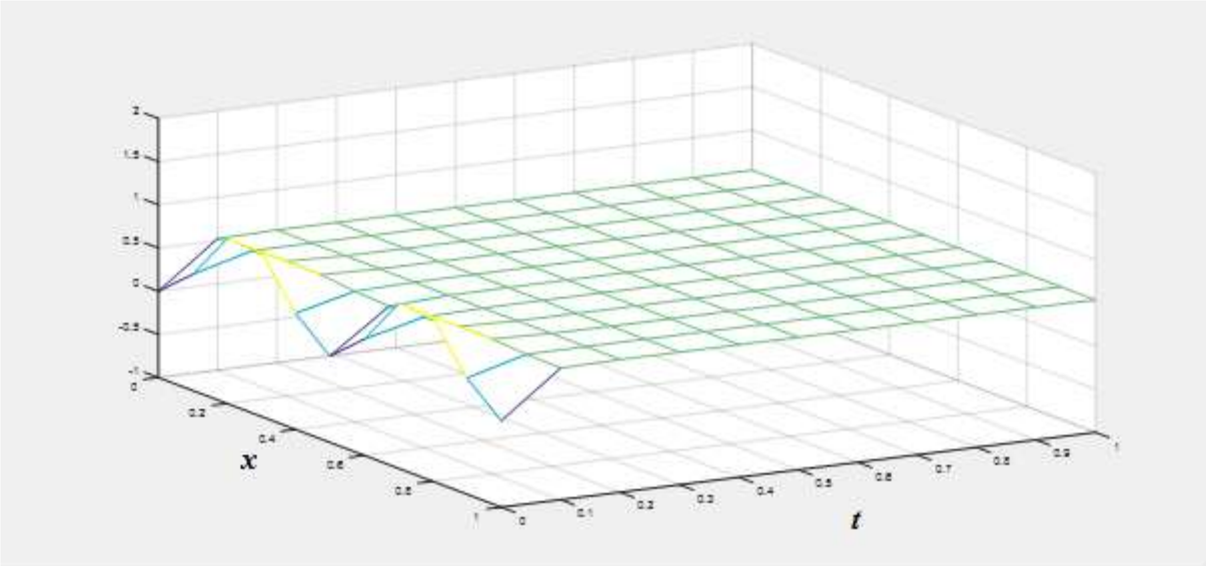

Fig. 3. Approximate solution of example 1 by the backward Difference Method.

The step sizes $h = k = 0.1$ and the diffusion coefficient D = 1.

Applying the backward difference method to the problem in 2. Above. With homogeneous Neumann boundary conditions.

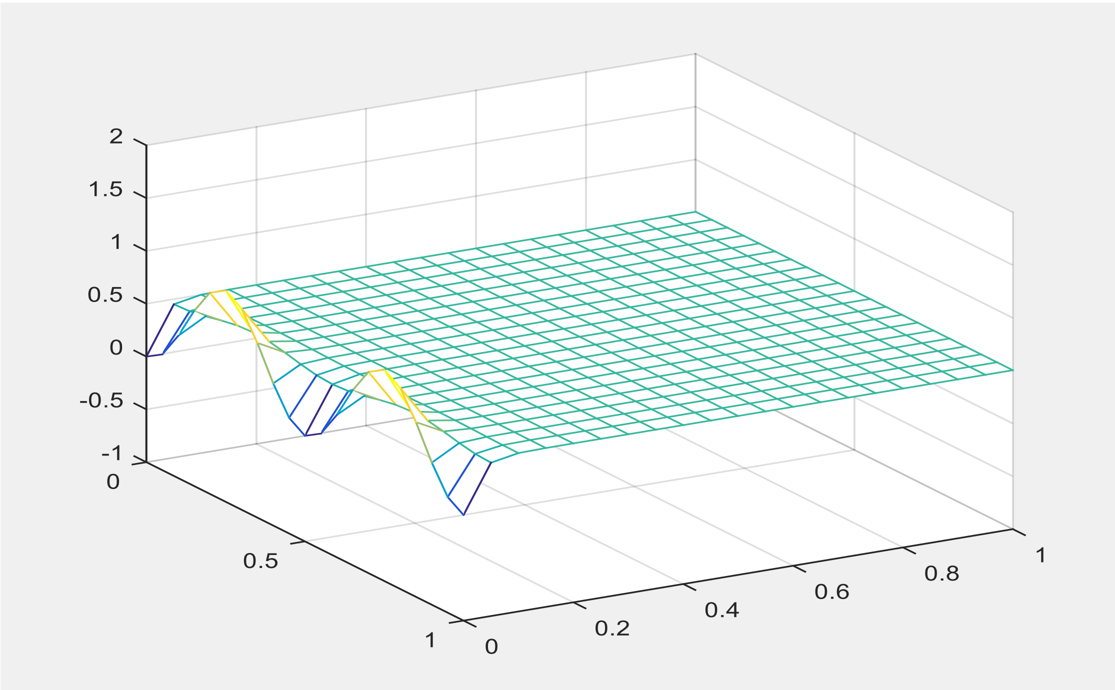

Fig. 4. Approximate solution of example (1) by Backward Difference Method with step sizes $h = k = 0.05$.

Fig.4 shows that the boundary conditions are no longer fixed at zero with Neumann condition.

4. CONCLUSION

In this research we study numerical techniques to solve partial differential equations. The finite difference method was implemented with forward time centred space (FTCS) scheme and numerical examples were also created on excel and Matlab and examined to investigate the accuracy and strength of the method.

The backward time space scheme implemented in Matlab and numerically analysed to investigate the strength and accuracy of this method.

The results revealed that with a numerical analysis, the backward difference method are preferred over the forward difference method since the forward difference method require a very small time step in comparison to space step sizes and will only converge if $\alpha^2 \frac{k}{h^2} \leq \frac{1}{2}$.

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