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Uniform Order Eleven of Eight-Step Hybrid Block Backward Differentiation Formulae for the Solution of Stiff Ordinary Differential Equations

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ABSTRACT
In this research, we developed a uniform order eleven of eight step Second derivative hybrid block backward differentiation formula for integration of stiff systems in ordinary differential equations. The single continuous formulation developed is evaluated at some grid point of $x = x_{n+j}, j = 0,1,2,3,4,5\ and 6$ and its first derivative was also evaluated at off-grid point $x = x_{n+j}, j = \frac{15}{2}$ and grid point $x = x_{n+j}, j = 8$. The method is suitable for the solution of stiff ordinary differential equations and the accuracy and stability properties of the newly constructed method are investigated and are shown to be A-stable. Our numerical results obtained are compared with the theoretical solutions as well as ODE23 solver.

KEYWORDS: Backward Differentiation Formula, Second Derivative, off-step points, Stiff ODEs

1. INTRODUCTION

In sciences, researchers are not desired in all solutions to a differential equation, but only in those extra conditions satisfied solutions. For instance, in the case of Newton's second law of motion for a point particle, one could be actually interested only in solutions such that the particle that specifically position at some initial time. Such condition is called an initial condition.

A first order ordinary differential equations (ODEs) on the unknown y is

$$y'(t) = f(t, y(t)) \text{ and } y(t_0) = y_0 \qquad a \le t \le b \qquad (1)$$

where f is continuous within the interval of integration. is sufficiently differentiable and satisfies a Lipschitz condition (1).

Moreover, Awoyemi and Idowu (2005) derived a class of hybrid collocation methods for third–order ordinary differential equations. Hybrid block scheme were constructed to defeat the Dahlquist (1956) barrier theorem in accordance with the conventional linear multistep method was changed to include off-step points of derivation process (see Lambert (1973) and Gear (1965)).

Similarly, Researchers like Butcher (2003), Zarina *et al.* (2005), Awoyemi *et al.* (2007), Areo *etal.* (2011), Ibijola *et al*. (2011) have all constructed linear multistep methods (LMMs) to generate numerical solution to (1).

Nasir *et al.* (2011), Zawawi *et al.* (2012), Akinfenwa et al. (2013), construct Block Backward Differentiation Formulas for Solving Ordinary Differential Equations. This method possesses the requirement for stiffly stable and suitable to solve stiff problems. The numerical results were presented to verify the efficiency of this method as compared to the Classical Backward Differentiation Formula (BDF) method and ode15s in Matlab. The method outperformed the BDF method and ode15s in terms of maximum error and execution time.

2. CONSTRUCTION OF UNIFORM ORDER ELEVEN OF EIGHT-STEP HYBRID

In this research, we constructed eight-step hybrid block backward differentiation formula. The method will be applied in block form by Onumanyi *et al* (1994,1999) where $k-step$ multistep collocation method with t interpolation points and m collocation points were obtained as follows:

$$y(x)=\sum_{j=0}^{r-1}\alpha_j(x)y_{n+j}+h\sum_{j=0}^{s-1}\beta_j(x)f_{n+j} \qquad (2)$$

and its extension it to the second derivative gives the general form of the continuous k –step 2nd derivative linear multistep method as:

$$y(x)=\sum_{j=0}^{r-1}\alpha_j(x)y_{n+j}+h\sum_{j=0}^{s-1}\beta_j(x)f_{n+j}+h^2\sum_{j=0}^{t-1}\gamma_j(x)g_{n+j} \qquad (3)$$

Here the coefficients $\alpha_j(x)$, $\beta_j(x)$ and $\gamma_j(x)$ are polynomials of degree $(p-1)$.

where

$$\alpha_j(x)=\sum_{i=0}^{r+s+t-1}\alpha_{j.i+1}x^i \qquad (4)$$

$$h\beta_j(x)=\sum_{i=0}^{r+s+t-1}h\beta_{j.i+1}x^i \qquad (5)$$

$$h^2\beta_j(x)=\sum_{i=0}^{r+s+t-1}h^2\gamma_{j.i+1}x^i \qquad (6)$$

2.1. Specification of the Hybrid Block Methods

From equation (2) - (6) we propose continuous k-step of extended block hybrid backward differentiation formula of the form

$$y(x)=\sum_{j}^{k}\alpha_j(x)y_{n+j}+h\beta_{k-1}(x)f_{n+k-1}+h\beta_{\frac{2k-1}{2}}f_{n+\frac{2k-1}{2}}+h\beta_k(x)f_{n+k}+h^2\gamma_{k-1}(x)g_{n+k-1}+\quad h^2\gamma_k(x)g_{n+k} \qquad (7)$$

2.1 .2 Eight Step Method with One Off-Step Point

In this case $=8$, $S=8$, $t=8$ and (7) becomes

$$y(x)=\alpha_0(x)y_n+\alpha_1(x)y_{n+1}+\alpha_2(x)y_{n+2}+\alpha_3(x)y_{n+3}+\alpha_4(x)y_{n+4}+\alpha_5(x)y_{n+5}+\alpha_6(x)f_{n+6}+\alpha_7(x)y_{n+7}=h[\beta_0(x)f_n+\beta_1(x)f_{n+1}+\beta_2(x)f_{n+2}+\beta_3(x)f_{n+3}+\beta_4(x)f_{n+4}+\beta_5(x)f_{n+5}+\beta_6(x)f_{n+6}+\beta_7(x)f_{n+7}+\beta_{\frac{15}{2}}(x)f_{n+\frac{15}{2}}+\beta_8(x)f_{n+8}]+h^2[\gamma_0(x)g_n+\gamma_1(x)g_{n+1}+\gamma_2(x)g_{n+2}+\gamma_3(x)g_{n+3}+\gamma_4(x)g_{n+4}+\gamma_5(x)g_{n+5}+\gamma_6(x)g_{n+6}+\gamma_7(x)g_{n+7}+\gamma_8(x)g_{n+8}] \qquad (8)$$

and interpolating (4)at $x=x_n,x_{n+1},x_{n+2}\ ,x_{n+3},x_{n+4},x_{n+5},x_{n+6},x_{n+7}$ and collocating (5) at $x=x_{n+7},x_{n+\frac{15}{2},}x_{n+8}$ to give the systems of equation.

$$y(x) = \sum_{j=0}^{6} a_j x^j \ = y_n \tag{9}$$

$$y'(x) = \sum_{j=1}^{6} j a_j x^{j-1} \ = f_{n+j} \tag{10}$$

$$y''(x) = \sum_{j=2}^{6} j(j-1) a_j x^{j-2} \ = f_{n+j} \tag{11}$$

Expressing the system of equations (9) -(11) in the form $AX = Y$ as:

$$A=\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} & x_n^{11} & x_n^{12} \\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 & x_{n+1}^{10} & x_{n+1}^{11} & x_{n+1}^{12} \\
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 & x_{n+2}^{10} & x_{n+2}^{11} & x_{n+2}^{12} \\
1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 & x_{n+3}^{10} & x_{n+3}^{11} & x_{n+3}^{12} \\
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 & x_{n+4}^9 & x_{n+4}^{10} & x_{n+4}^{11} & x_{n+4}^{12} \\
1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+5}^5 & x_{n+5}^6 & x_{n+5}^7 & x_{n+5}^8 & x_{n+5}^9 & x_{n+5}^{10} & x_{n+5}^{11} & x_{n+5}^{12} \\
1 & x_{n+6} & x_{n+6}^2 & x_{n+6}^3 & x_{n+6}^4 & x_{n+6}^5 & x_{n+6}^6 & x_{n+6}^7 & x_{n+6}^8 & x_{n+6}^9 & x_{n+6}^{10} & x_{n+6}^{11} & x_{n+6}^{12} \\
1 & x_{n+7} & x_{n+7}^2 & x_{n+7}^3 & x_{n+7}^4 & x_{n+7}^5 & x_{n+7}^6 & x_{n+7}^7 & x_{n+7}^8 & x_{n+7}^9 & x_{n+7}^{10} & x_{n+7}^{11} & x_{n+7}^{12} \\
0 & 1 & 2x_{n+7} & 3x_{n+7}^2 & 4x_{n+7}^3 & 5x_{n+7}^4 & 6x_{n+7}^5 & 7x_{n+7}^6 & 8x_{n+7}^7 & 9x_{n+7}^8 & 10x_{n+7}^9 & 11x_{n+7}^{10} & 12x_{n+7}^{11} \\
0 & 1 & 2x_{n+\frac{15}{2}} & 3x_{n+\frac{15}{2}}^2 & 4x_{n+\frac{15}{2}}^3 & 5x_{n+\frac{15}{2}}^3 & 6x_{n+\frac{15}{2}}^5 & 7x_{n+\frac{15}{2}}^6 & 8x_{n+\frac{15}{2}}^7 & 9x_{n+\frac{15}{2}}^8 & 10x_{n+\frac{15}{2}}^9 & 11x_{n+\frac{15}{2}}^{10} & 12x_{n+\frac{15}{2}}^{11} \\
0 & 1 & 2x_{n+8} & 3x_{n+8}^2 & 4x_{n+8}^3 & 5x_{n+8}^3 & 6x_{n+8}^5 & 7x_{n+8}^6 & 8x_{n+8}^7 & 9x_{n+8}^8 & 10x_{n+8}^9 & 11x_{n+8}^{10} & 12x_{n+8}^{11} \\
0 & 0 & 2 & 6x_{n+7} & 12x_{n+7}^2 & 15x_{n+7}^3 & 30x_{n+7}^3 & 42x_{n+7}^5 & 56x_{n+7}^6 & 72x_{n+7}^7 & 90x_{n+7}^8 & 110x_{n+7}^9 & 132x_{n+7}^{10} \\
0 & 0 & 2 & 6x_{n+8} & 12x_{n+8}^2 & 15x_{n+8}^3 & 30x_{n+8}^3 & 42x_{n+8}^5 & 56x_{n+8}^7 & 72x_{n+8}^7 & 90x_{n+8}^6 & 110x_{n+8}^9 & 132x_{n+8}^{10}
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \end{bmatrix}
=
\begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \\ f_{n+7} \\ f_{n+\frac{15}{2}} \\ f_{n+8} \\ g_{n+7} \\ g_{n+8} \end{bmatrix} \tag{12}$$

Using Maple software and inverting the matrix in (12) is yields the elements of the matrix A^{-1}`.

The columns of A^{-1}` give the continuous coefficients of (4)-(6) as:

$$\alpha_0(x) = 1 - \frac{95958313819129x}{29479449569430h} + \frac{111379502927091629x^2}{24762737638321200h^2} - \frac{147520826048566619623x^3}{4160139923237961600h^3} + \frac{597881727567689971 1x^4}{3328111938590369280h^4} - \frac{607983743988885997x^5}{978856452526579200h^5} + \frac{1006434668766110 9357x^6}{66562238771807385600Sh^6} - \frac{8316257821028201x^7}{316963041770511360h^7} + \frac{10247925246994591x^8}{3169630417705113600h^8} - \frac{1151437056841919x^9}{4160139923237961600h^9} + \frac{1042639590961571x^{10}}{66562238771807385600h^{10}} - \frac{17550288818501x^{11}}{33281119385903692800h^{11}} + \frac{31325547497x^{12}}{3915425810106316800h^{12}}$$

$$\alpha_1(x) = \frac{5999869653050x}{421134993849h} - \frac{162362039569331x^2}{5053619926188h^2} + \frac{19376892463308499x^3}{606434391142560h^3} - \frac{135118095989390719x^4}{7277212693710720h^4} + \frac{1002553554614321x^5}{142690444974720h^5} - \frac{53054436242457737x^6}{29108850774842880h^6} + \frac{11911219801561x^7}{35936852808448h^7} - \frac{82149948379561x^8}{1940590051656192h^8} + \frac{4520601070249x^9}{1212868782285120h^9} - \frac{6280178983223\,x^{10}}{29108850774842880h^{10}} + \frac{35892089683x^{11}}{4851475129140480h^{11}} - \frac{195148837x^{12}}{1712285339696640h^{12}}$$

$$\alpha_2(x) = -\frac{6875640670346x}{140378331283h} + \frac{5682835861917011x^2}{42113499384900h^2} - \frac{77232977678555219x^3}{505361992618800 0h^3} + \frac{589837246949700802 3x^4}{60643439114256000h^4} - \frac{14038171489246897 3x^5}{3567261124368000h^5} + \frac{2606002022753332429x^6}{242573756457024000h^6} - \frac{3288459697500179x^7}{16171583763801 60h^7} + \frac{289136258920171x^8}{1078105584253440h^8} - \frac{122500746419197x^9}{5053619926188000h^9} + \frac{347843988763331x^{10}}{242573756457024000h^{10}} - \frac{6073810272629x^{11}}{121286878228512000h^{11}} + \frac{11180057209x^{12}}{14269044497472000h^{12}}$$

$$\alpha_3(x) = \frac{124156819047625x}{842269987698h} - \frac{8705963111552455x^2}{20214479704752h^2} + \frac{84000083720233513x^3}{161715837638016h^3} - \frac{753668034392117 17x^4}{215621116850688h^4} + \frac{5641881259818595x^5}{38050785326592h^5} - \frac{108983688887548537x^6}{2587453402208256h^6} + \frac{3555263960924639x^7}{431242233701376h^7} - \frac{2894393499959887x^8}{2587453402208256h^8} + \frac{33503499322231x^9}{324331675276032h^9} - \frac{5396554991149\,x^{10}}{862484467402752h^{10}} + \frac{287842984985x^{11}}{1293726701104128h^{11}} - \frac{538239565x^{12}}{152203141306368h^{12}}$$

$$\alpha_4(x) = -\frac{324847870920125x}{842269987698h} + \frac{23427907792376395x^2}{20214479704752h^2} - \frac{700976440676708 79x^3}{485147512914048h^3} + \frac{5858213737831411585x^4}{5821770154968576h^4} - \frac{503607486682636 45x^5}{114152355979776h^5} + \frac{3008740914626408279x^6}{23287080619874304h^6} - \frac{33633230900175019x^7}{1293726701104128h^7} + \frac{28069177569117659x^8}{7762360206624768h^8} - \frac{166103439749567x^9}{485147512914048h^9} + \frac{491256957909977x^{10}}{23287080619874304h^{10}} - \frac{2963717454581x^{11}}{3881180103312384h^{11}} + \frac{16893432587x^{12}}{1369828271757312h^{12}}$$

$$\alpha_5(x) = \frac{698233548697762x}{701891656415h} - \frac{1279821905128554 83x^2}{42113499348900h^2} + \frac{3907504681261828703x^3}{1010723985237600h^3} - \frac{6679512172825498303x^4}{2425737564570240h^4} + \frac{881694260693648351x^5}{713452224873600h^5} - \frac{1797250740487240318 1x^6}{48514751291404800h^6} + \frac{1232943911807452 31x^7}{16171583763801 60h^7} - \frac{58404991154460107x^8}{5390527921267200h^8} + \frac{2115964283425693x^9}{2021447970475200h^9} - \frac{3188461778077843\,x^{10}}{48514751291404800h^{10}} + \frac{58725123939133x^{11}}{24257375645702400h^{11}} - \frac{x}{2853808899494400h^{12}}$$

$$\alpha_6(x) = -\frac{1591448100803050x}{421134993849h} + \frac{58973616807690011x^2}{5053619926188h^2} - \frac{9125523287132968219x^3}{606434391142560h^3} + \frac{2640794557478248 7893x^4}{2425737564570240h^4} - \frac{709173203422711231x^5}{142690444974720h^5} + \frac{147177660752751371 99x^6}{9702950258280960h^6} - \frac{10283700269670773 3x^7}{323431675276032h^7} + \frac{3307845701319711x^8}{71873705616896h^8} - \frac{2745685566667217x^9}{606434391142560h^9} + \frac{280733838723136x^{10}}{9702950258280960h^{10}} - \frac{52594481758343x^{11}}{4851475129140480h^{11}} + \frac{103239426419x^{12}}{570761779898880h^{12}}$$

$$\alpha_7(x) = \frac{18043919361064657x}{5895889913886h} - \frac{2344088895659759285 93x^2}{24762737638321200h^2} + \frac{25442689684224896387 2279x^3}{20800699616189808000h^3} - \frac{22140717769272678874 65043x^4}{24960839539427769600 0h^4} + \frac{220788937953081613150 9x^5}{543809140292544000h^5} - \frac{1240587114881513356098389x^6}{998433581577110784000h^6} + \frac{827893819304920783 19x^7}{31696304177051136 0h^7} - \frac{7210556382606736043 9x^8}{19017782506230681 60h^8} + \frac{1556063511770554403 29x^9}{41601399232379616000h^9} - \frac{2392039870645161857 71\,x^{10}}{998433581577110784000h^{10}} + \frac{499457292157396621x^{11}}{5546853230983948 8000h^{11}} - \frac{8845925643617369x^{12}}{5873138715159475 2000h^{12}}$$

$$\beta_7(x) = -\frac{30797607491730x}{140378331283} + \frac{12252449651037159x^2}{19652966379620h} - \frac{3924857667428473659x^3}{5502830586293600h^2} + \frac{28805166480229703303x^4}{66033967035523200h^3} - \frac{1824665680566703409x^5}{11653053006268800h^4} + \frac{26167954478105592707x^6}{792407604426278400h^5} - \frac{2623989574274443x^7}{754673908977408h^6} + \frac{64286272312943x^8}{1509347817954816h^7} - \frac{2052021323393473x^9}{33016983517761600h^8} + \frac{697442591071403x^{10}}{880452893806976 00h^9} - \frac{179476384395293x^{11}}{396203802213139200h^{10}} + \frac{479316757153x^{12}}{46612212025075200h^{11}}$$

$$\beta_{\frac{15}{2}}(x) = -\frac{428190539055104x}{140378331283} + \frac{6685811626819584x^2}{701891656415h} - \frac{7870814864065126 4x^3}{6317024907735h^2} + \frac{11583149024190976x^4}{1263404981547h^3} - \frac{1585503861215744x^5}{371589700455h^4} + \frac{8402119605809536x^6}{6317024907735h^5} - \frac{120118667889920x^7}{421134993849h^6} + \frac{89048602174336x^8}{2105674969245h^7} - \frac{26946704038912x^9}{6317024907735h^8} + \frac{1766885496448x^{10}}{63170249907735h^9} - \frac{67963410176x^{11}}{6317024907735h^{10}} + \frac{68496256x^{12}}{371589700455h^{11}}$$

$$\beta_8(x) = \frac{102709446821625x}{140378331283} - \frac{1285663550076535x^2}{561513325132h} + \frac{40470004012999603x^3}{13476319803168h^2} - \frac{35851135872260636 5x^4}{161715833763 8016h^3} + \frac{9851655119649403x^5}{9512696331648h^4} - \frac{209718963335245531x^6}{646863350552064h^5} + \frac{7530890237766749x^7}{107810558425344h^6} - \frac{748235136449605x^8}{71873705616896h^7} + \frac{14230248199867x^9}{13476319803168h^8} - \frac{45054856385429x^{10}}{646863350552064h^9} + \frac{872012211011x^{11}}{323431675276032h^{10}} - \frac{1769443135x^{12}}{38050785326592h^{11}}$$

$$\gamma_7(x) = \frac{252474550097700xh}{140378331283} - \frac{1570774572740333\ x^2}{280756662566} + \frac{1717354357900665499x^3}{235835596555440h} - \frac{15074283296663869543x^4}{2830027158665280h^2} + \frac{45540634791210947x^5}{18496909533760h^3} - \frac{8622045232427895329x^6}{11320108634661120h^4} + \frac{2910438299459199x^7}{17968426404224h^5} - \frac{2565992829113863x^8}{107810558425344h^6} + \frac{1121624842813669x^9}{471671193110880h^7} - \frac{1747468970243231x^{10}}{11320108634661120h^8} + \frac{3695515319961x^{11}}{628894924147840h^9} - \frac{66323138549x^{12}}{665888743215360h^{10}}$$

$$\gamma_8(x) = -\frac{16888540189500xh}{140378331283} + \frac{52898108169060\ x^2}{140378331283} - \frac{1111375015828707x^3}{2246053300528h} - \frac{3286416400864111x^4}{8984213202112h^2} + \frac{904578847981169x^5}{528483129536h^3} - \frac{1929273942531457x^6}{35936852808448h^4} + \frac{208278204400469x^7}{17968426404224h^5} - \frac{62226870194839x^8}{35936852808448h^6} + \frac{395507042623x^9}{2246053300528h^7} - \frac{418590251343x^{10}}{35936852808448h^8} + \frac{8126500505x^{11}}{17968426404224h^9} - \frac{16544781x^{12}}{2113932518144h^{10}}$$

Evaluating (8) at the following points at $x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}, x_{n+6}, x_{n+15/3}$ *and* x_{n+8} yields the following discrete methods which constitute the new eight step block method.

$$y_n - \frac{795573993889721900}{111379502927091629}y_{n+1} + \frac{3341507486807202468}{111379502927091629}y_{n+2} - \frac{10664804811651757375}{111379502927091629}y_{n+3} + \frac{28699187045661083875}{111379502927091629}y_{n+4} - \frac{75253528021559024004}{111379502927091629}y_{n+5} + \frac{288970722357681053900}{111379502927091629}y_{n+6} - \frac{234408889565975928593}{111379502927091629}y_{n+7} = \frac{1260h}{111379502927091629}\left[12252449651037159f_{n+7} + 187202725550948352f_{n+\frac{15}{2}} - 44998224252678725f_{n+8}\right] + \frac{88200h^2}{111379502927091629}[140378331283g_n + 1570774572740333g_{n+7} - 105796216338120g_{n+8}]$$

$$y_n + \frac{25231414254947640}{4965391873495939}y_{n+1} - \frac{918591857070157676}{24826959367479695}y_{n+2} + \frac{585503967350684025}{4965391873495939}y_{n+3} - \frac{4482510795788159225}{14896175620487817}y_{n+4} + \frac{3773303822956282636}{4965391873495939}y_{n+5} - \frac{14112172589850215860}{4965391873495939}y_{n+6} + \frac{171015849451373356153}{74480878102439085}y_{n+7} = \frac{28h}{4965391873495939}\left[32832731120396717f_{n+7} + 399847878563266560f_{n+\frac{15}{2}} - 95591127585482025f_{n+8}\right] + \frac{1960h^2}{496539187349 5939}[5615133251320g_{n+1} - 3388606915725319g_{n+7} + 224219376062145g_{n+8}]$$

$$y_n - \frac{7222751481023540}{206294902600341}y_{n+1} + \frac{8110158627648812}{5157372565008525}y_{n+2} + \frac{19088416297961325}{68764967533447}y_{n+3} - \frac{172698241297211525}{206294902600341}y_{n+4} + \frac{423413680921142884}{206294902600341}y_{n+5} - \frac{512403409953341620}{68764967533447}y_{n+6} + \frac{30898039789388169313}{5157372565008525}y_{n+7} = \frac{28h}{343824837667235}\left[7185554611961157f_{n+7} + 70180046374502400f_{n+\frac{15}{2}} - 16669945678353625f_{n+8}\right] - \frac{392h^2}{68764967533447}[7861186551848g_{n+2} + 601842327156471g_{n+7} - 38993163236825g_{n+8}]$$

$$y_n - \frac{2070320941916780}{88165482138851}y_{n+1} + \frac{180752455135885332}{440827410694255}y_{n+2} - \frac{415389786524529 25}{88165482138851}y_{n+3} - \frac{85609840632881075}{88165482138851}y_{n+4} + \frac{333063784390344924}{88165482138851}y_{n+5} - \frac{121032505604816722 0}{88165482138851}y_{n+6} + \frac{485120877687878579 3}{440827410694255}y_{n+7} = \frac{252h}{88165482138851}\left[440062565156559f_{n+7} + 3569446989660160f_{n+\frac{15}{2}} - 841909412290625f_{n+8}\right] + \frac{17640h^2}{88165482138851}[1965296637962g_{n+3} - 30992386067543g_{n+7} + 1963324978713g_{n+8}]$$

$$y_n - \frac{535551344432860}{26631670316133}y_{n+1} + \frac{10225338751074412}{44386117193555}y_{n+2} - \frac{827110478267325}{306111153059}y_{n+3} + \frac{1210127800927725}{306111153059}y_{n+4} + \frac{21449184720713764}{8877223438711}y_{n+5} - \frac{190583150657241580}{8877223438711}y_{n+6} + \frac{234226553226450363 9}{133158351580665}y_{n+7} = \frac{28h}{8877223438711}\left[667521725849171f_{n+7} + 5151530052157440f_{n+\frac{15}{2}} - 1207191039323325f_{n+8}\right] - \frac{1960h^2}{8877223438711}[9826483189810g_{n+4} + 45078599072657g_{n+7} - 2806256440575g_{n+8}]$$

$$y_n - \frac{28732889416700}{1558701718659}y_{n+1} + \frac{14647518007276}{82036932561}y_{n+2} - \frac{231791360980375}{173189079851}y_{n+3} + \frac{18773267530814125}{1558701718659}y_{n+4} - \frac{3928035699532}{219938157}y_{n+5} - \frac{3126320160999700}{173189079851}y_{n+6} + \frac{39036596515149631}{1558701718659}y_{n+7} = \frac{140h}{519567239553}\left[7012323610059f_{n+7} + 98186615783424f_{n+\frac{15}{2}} - 22800086843975f_{n+8}\right] + \frac{9800h^2}{519567239553}[462422738344g_{n+5} - 855181579233g_{n+7} + 52722651695g_{n+8}]$$

$$y_n - \frac{393848612015180}{22694743922483}y_{n+1} + \frac{17002139804752068}{113473719612415}y_{n+2} - \frac{21584615528729575}{22694743922483}y_{n+3} + \frac{120848531925713725}{22694743922483}y_{n+4} - \frac{931785054474996708}{22694743922483}y_{n+5} + \frac{669436923774516380}{22694743922483}y_{n+6} + \frac{799974701053192307}{113473719612415}y_{n+7} - \frac{252h}{22694743922483}\left[415729917509259f_{n+7} - 5120456609300480f_{n+\frac{15}{2}} + 1122075341897825f_{n+8}\right] - \frac{17640h^2}{22694743922483}[39305932759240g_{n+6} + 47323702151687g_{n+7} - 256515380385g_{n+8}]$$

$$y_{n+\frac{15}{2}} + \frac{1385420343021}{8243051549790568448}y_n - \frac{816784820775}{294394698206806016}y_{n+1} + \frac{6675149415753}{294394698206806016}y_{n+2} - \frac{148545520998875}{1177578792827224064}y_{n+3} + \frac{675021185821125}{1177578792827224064}y_{n+4} - \frac{779956749745173}{294394698206806016}y_{n+5} + \frac{6268503501535275}{294394698206806016}y_{n+6} - \frac{8400581608123449459}{8243051549790568448}y_{n+7} = \frac{6435h}{294394698206806016}\left[14072591027781f_{n+7} + 8522418946048f_{n+\frac{15}{2}} - 512303316525f_{n+8}\right] + \frac{289864575h^2}{1471973491034030 08}[24728719g_{n+7} + 768915g_{n+8}]$$

$$y_{n+8} - \frac{40335}{140378331283}y_n + \frac{654800}{140378331283}y_{n+1} - \frac{5230064}{140378331283}y_{n+2} + \frac{281410092}{140378331283}y_{n+3} - \frac{121250500}{140378331283}y_{n+4} + \frac{507886960}{140378331283}y_{n+5} - \frac{3191134800}{140378331283}y_{n+6} - \frac{137597358436}{140378331283}y_{n+7} = \frac{2520h}{140378331283}\left[12281666f_{n+7} + 32112640f_{n+\frac{15}{2}} + 12283227f_{n+8}\right] - \frac{352800h^2}{140378331283}[491g_{n+7} + 5692g_{n+8}] \quad (13)$$

3. CONVERGENCE AND STABILITY ANALYSIS OF THE METHOD

In this section the analysis of the newly constructed methods is carried by analyzing the order, error constant consistency, convergence and plotting the regions of absolute stability.

The new method in equation (3) is expressed in the form:

$$\sum_{j=0}^{k}(\alpha_j y_{n+j} - h\beta_j f_{n+j} - h^2\lambda_j f_{n+j}) = 0 \qquad (14)$$

Which is written symbolically as:

$$\rho(E)y_n - h^2\sigma(E) = 0 \qquad (15)$$

E being the shift operator and is defined as $E^i y_n = y_{n+j}$, $\rho(E)$and $\sigma(E)$ are respectively the first and second characteristics polynomial of the LMM such that

$$\rho(E) = \sum_{j=0}^{k} \alpha_j E^j, \qquad \sigma(E) = \sum_{j=0}^{k} \beta_j E^j \qquad (16)$$

Following (Fatunla, 1991) and (Lambert,1973) we defined the local truncation error associated with (3) to be linear difference operator

$$[y(x);h] = \sum_{j=0}^{k} \alpha_j y_{n+j} - h\beta_k f_{n+k} - h^2\gamma_k g_{n+k} \ . \qquad (17)$$

Assuming that $y(x)$ is sufficiently differentiable, we can we expand the terms in (17) as a Taylor series and comparing the coefficients of h gives

$$L[y(x);h] = c_0 y(x) + c_1 h y'^{(x)} + c_2 h^2 y''^{(x)} + \cdots + c_p h^p y^p(x) + \cdots \qquad (18)$$

Where the constants $C_p, p = 0, 1,2 \ldots., j = 1,2, \ldots k$.

Using the concept above, the hybrid block method are obtained with the help of MAPLE 18 SOFTWARE have the following uniform order and error constants.

3.1 Table 1 Order and Error Constants of the Eight -Step

y_n	11	$\frac{1109782068501692015}{7645089080915569414 56}$
y_{n+1}	11	$-\frac{1412674980352712279}{1022473494590283758 88}$
y_{n+2}	11	$-\frac{14601896648447213}{472002737149580208}$
y_{n+3}	11	$-\frac{29213909016714541}{605167869401073264}$
y_{n+4}	11	$-\frac{12085140283692917}{182799785049936912}$
y_{n+5}	11	$-\frac{922206057457405}{10698928596875376}$
y_{n+6}	11	$-\frac{1575458937383051}{14161520207629392}$
$y_{n+\frac{15}{2}}$	11	$-\frac{463663785585}{1884126068523558502 4}$
y_{n+8}	11	$\frac{11276615}{240889216481628}$

3.2 Zero Stability of the Eight -Step Method

The zero stability of the eight-step new method is determined using the approach of Ehigie *et al.* (2014) in which Maple18 software was used to yields the stability polynomial of the method as

$$\rho(r) = \det(rA - B) \qquad (19)$$

$$=\det\left[r\begin{bmatrix} \frac{7955739938897219 00}{111379502927091629} & \frac{3341507486807202468}{111379502927091629} & \frac{10664804811651757375}{111379502927091629} & \frac{28269918704566108 3875}{111379502927091629} & \frac{-7525352802155902 4004}{111379502927091629} & \frac{2889707223576810 53900}{111379502927091629} & \frac{23440888956597592 89593}{111379502927091629} & 0 & 0 \\ \frac{25231414254947460}{4965391873495939} & \frac{-918591857070157676}{24826959367479695} & \frac{585503967350684025}{4965391873495939} & \frac{-448251079578815922 5}{14896175620487817} & \frac{377330382295628263 6}{4965391873495939} & \frac{-1411217258985021586 0}{4965391873495939} & \frac{17101584945137335 6153}{74480878102439085} & 0 & 0 \\ \frac{-7222751481023540}{206294902600341} & \frac{8110158627648812}{5157372565008525} & \frac{19088416297961325}{68764967533447} & \frac{-172698241297211525}{206294902600341} & \frac{423413680921142884}{2062949A02600341} & \frac{-512403409953341620}{68764967533447} & \frac{30898039789388169313}{5157372565008525} & 0 & 0 \\ \frac{-2070320941916780}{88165482138851} & \frac{180752455135885332}{440827410694255} & \frac{-41538978652452925}{88165482138851} & \frac{85609840632881075}{88165482138851} & \frac{333063784390344924}{88165482138851} & \frac{-1210325056048167220}{88165482138851} & \frac{4851208776878785793}{440827410694255} & 0 & 0 \\ \frac{535551344432860}{26631670316133} & \frac{10225338751074412}{44386117193555} & \frac{827110478267325}{306111153059} & \frac{1210127809027725}{306111153059} & \frac{21449184720713764}{8877223438711} & \frac{-190583150657241580}{306111153059} & \frac{2342265532284503639}{133158351580665} & 0 & 0 \\ \frac{-28732889416700}{1558701718659} & \frac{14647518007276}{82036932561} & \frac{-231791360980375}{173189079851} & \frac{18773267530814125}{1558701718659} & \frac{3928035699532}{219938157} & \frac{-3126320160999700}{173189079851} & \frac{39036596515149631}{1558701718659} & 0 & 0 \\ \frac{-393848612015180}{22694743922483} & \frac{17302139804752068}{113473719612415} & \frac{-21584615528729575}{22694743922483} & \frac{120848531925713725}{22694743922483} & \frac{-931785054474996708}{22694743922483} & \frac{669436923774516380}{22694743922483} & \frac{799974701053192307}{113473719612415} & 0 & 0 \\ \frac{816784820775}{294394698206806016} & \frac{6675149415753}{294394698206806016} & \frac{-148545520998875}{1177578792827224064} & \frac{675021185821125}{1177578792827224064} & \frac{-779956749745173}{294394698206806016} & \frac{6268503501535275}{294394698206806016} & \frac{8400581608123449459}{8243051549790568448} & 1 & 0 \\ \frac{654800}{140378331283} & \frac{-5230064}{140378331283} & \frac{28141092}{140378331283} & \frac{-121250500}{140378331283} & \frac{507886960}{140378331283} & \frac{-3191134800}{140378331283} & \frac{-137597358436}{140378331283} & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1385420343021}{8243051549790568448} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{40335}{140378331283} \end{bmatrix}\right]$$

$$= det\begin{bmatrix} \frac{7955739938897219 00}{1113795029270916 29}r & \frac{33415074868072024 68}{1113795029270916 29}r & \frac{10664804811651757375}{1113795029270916 29}r & \frac{28269918704566108 3875}{1113795029270916 29}r & \frac{-75253528021559024004}{1113795029270916 29}r & \frac{28897072235768105 3900}{1113795029270916 29}r & \frac{23440888956597592 89593}{1113795029270916 29}r & 0 & 1 \\ \frac{2523141425494746 0}{4965391873495939}r & \frac{-9185918570701576 76}{24826959367479695}r & \frac{5855039673506840 25}{4965391873495939}r & \frac{-4482510795788159225}{14896175620487817}r & \frac{3773303822956282636}{4965391873495939}r & \frac{-14112172589850215860}{4965391873495939}r & \frac{17101584945137335 6153}{744808781024390 85}r & 0 & 1 \\ \frac{-722275148102354 0}{206294902600341}r & \frac{811015862764881 2}{5157372565008525}r & \frac{19088416297961325}{68764967533447}r & \frac{-17269824129721152 5}{206294902600341}r & \frac{4234136809211428 84}{206294902600341}r & \frac{-5124034099533416 20}{68764967533447}r & \frac{30898039789388169313}{5157372565008525}r & 0 & 1 \\ \frac{-207032094191678 0}{88165482138851}r & \frac{18075245513588533 2}{440827410694255}r & \frac{-41538978652452925}{88165482138851}r & \frac{85609840632881075}{88165482138851}r & \frac{333063784390344924}{88165482138851}r & \frac{-121032505604816722 0}{88165482138851}r & \frac{4851208776878785793}{440827410694255}r & 0 & 1 \\ \frac{535551344432860}{26631670316133}r & \frac{10225338751074412}{44386117193555}r & \frac{827110478267325}{306111153059}r & \frac{1210127809027725}{306111153059}r & \frac{21449184720713764}{8877223438711}r & \frac{-190583150657241580}{306111153059}r & \frac{2342265532284503639}{133158351580665}r & 0 & 1 \\ \frac{-28732889416700}{1558701718659}r & \frac{14647518007276}{82036932561}r & \frac{-231791360980375}{173189079851}r & \frac{18773267530814125}{1558701718659}r & \frac{3928035699532}{219938157}r & \frac{-3126320160999700}{173189079851}r & \frac{39036596515149631}{1558701718659}r & 0 & 1 \\ \frac{-393848612015180}{22694743922483}r & \frac{17302139804752068}{113473719612415}r & \frac{-21584615528729575}{22694743922483}r & \frac{120848531925713725}{22694743922483}r & \frac{-931785054474996708}{22694743922483}r & \frac{669436923774516380}{22694743922483}r & \frac{7999747010531923 07}{113473719612415}r & 0 & 1 \\ \frac{816784820775}{294394698206806016}r & \frac{6675149415753}{294394698206806016}r & \frac{-148545520998875}{1177578792827224064}r & \frac{675021185821125}{1177578792827224064}r & \frac{-779956749745173}{294394698206806016}r & \frac{6268503501535275}{294394698206806016}r & \frac{8400581608123449459}{8243051549790568448}r & r & \frac{-1385420343021}{8243051549790568448} \\ \frac{654800}{140378331283}r & \frac{-5230064}{140378331283}r & \frac{28141092}{140378331283}r & \frac{-121250500}{140378331283}r & \frac{507886960}{140378331283}r & \frac{-3191134800}{140378331283}r & \frac{-137597358436}{140378331283}r & 0 & r-\frac{40335}{140378331283} \end{bmatrix}$$

$$= r^9 - r^8 = 0$$

$$r_1 = 1, r_2 = r_3 = r_4 = r_5 = r_6 = r_7 = r_8 = r_9 = 0 < 1$$

Hence, the block method (13) is zero stable and is of order $P = 11$ and hence by Henrici (1962) it is convergent.

3.3 Absolute Stability Region of the Eight-Step Method

The coefficient of the block method (13) expressed in the form

$$A^{(1)}Y_{w+1} = B^{(0)}Y_w + hC^{(1)}F_w + h^2D^{(1)}G_w \qquad (20)$$

Where

$$y_{w+1} = (y_{n+1}, y_{n+2}, y_{n+3}, \dots \cdot y_{n-k-1}, y_{n+k})^T$$
$$y_w = (y_{n-k+1}, y_{n-k+2}, y_{n-k+3}, \dots \cdot y_{n-1}, y_n)^T$$
$$F_w = (f_{n+1}, f_{n+2}, y_{n+3}, \dots \cdot f_{n-k-1}, f_{n+k})^T$$

For $w = 0, \dots$ and $n = 0, \ k, \dots N - k$.

Applying the test problem

$$y' = \lambda y, \lambda < 0 \quad \text{to yield}$$
$$Y_{w+1} = D(z)Y_w, z = \lambda h$$

Where the matrix $D(z) = (r(A - Cz - D1z^2) - B)$ (21)

The matrix $D(z)$ has eigenvalues $\{\lambda_1, \lambda_2, \lambda_3, \dots \dots \dots, \lambda_k\} = \{0,0,0, \dots \dots \dots \lambda_k\}$

Where the dominant eigenvalue λ_k is the stability function $R(z): \mathbb{C} \to \mathbb{C}$ which is the rational function with real coefficients.

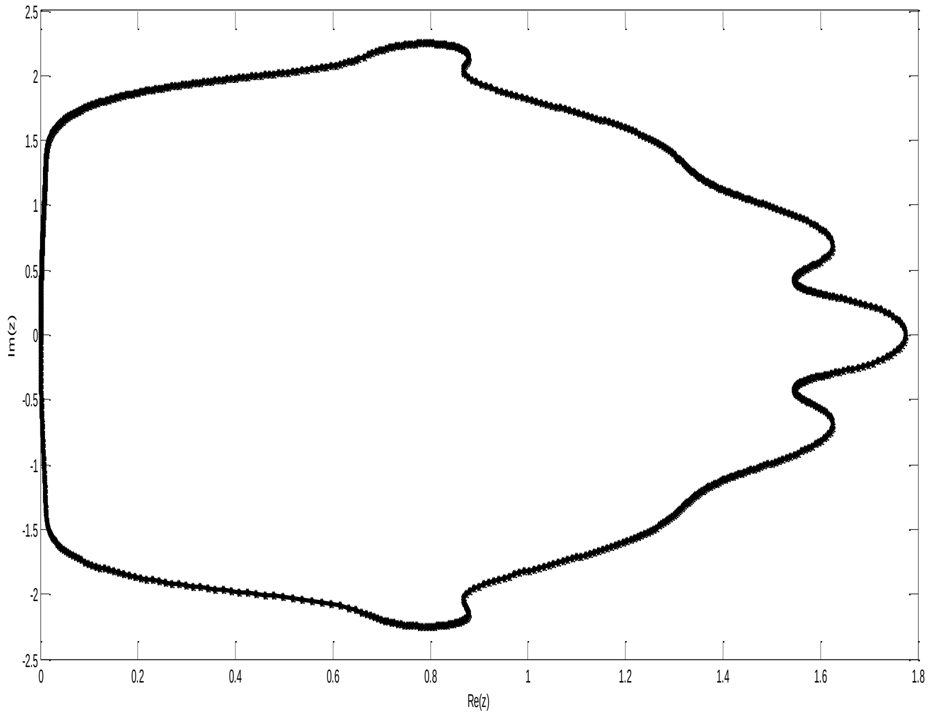

Figure 1. Region of absolute stability of newly constructed method for k=8

4. NUMERICAL EXPERIMENT

The Solution curves of stiff system plotted were compared with those of MATLAB ODE solver
Ode 23.

Problem 1

$$y_1' = 0.01y_1 - y_2 + y_3$$

$$y_2' = y_1 - 100.005y_2 + 99.995y_3$$

$$y_3' = 2y_1 + 99.995y_2 - 100.005y_3$$

Exact $y_{1(x)} = e^{-0.01x}(cos2x + sin2x)$ $\quad y_1(0) = 1$

$y_2(x) = e^{-0.01x}(cos2x + sin2x) + e^{-200x}$ $\quad y_2(0) = 1$

$y_2(x) = e^{-0.01x}(cos2x + sin2x)$ -e^{-200x} $\quad y_3(0) = 1$

$with\ h = 0.1$

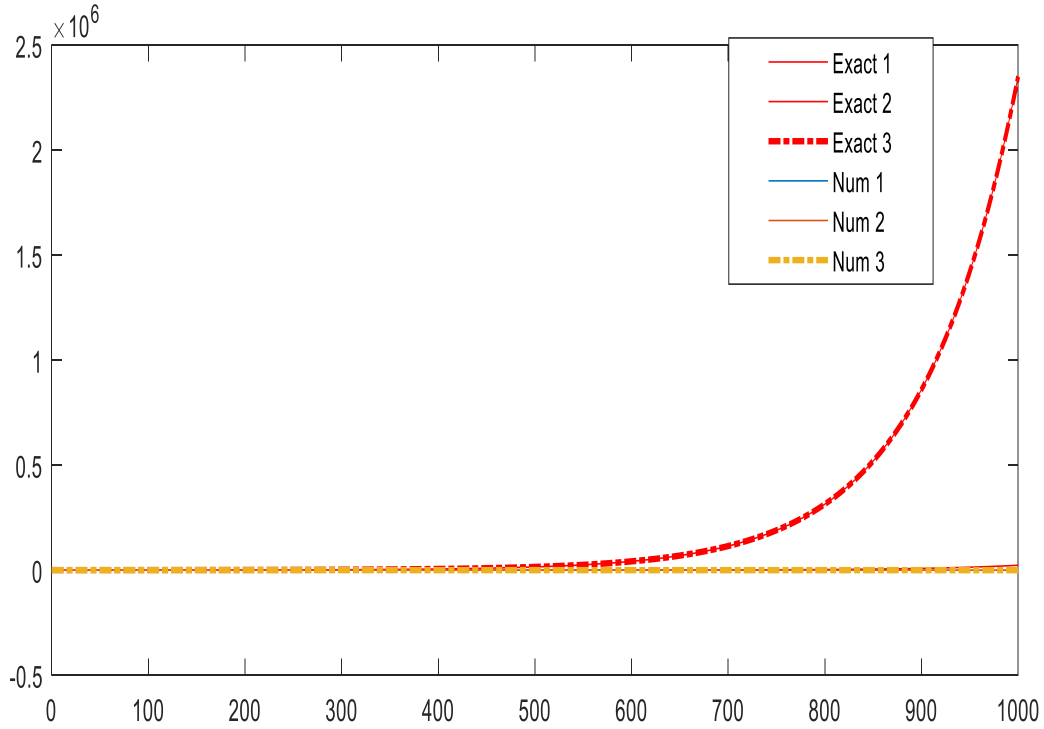

Figure 2: Solution Curve of Problem 1 Solved with eight-step

Problem 2

Robertson's Equations

$$y_1' = -0.04y_1 + 10000y_2y_3 \quad y_1(0) = 1$$

$$y_2' = 0.04y_1 - 10000y_2y_3 - 3000\,0000y_2^2 \quad y_2(0) = 0$$

$$y_3' = 3000\,0000y_2^2 \quad y_3(0) = 0$$

$$0 \le x \le 70 \text{ and } h = 0.1$$

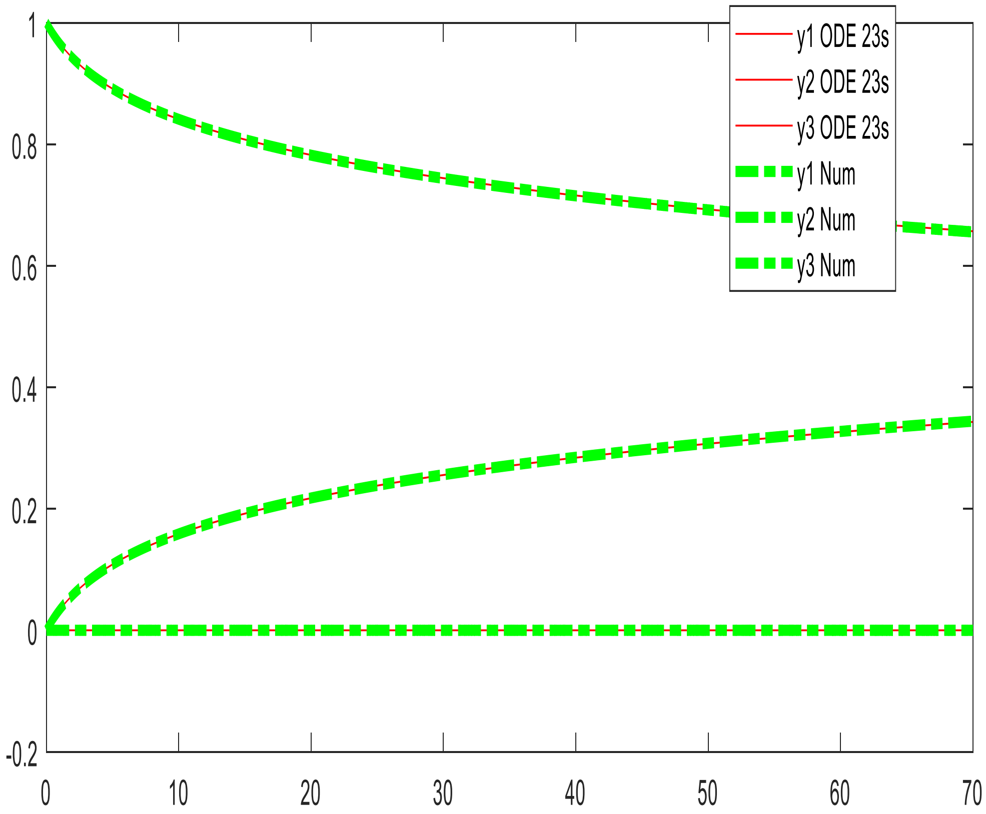

Figure 3: Solution Curve of Problem 2 Solved with eight-step

5. CONCLUSION

In this research, the uniform order eleven of eight-step hybrid block backward differentiation formulae for the solution of stiff ordinary differential equations was proposed and the scheme with one off-step point were analyzed and to be consistent and zero-stable and convergence. The newly constructed was tested on both linear and nonlinear stiff systems and the results obtained performed favorably compared to exact and ODE23 Solver.

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