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# Error Estimates for Galerkin Method to a Type of Partial Differential Equations

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**ABSTRACT**
Discuss Galerkin approximation to a type of second order nonlinear hyperbolic partial differential equations. Provide continuous time Galerkin approximation and fully discrete Galerkin approximations and derive optimal $L^2$-error estimates for continuous time and fully discrete Galerkin Approximations, separately.

**KEYWORDS:** Differential Equations, Galerkin, Approximation, Error Estimates

## I. INTRODUCTION

Galerkin method has been used as a numerical method to solve differential equations for long time [1]. There have been many researches on parabolic equations [2-5], but relatively fewer researches on hyperbolic equations. This paper will derive optimal $L^2$-error estimates for continuous time Galerkin approximation and fully discrete Galerkin approximations for second order nonlinear hyperbolic partial differential equations with nonzero initial and boundary conditions

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \nabla\cdot(a(\boldsymbol{x})\nabla u) + \displaystyle\sum_{i=1}^{n} b_i(\boldsymbol{x},u)u_x + f(\boldsymbol{x},u), \\ \qquad (\boldsymbol{x},t)\in\Omega\times(0,T) \\ u(\boldsymbol{x},0) = \varphi(\boldsymbol{x}), \qquad x\in\Omega \\ \dfrac{\partial u(\boldsymbol{x},0)}{\partial t} = \psi(\boldsymbol{x}), \qquad x\in\Omega \\ -a(\boldsymbol{x})\dfrac{\partial u}{\partial \boldsymbol{n}} = g(\boldsymbol{x},t) + \alpha_1(\boldsymbol{x})u + \alpha_2(\boldsymbol{x})\dfrac{\partial u}{\partial t}, \\ \qquad (\boldsymbol{x},t)\in\partial\Omega\times(0,T) \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded region in $R^n$, $\partial\Omega$ is its smooth boundary and $\boldsymbol{n}$is its exterior normal direction.

The remainder of the paper is organized as follows. Sec. II presents the notations and preconditions. In Sec. III and Sec. IV provide error estimates for continuous and discrete time Galerkin methods, respectively. Sec. V contains a short conclusion of the paper.

## II. PRELIMINARY

Let $H^s(\Omega)$ and $H^s(\partial\Omega)$ be Sobolev spaces [6] with norm $\|\cdot\|_s$ and $|\cdot|_s$ , respectively, and let $\|\cdot\| = \|\cdot\|_0$, $|\cdot| = |\cdot|_0$, $L^2(\Omega) = H^0(\Omega)$, $L^2(\partial\Omega) = H^0(\partial\Omega)$.

In addition, let

$$\|\cdot\|_{H_0^1}^2 = \sum_{i=1}^{n}\int_\Omega |f_{x_i}|^2 d\Omega = \|\Delta f\|^2$$

$$(f,g) = \int_\Omega f\cdot g d\Omega$$

$$\langle f,g\rangle = \int_{\partial\Omega} f\cdot g d\sigma$$

Let $X$ be a normal space with norm $\|\cdot\|_X$. For function $f:[0,T]\to X$, define

$$\|f\|_{L^2(X)}^2 = \int_0^T \|f(t)\|_X^2\, dt; \qquad \|f\|_{L^\infty(X)} = \sup_{0\le t\le T}\|f(t)\|_X$$

We assume condition (A) exists in following discussion.

Condition (A):

Let $d_0, K_i, i = 1,2,3,4,5,$ are constants.

$(i) 0 < d_0 \le a(\boldsymbol{x}) \le K_1, \quad \boldsymbol{x}\in\Omega;$

$\alpha_i(\boldsymbol{x}) \ge 0, \quad \boldsymbol{x}\in\partial\Omega, \quad i = 1,2$

$(ii)$ $a(\boldsymbol{x})$ is smooth enough so that

$(a)$ Bilinear form $a(w,v) = (a\nabla w,\nabla v) + (w,v)$ is 0-regular in $H^1(\Omega)$. That is if $f\in L^2(\Omega), w\in H^1(\Omega)$ and satisfy $a(w,v) = (f,v)$ for any $v\in H^1(\Omega)$, then $w\in H^2(\Omega)$

$(b)$ $\forall g\in H^{\frac{1}{2}}(\partial\Omega)$, the weak solution $v$of elliptic problem

$$\begin{cases} -\nabla\cdot(a\nabla v) + v = 0 & \boldsymbol{x}\in\Omega \\ -a\dfrac{\partial v}{\partial \boldsymbol{n}} = g & \boldsymbol{x}\in\partial\Omega \end{cases}$$

satisfies $\|v\|_2 \le K_2|g|_{\frac{1}{2}}$ .

$(iii)$ $a_i(\boldsymbol{x}), i = 1,2,$ is smooth enough so that

$$|a_i v|_{\frac{1}{2}} \le K_3|v|_{\frac{1}{2}} \qquad \forall\, v\in H^{\frac{1}{2}}(\partial\Omega)$$

$(iv)$ $\partial\Omega$ is smooth enough so that

$$|\cos(\boldsymbol{n},\boldsymbol{x}_i)\,v|_{\frac{1}{2}} \le K_4|v|_{\frac{1}{2}} \qquad \forall\, v \in H^{\frac{1}{2}}(\partial\Omega)$$
$$i = 1,2,\cdots,n.$$

where $\cos(\boldsymbol{n},\boldsymbol{x}_i)$ is directional cosines of exterior normal $\boldsymbol{n}$

$(v)$ For $(x,p) \in \Omega \times R, b_i(x,p), b_{ip}(x,p), b_{ix}(x,p),$ $b_{ipx}(x,p), b_{ipp}(x,p), i = 1,2,\cdots,n, f_p(x,p), f_{pp}(x,p)$ exist and are bounded by $K_5$. In addition, $f(x,0) \in L^2(\Omega)$.

$(vi)$ True solution $u$ of problem (1.1) satisfies

$$u_t, u_{x_i}, u_{tx_i}, u_{tt} \in L^\infty(\Omega \times [0,T]), \quad i = 1,2,\cdots,n,$$

$u, u_t \in L^\infty(H^k(\Omega))$ and $u_{tt} \in L^2(H^k(\Omega))$.

Let finite dimensional space $S^h$ be a $S^h_{1,m}(\Omega)$ $(m > 1)$ space. That means $S^h \in H^1(\Omega)$, and there exists constant $K_6$ such that for $\forall\, v \in H^j(\Omega)$,

$$\inf_{x\in S^h} \|v - x\|_l \le K_6 h^{j-l} \|v\|_j, \quad 0 \le l \le 1, l \le j \le m \quad (2.1)$$

For $\forall\, t \in [0,T]$, it is obvious that there exists a unique $w(x,t)$ that satisfies

$$(a\nabla(w-u),\nabla v) + (w-u,v) = 0, \qquad \forall\, v \in S^h \qquad (2.2)$$

and also $w(x,t)$ is a $[0,T] \to S^h$ twice differentiable mapping.

By (2.1) and [5], we obtain

***Lemma1.*** Let $u$ be the true solution of problem (1.1), $w(x,t)$ be determined by (2.2), and assume condition (A) exists, then there is a constant $K_7$independent of$h$ such that

$$\left\|\frac{\partial^r}{\partial t^r}(w-u)\right\|(t) \le K_7 h^s \left\|\frac{\partial^r u}{\partial t^r}\right\|_s (t),$$
$$s = \min(k,m), \quad r = 0,1,2$$

From [7], we have

***Lemma 2.*** Let $u$ be the true solution of problem (1.1), $w(x,t)$ be determined by (2.2), and assume condition (A) exists, then there is a constant $K_8$independent of$h$ such that

$$\left|\frac{\partial^r}{\partial t^r}(w-u)\right|_{-\frac{1}{2}}(t) \le K_8 h^s \left\|\frac{\partial^r u}{\partial t^r}\right\|_s (t),$$
$$s = \min(k,m), \quad r = 0,1,2$$

For convenience, we will use $C, C_i$, and $\varepsilon$ to represent constants independent of $h$ in the discussion below, and also, they can represent different values in different expressions.

## III. CONTINUOUS TIME GALERKIN METHOD

The continuous time Galerkin approximation is defined as a twice differentiable mapping $U$: $[0,T] \to S^h$ satisfying

$$\begin{cases} \left(\dfrac{\partial^2 U}{\partial t^2}, v\right) + (a\nabla U,\nabla v) + \langle g + \alpha_1 U + \alpha_2 \dfrac{\partial U}{\partial t}, v\rangle \\ = \displaystyle\sum_{i=1}^n \left(b_i(\boldsymbol{x},U)U_{x_i}, v\right) + (f(\boldsymbol{x},U),v) \quad \forall\, v \in S^h \quad (3.1) \\ U(x,0) = \varphi_0(\boldsymbol{x}) \\ \dfrac{\partial U(x,0)}{\partial t} = \psi_0(\boldsymbol{x}) \end{cases}$$

where $\varphi_0(\boldsymbol{x}), \psi_0(\boldsymbol{x}) \in S^h$ are approximations of $\varphi(\boldsymbol{x}), \psi(\boldsymbol{x})$, respectively.

***Lemma 3*** .Let condition (A) exist, and $u, U, w$be defined by (1.1), (3.1) and (2.2), respectively, then

$$\left\|\frac{\partial \xi}{\partial t}\right\|^2 + \|\xi\|^2$$
$$\le C\left[\|\xi(0)\|_1^2 + \left\|\frac{\partial \xi}{\partial t}(0)\right\|^2 + \|\eta\|^2_{L^\infty(L^2(\Omega))} + \left\|\frac{\partial \eta}{\partial t}\right\|^2_{L^\infty(L^2(\Omega))} + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|^2_{L^2(L^2(\Omega))} + \|\eta\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))} + \left\|\frac{\partial \eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))} + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega))}\right]$$

where $\xi = U - w$ and $\eta = w - u$.

**Proof.** By (1.1), $u$ satisfies

$$\left(\frac{\partial^2 u}{\partial t^2}, v\right) + (a\nabla u,\nabla v) + \langle g + \alpha_1 u + \alpha_2 \frac{\partial u}{\partial t}, v\rangle = \sum_{i=1}^n \left(b_i(\boldsymbol{x},u)u_{x_i}, v\right) + (f(\boldsymbol{x},u),v) \;\; \forall\, v \in H^1(\Omega) \quad (3.2)$$

Subtracting (3.2) from (3.1) and using (2.2), we have

$$\left(\frac{\partial^2 \xi}{\partial t^2}, v\right) + (a\nabla\xi,\nabla v) + \langle \alpha_1 \xi + \alpha_2 \frac{\partial \xi}{\partial t}, v\rangle = \sum_{i=1}^n \left(b_i(\boldsymbol{x},U)U_{x_i} - b_i(\boldsymbol{x},u)u_{x_i}, v\right) + (f(\boldsymbol{x},U) - f(\boldsymbol{x},u), v) + \left(\eta - \frac{\partial^2 \eta}{\partial t^2}, v\right) - \langle \alpha_1 \eta + \alpha_2 \frac{\partial \eta}{\partial t}, v\rangle, \forall\, v \in S^h \quad (3.3)$$

Taking$v = \frac{\partial \xi}{\partial t} \in S^h$ , by inequality

$$\frac{d}{dt}(\|\xi\|^2) \le \|\xi\|^2 + \left\|\frac{\partial \xi}{\partial t}\right\|^2$$

and considering

$$\left|\left(\eta - \frac{\partial^2 \eta}{\partial t^2}, v\right)\right| \le \|\eta\|^2 + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|^2 + \frac{1}{2}\|v\|^2$$

and

$$|(f(\boldsymbol{x},U) - f(\boldsymbol{x},u), v)| \le C_0(\|\xi\|^2 + \|\eta\|^2 + \|v\|^2)$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\|\xi\|^2 + \left\|\frac{\partial \xi}{\partial t}\right\|^2 + \left\|a^{\frac{1}{2}}\nabla\xi\right\|^2 + \left|\alpha_1^{\frac{1}{2}}\xi\right|^2\right] + \left|\alpha_2^{\frac{1}{2}}\frac{\partial \xi}{\partial t}\right|^2$$
$$\le C_1\left[\|\xi\|^2 + \left\|\frac{\partial \xi}{\partial t}\right\|^2 + \|\eta\|^2 + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|^2\right] + \sum_{i=1}^n \left(b_i(\boldsymbol{x},U)U_{x_i} - b_i(\boldsymbol{x},u)u_{x_i}, \frac{\partial \xi}{\partial t}\right) - \langle \alpha_1 \eta + \alpha_2 \frac{\partial \eta}{\partial t}, \frac{\partial \xi}{\partial t}\rangle$$

Integrating both sides, and noticing

$$|z|^2 \le |z|^2_{\frac{1}{2}} \le C_2\|z\|_1^2, \quad \forall\, z \in H^1(\Omega) \quad (3.4)$$

we have

$$\|\xi(t)\|_1^2 + \left\|\frac{\partial \xi}{\partial t}(t)\right\|^2$$
$$\le C_3\left[\|\xi(0)\|_1^2 + \left\|\frac{\partial \xi}{\partial t}(0)\right\|^2 + \|\eta\|^2_{L^\infty(L^2(\Omega))}\right.$$

$$+\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(L^2(\Omega))}+\int_0^t\left(\|\xi\|^2+\left\|\frac{\partial\xi}{\partial t}\right\|^2\right)ds$$
$$+\left|\sum_{i=1}^n\int_0^t\left(b_i(\boldsymbol{x},U)U_{x_i}-b_i(\boldsymbol{x},u)u_{x_i},\frac{\partial\xi}{\partial t}\right)ds\right|$$
$$+\left|\int_0^t\langle\alpha_1\eta+\alpha_2\frac{\partial\eta}{\partial t},\frac{\partial\xi}{\partial t}\rangle ds\right| \tag{3.5}$$

We estimate the last second term on the right side of (3.5).

$$\left|\sum_{i=1}^n\int_0^t\left(b_i(\boldsymbol{x},U)U_{x_i}-b_i(\boldsymbol{x},u)u_{x_i},\frac{\partial\xi}{\partial t}\right)ds\right|$$
$$=\left|\sum_{i=1}^n\int_0^t\left(b_i(\boldsymbol{x},U)\xi_{x_i},\frac{\partial\xi}{\partial t}\right)ds\right.$$
$$+\sum_{i=1}^n\int_0^t\left([b_i(\boldsymbol{x},U)-b_i(\boldsymbol{x},u)]w_{x_i},\frac{\partial\xi}{\partial t}\right)ds$$
$$\left.+\sum_{i=1}^n\int_0^t\left(b_i(\boldsymbol{x},u)\eta_{x_i},\frac{\partial\xi}{\partial t}\right)ds\right|$$
$$\le C_4\int_0^t\left(\|\xi\|^2_{H_0^1}+\left\|\frac{\partial\xi}{\partial t}\right\|^2\right)ds$$
$$+C_5\left[\int_0^t\left(\|\xi\|^2+\left\|\frac{\partial\xi}{\partial t}\right\|^2\right)ds+\|\eta\|^2_{L^\infty(L^2(\Omega))}\right]$$
$$+\left|\sum_{i=1}^n\int_0^t\left(b_i(\boldsymbol{x},u)\eta_{x_i},\frac{\partial\xi}{\partial t}\right)ds\right| \tag{3.6}$$

$$\left|\sum_{i=1}^n\int_0^t\left(b_i(\boldsymbol{x},u)\eta_{x_i},\frac{\partial\xi}{\partial t}\right)ds\right|$$
$$\le\left|\sum_{i=1}^n\int_0^t\left(\left[b_i(\boldsymbol{x},u)\frac{\partial\xi}{\partial t}\right]_{x_i},\eta\right)ds\right|$$
$$+\left|\sum_{i=1}^n\int_0^t\langle b_i(\boldsymbol{x},u)\eta\cos(\boldsymbol{n},\boldsymbol{x}_i),\frac{\partial\xi}{\partial t}\rangle ds\right| \tag{3.7}$$

$$\left|\sum_{i=1}^n\int_0^t\left(\left[b_i(\boldsymbol{x},u)\frac{\partial\xi}{\partial t}\right]_{x_i},\eta\right)ds\right|$$
$$\le\left|\sum_{i=1}^n\int_0^t\left(\left[b_{ix_i}(\boldsymbol{x},u)+b_{iu}(\boldsymbol{x},u)u_{x_i}\right]\frac{\partial\xi}{\partial t},\eta\right)ds\right|$$
$$+\left|\sum_{i=1}^n\int_0^t\left(b_i(\boldsymbol{x},u)\left(\frac{\partial\xi}{\partial t}\right)_{x_i},\eta\right)ds\right|$$
$$\le C_6\left[\int_0^t\left\|\frac{\partial\xi}{\partial t}\right\|^2ds+\|\eta\|^2_{L^\infty(L^2(\Omega))}\right]+\left|\sum_{i=1}^n\left(b_i(\boldsymbol{x},u)\xi_{x_i},\eta\right)\right|_0^t$$
$$+\left|\sum_{i=1}^n\int_0^t([b_i(\boldsymbol{x},u)\eta]_t,\xi_{x_i})ds\right|$$
$$\le C_7\left[\int_0^t\left\|\frac{\partial\xi}{\partial t}\right\|^2ds+\|\eta\|^2_{L^\infty(L^2(\Omega))}+\|\xi(0)\|^2_{H_0^1}\right]$$
$$+\varepsilon\|\xi(t)\|^2_{H_0^1}+\left|\sum_{i=1}^n\int_0^t(b_{iu}(\boldsymbol{x},u)u_t\eta+b_i(\boldsymbol{x},u)\eta_t,\xi_{x_i})ds\right|$$
$$\le C_8\left[\int_0^t\left(\left\|\frac{\partial\xi}{\partial t}\right\|^2+\|\xi\|^2_{H_0^1}\right)ds+\|\xi(0)\|^2_{H_0^1}+\|\eta\|^2_{L^\infty(L^2(\Omega))}\right.$$
$$\left.+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(L^2(\Omega))}\right]+\varepsilon\|\xi(t)\|^2_{H_0^1} \tag{3.8}$$

By condition (A) (iv), (v) and (3.4),

$$\left|\sum_{i=1}^n\int_0^t\langle b_i(\boldsymbol{x},u)\eta\cos(\boldsymbol{n},\boldsymbol{x}_i),\frac{\partial\xi}{\partial t}\rangle ds\right|$$
$$\le\left|\sum_{i=1}^n\langle b_i(\boldsymbol{x},u)\eta\cos(\boldsymbol{n},\boldsymbol{x}_i),\xi\rangle|_0^t\right|$$
$$+\left|\sum_{i=1}^n\int_0^t\langle b_i(\boldsymbol{x},u)\cos(\boldsymbol{n},\boldsymbol{x}_i)\frac{\partial\eta}{\partial t},\xi\rangle ds\right|$$
$$+\left|\sum_{i=1}^n\int_0^t\langle b_{iu}(\boldsymbol{x},u)u_t\cos(\boldsymbol{n},\boldsymbol{x}_i)\eta,\xi\rangle ds\right|$$
$$\le C_9\left[|\eta(t)|_{-\frac12}|\xi(t)|_{\frac12}+|\eta(0)|_{-\frac12}|\xi(0)|_{\frac12}\right.$$
$$\left.+\int_0^t\left(\left|\frac{\partial\eta}{\partial t}\right|_{-\frac12}|\xi|_{\frac12}+|\eta|_{-\frac12}|\xi|_{\frac12}\right)ds\right]$$
$$\le C_{10}\left[\|\eta\|^2_{L^\infty(H^{-\frac12}(\partial\Omega))}+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac12}(\partial\Omega))}+\|\xi(0)\|_1^2\right.$$
$$\left.+\int_0^t\|\xi\|_1^2\,ds\right]+\varepsilon\|\xi(t)\|_1^2 \tag{3.9}$$

By (3.7), (3.8) and (3.9), we have

$$\left|\sum_{i=1}^n\int_0^t\left(b_i(\boldsymbol{x},u)\eta_{x_i},\frac{\partial\xi}{\partial t}\right)ds\right|$$
$$\le 2\varepsilon\|\xi(t)\|_1^2+C_{11}\left[\|\eta\|^2_{L^\infty(L^2(\Omega))}+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(L^2(\Omega))}\right.$$
$$+\|\xi(0)\|_1^2+\|\eta\|^2_{L^\infty(H^{-\frac12}(\partial\Omega))}+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac12}(\partial\Omega))}$$
$$\left.+\int_0^t\left(\|\xi\|_1^2+\left\|\frac{\partial\xi}{\partial t}\right\|^2\right)ds\right] \tag{3.10}$$

Now we obtain the estimate of the last second term on the right side of (3.5) from (3.6) and (3.10)

$$\left|\sum_{i=1}^n\int_0^t\left(b_i(\boldsymbol{x},U)U_{x_i}-b_i(\boldsymbol{x},u)u_{x_i},\frac{\partial\xi}{\partial t}\right)ds\right|$$
$$\le 2\varepsilon\|\xi(t)\|_1^2+C_{12}\left[\|\eta\|^2_{L^\infty(L^2(\Omega))}+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(L^2(\Omega))}\right.$$
$$+\|\xi(0)\|_1^2+\|\eta\|^2_{L^\infty(H^{-\frac12}(\partial\Omega))}+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac12}(\partial\Omega))}$$
$$\left.+\int_0^t\left(\|\xi\|_1^2+\left\|\frac{\partial\xi}{\partial t}\right\|^2\right)ds\right] \tag{3.11}$$

Similarly, we can obtain the estimate of the last term on the right side of (3.5)

$$\left|\int_0^t\langle\alpha_1\eta+\alpha_2\frac{\partial\eta}{\partial t},\frac{\partial\xi}{\partial t}\rangle ds\right|$$
$$\le\varepsilon\|\xi(t)\|_1^2+C_{13}\left[\|\eta\|^2_{L^\infty(H^{-\frac12}(\partial\Omega))}+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac12}(\partial\Omega))}\right.$$
$$\left.+\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(H^{-\frac12}(\partial\Omega))}+\|\xi(0)\|_1^2+\int_0^t\|\xi\|_1^2ds\right] \tag{3.12}$$

Taking sufficient small $\varepsilon$, we obtain following estimate from (3.5), (3.11) and (3.12)

$$\|\xi(t)\|_1^2+\left\|\frac{\partial\xi}{\partial t}(t)\right\|^2$$
$$\le C_{14}\Bigg[\|\xi(0)\|_1^2+\left\|\frac{\partial\xi}{\partial t}(0)\right\|^2+\|\eta\|^2_{L^\infty(L^2(\Omega))}$$
$$+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(L^2(\Omega))}+\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(L^2(\Omega))}+\|\eta\|^2_{L^\infty(H^{-\frac12}(\partial\Omega))}$$
$$+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac12}(\partial\Omega))}+\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(H^{-\frac12}(\partial\Omega))}$$
$$+\int_0^t\left(\|\xi\|_1^2+\left\|\frac{\partial\xi}{\partial t}\right\|^2\right)ds\Bigg] \tag{3.13}$$

From (3.13) and using Gronwall inequality, we get the conclusion of lemma 3

Then we immediately get the following optimal $L^2$-error estimates for continuous time Galerkin approximation.

***Theorem 1.*** Let $u$ be the true solution of problem (1.1), $U$ be its Galerkin approximation determined by (3.1), $w$ be determined by (2.2). Assume condition (A) exists, and also $\|(U-w)(0)\|_1\le C_{15}h^s$, $\left\|\frac{\partial}{\partial t}(U-w)(0)\right\|\le C_{16}h^s$ , then there is a constant $C$independent of$h$ such that

$$\|U-u\|_{L^\infty(L^2(\Omega))}+\left\|\frac{\partial}{\partial t}(U-u)\right\|_{L^\infty(L^2(\Omega))}\le Ch^s$$

where $s=\min(k,m)$.

## IV. FULLY DISCRETE GALERKIN METHOD

Dividing the interval $[0,T]$ into $J$ subintervals of equal length: $0=t_0<t_1<\cdots<t_J=T$ , denoting $\Delta t=t_{j+1}-t_j,\ j=0,1,\cdots,J-1$.

For any function $g$defined at points $t_j=j\Delta t, j=0,1,\cdots,J$, denoting $g(t_j)$ by $g^j$. In addition, introducing following notations:

$$\begin{cases} g^{j+\frac12}=\frac12(g^{j+1}+g^j);\\ g_{j+\frac14}=\frac14(g^{j+1}+2g^j+g^{j-1})=\frac12\left(g^{j+\frac12}+g^{j-\frac12}\right);\\ \partial_t g^{j+\frac12}=\frac{1}{\Delta t}(g^{j+1}-g^j);\\ \delta_t g^j=\frac{1}{2\Delta t}(g^{j+1}-g^{j-1})=\frac{1}{\Delta t}\left(g^{j+\frac12}-g^{j-\frac12}\right)\\ \qquad=\frac12\left(\partial_t g^{j+\frac12}+\partial_t g^{j-\frac12}\right);\\ \partial_t^2 g^j=\frac{1}{(\Delta t)^2}(g^{j+1}-2g^j+g^{j-1})\\ \qquad=\frac{1}{\Delta t}\left(\partial_t g^{j+\frac12}-\partial_t g^{j-\frac12}\right) \end{cases} \tag{4.1}$$

Defining discrete Galerkin approximation as a sequence $\{U^j\}_{j=0}^J$ in $S^h$, which satisfies

$$\begin{cases} (\partial_t^2U^j,v)+\left(a\nabla U_{j+\frac14},\nabla v\right)+\langle g_{j+\frac14}+\alpha_1U_{j+\frac14}+\alpha_2\delta_tU^j,v\rangle\\ =\sum_{i=1}^n\left(b_i\left(\boldsymbol{x},U_{j+\frac14}\right)\left(U_{x_i}\right)_{j+\frac14},v\right)+\left(f\left(\boldsymbol{x},U_{j+\frac14}\right),v\right),\\ \quad\text{for }\forall v\in S^h \qquad (4.2)\\ U^0=\varphi_1(\boldsymbol{x})\\ \partial_tU^{\frac12}=\psi_1(\boldsymbol{x}) \end{cases}$$

where $j=1,2,\cdots J-1$. Functions $\varphi_1(\boldsymbol{x})$, $\psi_1(\boldsymbol{x})\in S^h$ are approximations of $\varphi(\boldsymbol{x})$and $\psi(\boldsymbol{x})$, respectively.

The existence of the solution of (4.2) can be obtained by using Brouwer's fixed point theorem.

***Lemma 4.*** Let $u$ be the true solution of problem (1.1), $\{U^j\}_{j=0}^J$ be its discrete Galerkin approximation determined by (4.2), $w$ be determined by (2.2). Assume condition (A) exists, and $u_t^{(3)},u_t^{(4)}\in L^\infty(L^2(\partial\Omega)),u_t^{(4)}\in L^2(L^2(\Omega))$, then there is a constant $C$ independent of $h,\Delta t$ such that

$$\|\partial_t\xi\|^2_{\bar L^\infty(L^2(\Omega))}+\|\xi\|^2_{\bar L^\infty(L^2(\Omega))}$$
$$\le C\Bigg[\left\|\xi^{\frac12}\right\|_1^2+\left\|\partial_t\xi^{\frac12}\right\|^2+\|\eta\|^2_{L^\infty(L^2(\Omega))}+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(L^2(\Omega))}$$
$$+\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(L^2(\Omega))}+\|\eta\|^2_{L^\infty(H^{-\frac12}(\partial\Omega))}+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac12}(\partial\Omega))}$$
$$+\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(H^{-\frac12}(\partial\Omega))}+(\Delta t)^4\Bigg]$$

where$\xi^j=U^j-w^j,\ \eta^j=w^j-u^j$and

$$\|\xi\|^2_{\bar L^\infty(L^2(\Omega))}=\max_{1\le j\le J-1}\left\|\xi^{j+\frac12}\right\|^2$$
$$\|\partial_t\xi\|^2_{\bar L^\infty(L^2(\Omega))}=\max_{1\le j\le J-1}\left\|\partial_t\xi^{j+\frac12}\right\|^2$$

**Proof.** By (1.1), we have

$$\left((u_{tt})_{j+\frac14},v\right)+\left(a\nabla u_{j+\frac14},\nabla v\right)+\Big\langle g_{j+\frac14}+\alpha_1u_{j+\frac14}+\alpha_2(u_t)_{j+\frac14},v\Big\rangle$$

$$=\sum_{i=1}^n\left(\left[b_i(\boldsymbol{x},u)u_{x_i}\right]_{j+\frac14},v\right)+\left([f(\boldsymbol{x},u)]_{j+\frac14},v\right),$$

for $\forall v\in H^1(\Omega)$(4.3)

Introducing

$$\gamma_j=(u_{tt})_{j+\frac14}-\partial_t^2u^j;\ S_j=(u_t)_{j+\frac14}-\delta_tu^j \tag{4.4}$$

Subtracting (4.3) from (4.2), and using (2.2), we get

$$(\partial_t^2\xi^j,v)+\left(a\nabla\xi_{j+\frac14},\nabla v\right)+\langle\alpha_1\xi_{j+\frac14}+\alpha_2\delta_t\xi^j,v\rangle$$
$$=\sum_{i=1}^n\left(b_i\left(\boldsymbol{x},U_{j+\frac14}\right)\left(U_{x_i}\right)_{j+\frac14}-\left[b_i(\boldsymbol{x},u)u_{x_i}\right]_{j+\frac14},v\right)$$
$$+\left(f\left(\boldsymbol{x},U_{j+\frac14}\right)-[f(\boldsymbol{x},u)]_{j+\frac14},v\right)+\left(\gamma_j+\eta_{j+\frac14}-\partial_t^2\eta^j,v\right)$$
$$-\langle\alpha_1\eta_{j+\frac14}+\alpha_2(\delta_t\eta^j-S_j),v\rangle,\qquad\forall v\in S^h \tag{4.5}$$

Taking$v=\delta_t\xi^j$, and multiplying both side of (4.5) by $2\Delta t$, then adding$j$ from 1 to $m$, considering

$$\left\|\xi^{m+\frac{1}{2}}\right\|^2 \leq \left\|\xi^{\frac{1}{2}}\right\|^2 + \Delta t\sum_{j=0}^{m}\left(\left\|\xi^{j+\frac{1}{2}}\right\|^2 + \left\|\partial_t\xi^{j+\frac{1}{2}}\right\|^2\right)$$

We have

$$\left\|\xi^{m+\frac{1}{2}}\right\|_1^2 + \left\|\partial_t\xi^{m+\frac{1}{2}}\right\|^2$$

$$\leq C_0\left\{\left\|\xi^{\frac{1}{2}}\right\|_1^2 + \left\|\partial_t\xi^{\frac{1}{2}}\right\|^2 + \Delta t\left[\sum_{j=1}^{m}\left(\left\|\xi^{j+\frac{1}{2}}\right\|^2 + \left\|\partial_t\xi^{j+\frac{1}{2}}\right\|^2\right)\right.\right.$$

$$+\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},U_{j+\frac{1}{4}}\right)\left(U_{x_i}\right)_{j+\frac{1}{4}} - \left[b_i(\boldsymbol{x},u)u_{x_i}\right]_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$+\left|\sum_{j=1}^{m}\left(f\left(\boldsymbol{x},U_{j+\frac{1}{4}}\right) - [f(\boldsymbol{x},u)]_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$+\left|\sum_{j=1}^{m}\left(\gamma_j + \eta_{j+\frac{1}{4}} - \partial_t^2\eta^j,\delta_t\xi^j\right)\right|$$

$$\left.\left.+\left|\sum_{j=1}^{m}\left\langle\alpha_1\eta_{j+\frac{1}{4}} + \alpha_2\left(\delta_t\eta^j - S_j\right),\delta_t\xi^j\right\rangle\right|\right]\right\} \tag{4.6}$$

In the following we will estimate the right side of (4.6) term by term.

$$\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},U_{j+\frac{1}{4}}\right)\left(U_{x_i}\right)_{j+\frac{1}{4}} - \left[b_i(\boldsymbol{x},u)u_{x_i}\right]_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$\leq \Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},U_{j+\frac{1}{4}}\right)\left(\xi_{x_i}\right)_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$+\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left[b_i\left(\boldsymbol{x},U_{j+\frac{1}{4}}\right) - b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)\right]\left(w_{x_i}\right)_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$+\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)\left(\eta_{x_i}\right)_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$+\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)\left(u_{x_i}\right)_{j+\frac{1}{4}} - \left[b_i(\boldsymbol{x},u)u_{x_i}\right]_{j+\frac{1}{4}},\delta_t\xi^j\right)\right| \tag{4.7}$$

Now we estimate four terms on the right side of (4.7), separately.

$$\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},U_{j+\frac{1}{4}}\right)\left(\xi_{x_i}\right)_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$\leq C_1\Delta t\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left|\xi_{x_i}^{j+\frac{1}{2}}\right| + \left|\xi_{x_i}^{j-\frac{1}{2}}\right|,\left|\partial_t\xi^{j+\frac{1}{2}}\right| + \left|\partial_t\xi^{j-\frac{1}{2}}\right|\right)$$

$$\leq C_2\Delta t\sum_{j=0}^{m}\left(\left\|\xi^{j+\frac{1}{2}}\right\|_{H_0^1}^2 + \left\|\partial_t\xi^{j+\frac{1}{2}}\right\|^2\right) \tag{4.8}$$

$$\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left[b_i\left(\boldsymbol{x},U_{j+\frac{1}{4}}\right) - b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)\right]\left(w_{x_i}\right)_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$\leq C_3\left[\|\eta\|_{L^\infty\left(L^2(\Omega)\right)}^2 + \Delta t\sum_{j=0}^{m}\left(\left\|\xi^{j+\frac{1}{2}}\right\|^2 + \left\|\partial_t\xi^{j+\frac{1}{2}}\right\|^2\right)\right] \tag{4.9}$$

$$\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)\left(\eta_{x_i}\right)_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$\leq \Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)\eta_{j+\frac{1}{4}}\cos(\boldsymbol{n},\boldsymbol{x}_i),\delta_t\xi^j\right\rangle\right|$$

$$+\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left[b_{ix_i}\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right) + b_{iu}\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)\left(u_{x_i}\right)_{j+\frac{1}{4}}\right]\delta_t\xi^j,\eta_{j+\frac{1}{4}}\right)\right|$$

$$+\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)\delta_t\left(\xi_{x_i}^j\right),\eta_{j+\frac{1}{4}}\right)\right| \tag{4.10}$$

For the right side of (4.10)

$$\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)\eta_{j+\frac{1}{4}}\cos(\boldsymbol{n},\boldsymbol{x}_i),\delta_t\xi^j\right\rangle\right|$$

$$= \left|\sum_{j=1}^{m-1}\sum_{i=1}^{n}\left\langle\left[b_i\left(\boldsymbol{x},u_{j+1+\frac{1}{4}}\right)\eta_{j+1+\frac{1}{4}} - b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)\eta_{j+\frac{1}{4}}\right]\cos(\boldsymbol{n},\boldsymbol{x}_i),\xi^{j+\frac{1}{2}}\right\rangle\right.$$

$$+\sum_{i=1}^{n}\left\langle b_i\left(\boldsymbol{x},u_{1+\frac{1}{4}}\right)\eta_{1+\frac{1}{4}}\cos(\boldsymbol{n},\boldsymbol{x}_i),\xi^{\frac{1}{2}}\right\rangle$$

$$\left.-\sum_{i=1}^{n}\left\langle b_i\left(\boldsymbol{x},u_{m+\frac{1}{4}}\right)\eta_{m+\frac{1}{4}}\cos(\boldsymbol{n},\boldsymbol{x}_i),\xi^{m+\frac{1}{2}}\right\rangle\right|$$

$$\leq C_4\left[\Delta t\sum_{j=1}^{m}\left|\delta_t\eta^j\right|_{-\frac{1}{2}}^2 + \|\eta\|_{L^\infty\left(H^{-\frac{1}{2}}(\partial\Omega)\right)}^2 + \Delta t\sum_{j=1}^{m-1}\left\|\xi^{j+\frac{1}{2}}\right\|_1^2 + \left\|\xi^{\frac{1}{2}}\right\|_1^2\right] + \varepsilon\left\|\xi^{m+\frac{1}{2}}\right\|_1^2 \tag{4.11}$$

$$\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left[b_{ix_i}\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right) + b_{iu}\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)\left(u_{x_i}\right)_{j+\frac{1}{4}}\right]\delta_t\xi^j,\eta_{j+\frac{1}{4}}\right)\right|$$

$$\leq C_5\left[\|\eta\|^2_{L^\infty\left(L^2(\Omega)\right)}+\Delta t\sum_{j=0}^{m}\left\|\partial_t\xi^{j+\frac{1}{2}}\right\|^2\right] \qquad (4.12)$$

$$\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)\delta_t(\xi^j_{x_i}),\eta_{j+\frac{1}{4}}\right)\right|$$

$$\leq\left|\sum_{j=1}^{m-1}\sum_{i=1}^{n}\left(\left[b_i\left(\boldsymbol{x},u_{j+1+\frac{1}{4}}\right)\eta_{j+1+\frac{1}{4}}-b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)\eta_{j+\frac{1}{4}}\right],\xi^{j+\frac{1}{2}}_{x_i}\right)\right|$$

$$+\left|\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},u_{m+\frac{1}{4}}\right)\xi^{m+\frac{1}{2}}_{x_i},\eta_{m+\frac{1}{4}}\right)\right|$$

$$+\left|\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},u_{1+\frac{1}{4}}\right)\xi^{\frac{1}{2}}_{x_i},\eta_{1+\frac{1}{4}}\right)\right|$$

$$\leq C_6\left[\Delta t\sum_{j=1}^{m}\|\delta_t\eta^j\|^2+\|\eta\|^2_{L^\infty(L^2(\Omega))}+\Delta t\sum_{j=1}^{m-1}\left\|\xi^{j+\frac{1}{2}}\right\|^2_{H_0^1}+\left\|\xi^{\frac{1}{2}}\right\|^2_{H_0^1}\right]+\varepsilon\left\|\xi^{m+\frac{1}{2}}\right\|^2_{H_0^1} \qquad (4.13)$$

By (4.10)-(4.13), we get

$$\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)(\eta_{x_i})_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$\leq C_7\left[\Delta t\sum_{j=1}^{m}\left(\|\delta_t\eta^j\|^2+|\delta_t\eta^j|^2_{-\frac{1}{2}}\right)+\|\eta\|^2_{L^\infty\left(L^2(\Omega)\right)}+\|\eta\|^2_{L^\infty\left(H^{-\frac{1}{2}}(\partial\Omega)\right)}+\left\|\xi^{\frac{1}{2}}\right\|^2_1+\Delta t\sum_{j=0}^{m}\left(\left\|\xi^{j+\frac{1}{2}}\right\|^2_1+\left\|\partial_t\xi^{j+\frac{1}{2}}\right\|^2\right)\right]+2\varepsilon\left\|\xi^{m+\frac{1}{2}}\right\|^2_1 \qquad (4.14)$$

For last term in (4.7)

$$\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)(u_{x_i})_{j+\frac{1}{4}}-[b_i(\boldsymbol{x},u)u_{x_i}]_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$\leq\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left[b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)-(b_i(\boldsymbol{x},u))_{j+\frac{1}{4}}\right]u^j_{x_i},\delta_t\xi^j\right)\right|$$

$$+(\Delta t)^2\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{1}{4}\left[b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)-b_i(\boldsymbol{x},u^{j+1})\right]u_{x_it}(\bar{t}_j),\delta_t\xi^j\right)\right|$$

$$+(\Delta t)^2\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{1}{4}\left[b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)-b_i(\boldsymbol{x},u^{j-1})\right]u_{x_it}(\bar{t}_{j-1}),\delta_t\xi^j\right)\right| \qquad (4.15)$$

where$t_l\leq\bar{t}_l\leq t_{l+1}$. Below we will use the same notation, which can represent different values in different places.

Since

$$u_{j+\frac{1}{4}}=u^j+\frac{(\Delta t)^2}{8}\left[u_{tt}(\bar{t}_{j-1})+u_{tt}(\bar{t}_j)\right],$$

$$b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)=b_i(\boldsymbol{x},u^j)+b_{iu}(\boldsymbol{x},\hat{u}_j)\frac{(\Delta t)^2}{8}\left[u_{tt}(\bar{t}_{j-1})+u_{tt}(\bar{t}_j)\right] \qquad (4.16)$$

where $\hat{u}_j\in\left[u^j,u_{j+\frac{1}{4}}\right]$.

Since

$$b_i(\boldsymbol{x},u^{j+1})=b_i(\boldsymbol{x},u^j)+b_{iu}(\boldsymbol{x},u^j)(u^{j+1}-u^j)+b_{iuu}(\boldsymbol{x},\bar{u}_j)\frac{(u^{j+1}-u^j)^2}{2}$$

and

$$b_i(\boldsymbol{x},u^{j-1})=b_i(\boldsymbol{x},u^j)+b_{iu}(\boldsymbol{x},u^j)(u^{j-1}-u^j)+b_{iuu}(\boldsymbol{x},\bar{u}_{j-1})\frac{(u^{j-1}-u^j)^2}{2}$$

where $\bar{u}_l\in[u^l,u^{l+1}]$, there is

$$[b_i(\boldsymbol{x},u)]_{j+\frac{1}{4}}$$

$$=b_i(\boldsymbol{x},u^j)+\frac{1}{4}b_{iu}(\boldsymbol{x},u^j)(u^{j+1}-2u^j+u^{j-1})+\frac{1}{4}\left[b_{iuu}(\boldsymbol{x},\bar{u}_j)\frac{(u^{j+1}-u^j)^2}{2}+b_{iuu}(\boldsymbol{x},\bar{u}_{j-1})\frac{(u^{j-1}-u^j)^2}{2}\right]$$

$$=b_i(\boldsymbol{x},u^j)+\frac{(\Delta t)^2}{8}b_{iu}(\boldsymbol{x},u^j)\left[u_{tt}(\bar{t}_{j-1})+u_{tt}(\bar{t}_j)\right]+\frac{(\Delta t)^2}{8}\left[b_{iuu}(\boldsymbol{x},\bar{u}_j)\left(u_t(\bar{t}_j)\right)^2+b_{iuu}(\boldsymbol{x},\bar{u}_{j-1})\left(u_t(\bar{t}_{j-1})\right)^2\right] \qquad (4.17)$$

By (4.16) and (4.17), we have

$$\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left[b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)-(b_i(\boldsymbol{x},u))_{j+\frac{1}{4}}\right]u^j_{x_i},\delta_t\xi^j\right)\right|$$

$$\leq C_8\left[\Delta t\sum_{j=0}^{m}\left\|\partial_t\xi^{j+\frac{1}{2}}\right\|^2+(\Delta t)^4\right] \qquad (4.18)$$

and

$$(\Delta t)^2\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{1}{4}\left[b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)-b_i(\boldsymbol{x},u^{j+1})\right]u_{x_it}(\bar{t}_j),\delta_t\xi^j\right)\right|$$

$$+(\Delta t)^2\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{1}{4}\left[b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)\right.\right.\right.$$

$$\left.\left.\left.-b_i(\boldsymbol{x},u^{j-1})\right]u_{x_it}(\bar{t}_{j-1}),\delta_t\xi^j\right)\right|$$

$$\leq C_9\left[\Delta t\sum_{j=0}^{m}\left\|\partial_t\xi^{j+\frac{1}{2}}\right\|^2+(\Delta t)^4\right] \quad (4.19)$$

From (4.15), (4.18) and (4.19), we get

$$\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)(u_{x_i})_{j+\frac{1}{4}}-[b_i(\boldsymbol{x},u)u_{x_i}]_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$\leq C_{10}\left[\Delta t\sum_{j=0}^{m}\left\|\partial_t\xi^{j+\frac{1}{2}}\right\|^2\right.$$

$$\left.+(\Delta t)^4\right] \quad (4.20)$$

From (4.7)-(4.9), (4.14) and (4.20), we get estimate of the last fourth term onthe right side of (4.6)

$$\Delta t\left|\sum_{j=1}^{m}\sum_{i=1}^{n}\left(b_i\left(\boldsymbol{x},U_{j+\frac{1}{4}}\right)(U_{x_i})_{j+\frac{1}{4}}-[b_i(\boldsymbol{x},u)u_{x_i}]_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$\leq C_{11}\left[\Delta t\sum_{j=1}^{m}\left(\left\|\delta_t\eta^j\right\|^2+\left|\delta_t\eta^j\right|_{-\frac{1}{2}}^2\right)+\|\eta\|_{L^\infty(L^2(\Omega))}^2\right.$$

$$+\|\eta\|_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))}^2+\Delta t\sum_{j=0}^{m}\left(\left\|\xi^{j+\frac{1}{2}}\right\|_1^2+\left\|\partial_t\xi^{j+\frac{1}{2}}\right\|^2\right)$$

$$\left.+\left\|\xi^{\frac{1}{2}}\right\|_1^2+(\Delta t)^4\right]$$

$$+2\varepsilon\left\|\xi^{m+\frac{1}{2}}\right\|_1^2 \quad (4.21)$$

Now we estimate the last third term on the right side of (4.6)

$$\Delta t\left|\sum_{j=1}^{m}\left(f\left(\boldsymbol{x},U_{j+\frac{1}{4}}\right)-[f(\boldsymbol{x},u)]_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$\leq\Delta t\left|\sum_{j=1}^{m}\left(f\left(\boldsymbol{x},U_{j+\frac{1}{4}}\right)-f\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right),\delta_t\xi^j\right)\right|$$

$$+\Delta t\left|\sum_{j=1}^{m}\left(f\left(\boldsymbol{x},u_{j+\frac{1}{4}}\right)-[f(\boldsymbol{x},u)]_{j+\frac{1}{4}},\delta_t\xi^j\right)\right|$$

$$\leq C_{12}\Delta t\sum_{j=0}^{m}\left(\left\|\xi^{j+\frac{1}{2}}\right\|_1^2+\left\|\partial_t\xi^{j+\frac{1}{2}}\right\|^2\right)$$

$$+C_{12}\left[\|\eta\|_{L^\infty(L^2(\Omega))}^2+(\Delta t)^4\right] \quad (4.22)$$

Since

$$(u_{tt})_{j+\frac{1}{4}}=(u_{tt})^j+\frac{1}{4}\left[\int_{t_j}^{t_{j+1}}u_t^{(4)}(\tau)(t_{j+1}-\tau)d\tau\right.$$

$$\left.+\int_{t_j}^{t_{j-1}}u_t^{(4)}(\tau)(t_{j-1}-\tau)d\tau\right]$$

and

$$\partial_t^2u^j=(u_{tt})^j+\frac{1}{6(\Delta t)^2}\left[\int_{t_j}^{t_{j+1}}u_t^{(4)}(\tau)(t_{j+1}-\tau)^3d\tau\right.$$

$$\left.+\int_{t_j}^{t_{j-1}}u_t^{(4)}(\tau)(t_{j-1}-\tau)^3d\tau\right],$$

there is

$$\Delta t\sum_{j=1}^{J-1}\|\gamma_j\|^2\leq C_{13}(\Delta t)^4\left\|u_t^{(4)}\right\|_{L^2(L^2(\Omega))}^2.$$

Thus, we get the estimate of the last second term on the right side of (4.6)

$$\Delta t\left|\sum_{j=1}^{m}\left(\gamma_j+\eta_{j+\frac{1}{4}}-\partial_t^2\eta^j,\delta_t\xi^j\right)\right|$$

$$\leq C_{14}\left[\Delta t\sum_{j=1}^{m}\left\|\partial_t^2\eta^j\right\|^2+\|\eta\|_{L^\infty(L^2(\Omega))}^2\right.$$

$$+\Delta t\sum_{j=0}^{m}\left\|\partial_t\xi^{j+\frac{1}{2}}\right\|^2$$

$$\left.+(\Delta t)^4\right] \quad (4.23)$$

Next, we estimate the last term on the right side of (4.6). Introducing

$$R_j=\frac{S_{j+1}-S_j}{\Delta t}$$

and considering

$$\frac{\delta_t\eta^{j+1}-\delta_t\eta^j}{\Delta t}=\frac{1}{2}(\partial_t^2\eta^{j+1}+\partial_t^2\eta^j)$$

and

$$\frac{\eta_{j+1+\frac{1}{4}}-\eta_{j+\frac{1}{4}}}{\Delta t}=\frac{1}{2}(\delta_t\eta^{j+1}+\delta_t\eta^j),$$

hence

$$\Delta t\left|\sum_{j=1}^{m}\langle\alpha_1\eta_{j+\frac{1}{4}}+\alpha_2(\delta_t\eta^j-S_j),\delta_t\xi^j\rangle\right|$$

$$\leq C_{15}\left[\Delta t\sum_{j=1}^{m}\left(\left|\delta_t\eta^j\right|_{-\frac{1}{2}}^2+\left|\partial_t^2\eta^j\right|_{-\frac{1}{2}}^2\right)+\Delta t\sum_{j=1}^{m-1}\left\|\xi^{j+\frac{1}{2}}\right\|_1^2\right.$$

$$+[\![\eta]\!]_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))}^2+\left|\delta_t\eta^m\right|_{-\frac{1}{2}}^2+\left|\delta_t\eta^1\right|_{-\frac{1}{2}}^2+\left\|\xi^{\frac{1}{2}}\right\|_1^2$$

$$\left.+|S_m|_{-\frac{1}{2}}^2+|S_1|_{-\frac{1}{2}}^2+\Delta t\sum_{j=1}^{m-1}\left|R_j\right|_{-\frac{1}{2}}^2\right]+\varepsilon\left\|\xi^{m+\frac{1}{2}}\right\|_1^2 \quad (4.24)$$

Since

$$(u_t)_{j+\frac{1}{4}}=(u_t)^j+\frac{1}{4}\left[\int_{t_j}^{t_{j+1}}u_t^{(3)}(\tau)(t_{j+1}-\tau)d\tau\right.$$

$$\left.+\int_{t_j}^{t_{j-1}}u_t^{(3)}(\tau)(t_{j-1}-\tau)d\tau\right]$$

and

$$\delta_t u^j = (u_t)^j + \frac{1}{4\Delta t}\left[\int_{t_j}^{t_{j+1}} u_t^{(3)}(\tau)\left(t_{j+1}-\tau\right)^2 d\tau + \int_{t_j}^{t_{j-1}} u_t^{(3)}(\tau)\left(t_{j-1}-\tau\right)^2 d\tau\right],$$

there is

$$|S_j|_{-\frac{1}{2}}^2 \le |S_j|^2 \le C_{16}(\Delta t)^4 \|u_t^{(3)}\|_{L^\infty(L^2(\partial\Omega))}^2 \quad (4.25)$$

Since

$$R_j = \frac{S_{j+1}-S_j}{\Delta t} = \frac{1}{4}\left[\int_{t_{j+1}}^{t_{j+2}} u_t^{(4)}(\bar{\tau}_{j+1})\left(t_{j+2}-\tau\right)d\tau + \int_{t_{j+1}}^{t_j} u_t^{(4)}(\bar{\tau}_j)\left(t_j-\tau\right)d\tau\right] - \frac{1}{4\Delta t}\left[\int_{t_{j+1}}^{t_{j+2}} u_t^{(4)}(\tilde{\tau}_{j+1})\left(t_{j+2}-\tau\right)^2 d\tau + \int_{t_{j+1}}^{t_j} u_t^{(4)}(\tilde{\tau}_j)\left(t_j-\tau\right)^2 d\tau\right]$$

where $\bar{\tau}_l, \tilde{\tau}_l \in [t_{l-1}, t_{l+1}]$,there is

$$\Delta t \sum_{j=1}^{J-2} |R_j|_{-\frac{1}{2}}^2 \le \sum_{j=1}^{J-2} |R_j|^2 \le C_{17}(\Delta t)^4 \|u_t^{(4)}\|_{L^\infty(L^2(\Omega))}^2 \quad (4.26)$$

Then we obtain the estimate of the last term on the right side of (4.6) from (4.24), (4.25) and (4.26)

$$\Delta t \left|\sum_{j=1}^{m} \langle \alpha_1 \eta_{j+\frac{1}{4}} + \alpha_2(\delta_t \eta^j - S_j), \delta_t \xi^j \rangle\right| \le C_{18}\left[\Delta t \sum_{j=1}^{m}\left(|\delta_t \eta^j|_{-\frac{1}{2}}^2 + |\partial_t^2 \eta^j|_{-\frac{1}{2}}^2\right) + \|\eta\|_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))}^2 + |\delta_t \eta^m|_{-\frac{1}{2}}^2 + |\delta_t \eta^1|_{-\frac{1}{2}}^2 + \left\|\xi^{\frac{1}{2}}\right\|_1^2 + (\Delta t)^4 + \Delta t \sum_{j=1}^{m-1} \left\|\xi^{j+\frac{1}{2}}\right\|_1^2\right] + \varepsilon \left\|\xi^{m+\frac{1}{2}}\right\|_1^2 \quad (4.27)$$

In addition, from

$$\partial_t^2 \eta^j = \frac{1}{(\Delta t)^2}\left[\int_{t_j}^{t_{j+1}} \frac{\partial^2 \eta}{\partial t^2}(\tau)\left(t_{j+1}-\tau\right)d\tau + \int_{t_j}^{t_{j-1}} \frac{\partial^2 \eta}{\partial t^2}(\tau)\left(t_{j-1}-\tau\right)d\tau\right]$$

we can deduce

$$\Delta t \sum_{j=1}^{J-1} \|\partial_t^2 \eta^j\|^2 \le C_{19} \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|_{L^2(L^2(\Omega))}^2 \quad (4.28)$$

$$\Delta t \sum_{j=1}^{J-1} |\partial_t^2 \eta^j|_{-\frac{1}{2}}^2 \le C_{20} \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|_{L^2(H^{-\frac{1}{2}}(\partial\Omega))}^2 \quad (4.29)$$

and from

$$\delta_t \eta^j = \frac{\partial \eta}{\partial t}(\tilde{t}_j)$$

we deduce

$$\|\delta_t \eta^j\|^2 = \left\|\frac{\partial \eta}{\partial t}\right\|_{L^\infty(L^2(\Omega))}^2 \quad (4.30)$$

$$|\delta_t \eta^j|_{-\frac{1}{2}}^2 \le \left\|\frac{\partial \eta}{\partial t}\right\|_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))}^2 \quad (4.31)$$

Taking $\varepsilon$ small enough, then when $\Delta t$ is appropriately small, we can obtain following estimate from (4.6), (4.21)-(4.23) and (4.27)-(4.31)

$$\left\|\xi^{m+\frac{1}{2}}\right\|_1^2 + \left\|\partial_t \xi^{m+\frac{1}{2}}\right\|^2 \le C_{21}\left[\left\|\xi^{\frac{1}{2}}\right\|_1^2 + \left\|\partial_t \xi^{\frac{1}{2}}\right\|^2 + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|_{L^2(L^2(\Omega))}^2 + \left\|\frac{\partial \eta}{\partial t}\right\|_{L^\infty(L^2(\Omega))}^2 + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|_{L^2(H^{-\frac{1}{2}}(\partial\Omega))}^2 + \left\|\frac{\partial \eta}{\partial t}\right\|_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))}^2 + \|\eta\|_{L^\infty(L^2(\Omega))}^2 + \|\eta\|_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))}^2 + (\Delta t)^4 + \Delta t \sum_{j=1}^{m-1}\left(\left\|\xi^{j+\frac{1}{2}}\right\|_1^2 + \left\|\partial_t \xi^{j+\frac{1}{2}}\right\|^2\right)\right] \quad (4.32)$$

From (4.32) and using Gronwall inequality, we deduced conclusion of lemma4.

■

By lemma1, lemma2, lemma4, and considering

$$\|\eta\|_{\tilde{L}^\infty(L^2(\Omega))}^2 \le \|\eta\|_{L^\infty(L^2(\Omega))}^2$$

and $\|\partial_t \eta\|_{\tilde{L}^\infty(L^2(\Omega))}^2 \le \left\|\frac{\partial \eta}{\partial t}\right\|_{L^\infty(L^2(\Omega))}^2$

we get the following optimal $L^2$-error estimates for the fully discrete Galerkin approximation.

***Theorem 2.*** Let $u$ be the true solution of problem (1.1), $\{U^j\}_{j=0}^J$ be its fully discrete Galerkin approximation determined by (4.2), $w$ be determined by (2.2). Assume condition (A) exists, and also $u_t^{(3)}, u_t^{(4)} \in L^\infty(L^2(\partial\Omega)), u_t^{(4)} \in L^2(L^2(\Omega))$,and

$$\left\|U^{\frac{1}{2}} - w^{\frac{1}{2}}\right\|_1 \le C_{22}(h^s + (\Delta t)^2),$$

$$\left\|\partial_t (U-w)^{\frac{1}{2}}\right\| \le C_{23}(h^s + (\Delta t)^2),$$

then there is a constant $C$independent of $h$ and $\Delta t$ such that

$$\|U-u\|_{\tilde{L}^\infty(L^2(\Omega))} + \|\partial_t(U-u)\|_{\tilde{L}^\infty(L^2(\Omega))} \le C(h^s + (\Delta t)^2)$$

where $s = \min(k, m)$.

## V. CONCLUSION

Galerkin approximation is an important numerical method for differential equations. The main focus of this paper was to provide the optimal $L^2$-error estimates for continuous time and fully discrete Galerkin approximations for a type of second order nonlinear hyperbolic equations and give detail proof. These are theoretical estimates. A possible further research is practical numerical calculation to verify the theoretical results.

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