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Some Double Sequence Spaces Generated by Four-Dimensional Pascal Matrix

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ABSTRACT

The purpose of this paper is to discover and examine a four-dimensional Pascal matrix domain on Pascal sequence spaces. We show that they are BK spaces and also establish their Schauder basis, topological properties, isomorphism and some inclusions.

KEYWORDS: Four-Dimensional, Pascal Matrix, BK spaces

1. BASIC NOTATIONS AND BACKGROUND

Let $\iota: \mathbb{N} \times \mathbb{N} \to \tau$ be a function, where τ may stand for any nonempty set and $\mathbb{N}$ a set of counting numbers. Then $(j,k) \to \iota(j,k) = x_{jk}$ can be termed to be a double sequence.

In 2018, see (Polat, 2018), the Pascal sequence spaces p_∞, p_c and p_0 were introduced. If P denote the Pascal means, then p_∞, p_c and p_0 are sets of all sequences whose P −transforms were in l_∞, c and c_0, the spaces of bounded, convergent and null sequences respectively. In 1991 (see Moricz, 1991), the sequence spaces c and c_0 were extended to double sequence spaces. Motivated by the work of Moricz, this paper will try to extend the sequence spaces of Pascal, which are, p_∞, p_c and p_0 to double sequence spaces; and study their properties. However, we need to fix some notations.

Let ω^2 be a vector space of all complex valued double sequences for which coordinatewise addition and scalar multiplications are defined. Further, a vector subspace of ω^2 is termed as a double sequence space. The space l_∞^2 denotes the space of all bounded sequences with norm$\|x\|_\infty = \sup_{j,k\in\mathbb{N}} |x_{jk}| < \infty, N = \{1,2,3,\dots\}$. If $x = x_{jk} \in \mathbb{C}$, then x is convergent to a number l in Pringsheim's sense if for every $\varepsilon > 0$, there exists a number $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{C}$ such that $|x_{jk} - l| < \varepsilon \ \forall\, j,k \geq n_0$, and we write $P - \lim_{j,k\to\infty} x_{jk} = l$, $\mathbb{C}$ being the complex field, see (Pringsheim, 1900). c^2 is used to denote the space of all convergent double sequences in Pringsheim's sense and c^2 not need to be bounded. Also, c_b^2 is the space of all double sequences which are both convergent in Pringsheim's sense and bounded; that is, $c_b^2 = c^2 \cap l_\infty^2$. c_0^2 is the space of all double sequences converging to zero in Pringsheim's sense, that is, $c_{0b}^2 = c_0^2 \cap l_\infty^2$. The following sequence spaces, see (Moricz, 1991) will be useful in the sequel:

$$l_\infty^2 = \left\{ x = (x_{jk}) \in \omega^2 : \sup_{j,k\geq 1} |x_{jk}| < \infty \right\},$$

$$c^2 = \left\{ x = (x_{jk}) \in \omega^2 : \exists\, l \in \mathbb{C} \ni \lim_{j,k\to\infty} |x_{jk}| = l \right\},$$

$$c_0^2 = \left\{ x = (x_{jk}) \in \omega^2 : \lim_{j,k\to\infty} |x_{jk}| = 0 \right\},$$

$$c_b^2 = \{ x = (x_{jk}) \in \omega^2 : x \in c^2 \cap l_\infty^2 \},$$

$$c_{0b}^2 = \{ x = (x_{jk}) \in \omega^2 : x \in c_0^2 \cap l_\infty^2 \}.$$

Let X and Y be two double sequence spaces and $A = (a_{mnjk})$ be any four-dimensional infinite matrix of complex numbers. Then A is said to define a matrix mapping from X into Y and write $A: X \to Y$, if for every $x = (x_{jk}) \in X$, the A −transform $Ax = \{(Ax)_{mn}\}_{mn}$ of x exists and is in Y, that is,

$$(Ax)_{mn} = P - \sum_{j,k}^{m,n} a_{mnjk} x_{jk} \qquad (1)$$

for each $m, n \in \mathbb{N}$, exists. The v −matrix domain $\chi_A^{(v)}$ of A in X is defined by

$$\chi_A^{(v)} = \left\{ x = (x_{jk}) \in \omega^2 : P - \sum_{j,k}^{m,n} a_{mnjk} x_{jk} \ \textit{exists and is in } Y \right\}.$$

Clearly, (1) suggests that A maps X into Y if $X \subset Y_A^{(v)}$; and $(X:Y)$ can denote the set of all four-dimensional matrices transforming X into Y. $A = (a_{mnjk}) \in (X:Y)$ if, and only if the double series on the right of (1) converges in Pringsheim's sense for each $m,n \in \mathbb{N}$, that is, $A_{mn} \in \chi^{\beta(v)}$ for all $j,k \in \mathbb{N}$ and any $x \in X$ to have $Ax \in Y$ for all $x \in X$.

It is well known that $A = (a_{mnjk})$ is a triangular matrix if $a_{mnjk} = 0$ for $j > m, k > m$ or both, and $a_{mnjk} \neq 0$ for all $m,n \in \mathbb{N}$. Every triangular matrix has a unique inverse which also happens to be a triangular matrix too (see Cooke, 1950).

2. Pascal Double Sequence Spaces

Pascal matrix existed for a very long time. It used to be of a finite order, not until 2002, see (Aggarwala & Lamoureux, 2002) where the authors declared that there was no reason, whatsoever, to stop at a finite matrix of this type for, one can extend the Pascal matrix of finite order to an infinite lower triangular matrix. We felt that this extension aroused Polat in his paper (Polat, 2018) to introduce some Pascal sequence spaces, each which is a matrix domain via infinite Pascal matrix as follows:

$$(l_\infty)_P = p_\infty = \left\{x = (x_k) \in \omega:\ \sup_n \left|\sum_k^n \binom{n}{n-k} x_k\right| < \infty\right\}$$

$$(c)_P = p_c = \left\{x = (x_k) \in \omega:\ \lim_{n\to\infty} \sum_k^n \binom{n}{n-k} x_k \ exists\right\}$$

$$(c_0)_P = p_0 = \left\{x = (x_k) \in \omega:\ \lim_{n\to\infty} \sum_k^n \binom{n}{n-k} x_k = 0\right\}$$

This paper will therefore wish to introduce Pascal double sequence spaces, $p_\infty^2, p_c^2, p_{bc}^2$ and p_0^2 as matrix domains of four-dimensional Pascal matrix, but, first we define the four-dimensional Pascal matrix $P = (p_{mn}^{jk})$ as follows:

$$p_{mn}^{jk} = \begin{cases} \binom{m}{m-j}\binom{n}{n-k} & , 0 \le j \le m, 0 \le k \le n \\ 0 & , j > m \ and \ k > n. \end{cases} \qquad (2)$$

with inverse $P^{-1} = Q = (q_{mn}^{jk})$ defined by

$$q_{mn}^{jk} = \begin{cases} (-1)^{(m-j)+(n-k)} \binom{m}{m-j}\binom{n}{n-k} & , 0 \le j \le m, 0 \le k \le n \\ 0 & , j > m \ and \ k > n. \end{cases} \qquad (3)$$

Now, we introduce the extensions of the spaces $p_\infty, p_c, and\ p_0$ denoted by $p_\infty^2, p_c^2, p_{bc}^2$ and p_0^2 as the collections of all double sequences such that each P –transform of them are in the spaces l_∞^2, c^2, c_b^2 and c_0^2 respectively; as follows:

$$p_\infty^2 = \left\{x = (x_{jk}) \in \omega^2: \sup_{m,n} \left|\sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}\right| < \infty\right\} \qquad (4)$$

$$p_c^2 = \left\{x = (x_{jk}) \in \omega^2: \exists l \in \mathbb{C} \ni \lim_{m,n\to\infty} \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} = l\right\} \qquad (5)$$

$$p_{bc}^2 = \left\{x = (x_{jk}) \in \omega^2: \left(\sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}\right) \in c_b^2\right\} \qquad (6)$$

$$p_0^2 = \left\{x = (x_{jk}) \in \omega^2: \lim_{m,n\to\infty} \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} = 0\right\} \qquad (7)$$

Let $\chi_A = \{x = (x_{jk}) \in \omega^2: Ax \in X\}$ be a matrix domain of a four-dimensional matrix A, then the Pascal sequence spaces in (4), (5), (6) and (7) are also matrix domains, as $p_\infty^2 = (l_\infty^2)_P$, $p_c^2 = (c^2)_P$, $p_{bc}^2 = (c_b^2)_P$ and $p_0^2 = (c_0^2)_P$, respectively; while P –transform of a double sequence space $x = (x_{jk})$ in (4), (5), (6) or (7) can be defined as

$$y_{mn} = (Px)_{mn} = \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} \qquad (8)$$

Each of the spaces $p_\infty^2, p_c^2, p_{bc}^2$ and p_0^2 is linear and can be normed with a norm given as $\|Px\|_{l_\infty^2}$ and defined by

$$\|Px\|_\infty = \sup_{m,n} \left|\sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}\right| \qquad (9)$$

Theorem 1: The double sequence spaces $p_\infty^2, p_c^2, p_{bc}^2$ and p_0^2 are BK spaces.

Before the proof, we look at the following:

Lemma 1: Malkowsky & Racocevic, 2000, P. 178: Let T be a triangle and $(X, \|\cdot\|)$ be a BK space with $\|x\|_T = \|T(x)\|$ with norm defined in (9).

We can now proof Theorem 1.

Proof: It is sufficient to prove the theorem for p_∞^2. Since l_∞^2 is a BK space, we define a map $A_P: p_\infty^2 \to l_\infty^2$ by $A_P(x) = P(x)\ \forall\ x \in p_\infty^2$. Since P is a triangular matrix, then A_P is linear, 0ne-to-one and onto. If A_P^{-1} is the inverse of A_P, it is also linear, one-to-one and onto, so that $p_\infty^2 = A_P^{-1}(\, l_\infty^2)$ is a Banach space. It remains to show the coordinates are continuous in l_∞^2. To do this, let $x_{jk} \to x$ in p_∞^2. Then $y_{jk}^{[r,s]} = P(x^{[r,s]}) \Longrightarrow y_{jk} = P(x)$, since l_∞^2 is a BK space. Let $P^{-1} = Q$ be the inverse of P, which is also a triangle. Then $x_{mn}^{[r,s]} = \sum_{j,k=0}^{m,n} Q y_{mn}^{[r,s]} \to \sum_{j,k=0}^{m,n} Q y_{mn} = x$. This shows that the coordinates are continuous on p_∞^2. Hence p_∞^2 is a BK space.

Definition 1 [Loganathan & Moorthy, 2016]: A double sequence (x_{jk}) in an infinite dimensional space X is called a double basis in X if for every $x \in X$, there exists a unique double sequence of scalars (α_{jk}) such that $\sum(\alpha_{jk})(x_{jk}) \to x$, that is $x = \sum \alpha_{jk} x_{jk}$.

Definition 2 [Loganathan & Moorthy, 2016]: A double sequence $(x_{jk})_{j,k=0}^{\infty}$ in a double sequence space X is called a Schauder double basis if, for every $x \in X$, there exists a unique double sequence of scalars $(\lambda_{jk})_{j,k=0}^{\infty}$ such that $x = \sum_{j,k=0}^{\infty} \lambda_{jk} x_{jk}$.

Definition 3 [Rao & Subramanian, 2004]: An FK space (or a metric space) is said to have an AK –property if $(\delta_{jk}^{mn})$is a Schauder double basis for X or equivalently $x^{[m,n]} \to x$, where $x^{[m,n]} = \sum_{j,k=0}^{m,n} x_{jk}\delta_{jk}\ \ \forall\ m,n,j,k \in \mathbb{N}$, and

$$(\delta_{jk}^{mn}) = \delta_{jk} = \begin{bmatrix} 0, & 0, & \cdots, & 0, & 0, & \cdots \\ 0, & 0, & \cdots, & 0, & 0, & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0, & 0, & \cdots, & 1, & 0, & \cdots \\ 0, & 0, & \cdots, & 0, & 0, & \cdots \end{bmatrix}$$

with 1 in the $(m,n)^{\text{th}}$ position and 0 elsewhere.

Theorem 2: Let $m,n,j,k \in \mathbb{N}$ and define $q^{(jk)} = \{q^{(jk)}\}_{mn}$ by

$$q_{mn}^{jk} = \begin{cases} 0 & ,j > m \text{ and } k > n \text{ or both}, \\ (-1)^{(m-j)+(n-k)}\binom{m}{m-j}\binom{n}{n-k} & ,0 \le k \le n \text{ and } 0 \le j \le m. \end{cases}$$

Then the following statements are correct:

a) The set $\{q^{(jk)}\}$ is a double basis for the double sequence space p_0^2 such that any $x \in p_0^2$ has a unique representation of the form

$$x = \sum_{jk} \zeta_{jk} q^{(jk)}; \qquad (10)$$

where $\zeta_{jk} = (Px)_{jk}\ \ \forall\ j,k \in \mathbb{N}$.

b) The set $\{e, q^{(jk)}\}$ is a double basis for the double sequence space p_c^2, such that any $x \in p_c^2$ has a unique representation of the form

$$x = le + \sum_{jk}(\zeta_{jk} - l)q^{(jk)}$$

where $l = \lim_{j,k\to\infty}(Px)_{jk}$ (12)

c) p_0^2 has AK property.

Proof:

a) We want to show that $\{q^{(jk)}\} \subset p_0^2$. Since $Pq^{jk} = e^{jk} \in c_0^2$ for $j,k = 0,1,2,\ldots$, e^{jk} is a double sequence whose non-zero term is 1 in the $(j,k)^{th}$ place for each j,k. Now, let $x \in p_0^2$. For every r and s, we write $x^{[r,s]} = \sum_{j,k=0}^{r,s} \zeta_{jk} Pq^{(jk)}$

P is continuous. So, we can apply P to (10) to have

$$x^{[r,s]} = \sum_{j,k=0}^{r,s} \zeta_{jk} Pq^{(jk)} = x^{[r,s]}$$

$$= \sum_{j,k=0}^{r,s} (Px)_{jk} e^{jk} \qquad (14)$$

and $\{P(x - x^{[r,s]})\}_{it} = \begin{cases} 0\ , 0 \le i \le r\ \&\ 0 \le t \le s \\ (Px)_{it} \quad\ , i > r\ \&\ t > s \end{cases}$. Let $\varepsilon > 0$ be given. Then there exist r_0 and s_0 such that $|(Px)_{r,s}| < \frac{\varepsilon}{2}\ \forall\ r > r_0$ and $s > s_0$. Therefore,

$$\|x - x^{[r,s]}\|_{p_0^2} = \sup_{i,t>r,s} |(Px)_{jm}|$$

$$\le \sup_{m,n>r_0,s_0} |(Px)_{jk}| \le \frac{\varepsilon}{2} < \varepsilon$$

$\forall\ r > r_0$ and $s > s_0$. Clearly, this shows that $x \in p_0^2$ as in (10).

Next, we show the uniqueness of the representation of x in (10). For this, suppose on the contrary that there exists another representation of $x = \sum_{jk} \xi_{jk} q^{(jk)}$. The linear transformation $P: p_0^2 \to c_0^2$ is continuous. It implies that we can have

$$(P(x))_{mn} = \sum_{j,k} \xi_{jk} (Pq^{(jk)})_{mn} = \sum_{j,k} \xi_{jk} e^{jk}$$

$$= \xi_{jk} \qquad (j,k \in \mathbb{N}),$$

which is a clear contradiction to the fact that $(P(x))_{mn} = \zeta_{jk}$. So, the representation (10) is unique. This completes the proof of (a).

b) Clearly, $\{q^{(jk)}\} \subset p_0^2$ and $Pq = e \in c^2$. Hence, the inclusion $\{e, q^{(jk)}\} \subset p_c^2$. Next, we take $x \in p_c^2$ arbitrary. Then there exists a unique l satisfying (12). Let us set $z = \sum_{jk}(\zeta_{jk} - l)q^{(jk)}$, then $z \in p_0^2$ whenever $z = x - lq$. Thus, the representation of z is also unique like x in (10).

c) Let $x = (x_{jk}) \in p_0^2$ and take the $(r,s)^{th}$ sectional sequence of x, i.e.

$$x^{[r,s]} = \sum_{j,j=0}^{r,s} x_{jk}\delta_{jk}\ \ \forall\ r,s \in \mathbb{N}.$$

Then we have $\|x - x^{[r,s]}\|_{p_0^2} = \sup_{r,s}|x_{jk}| \to 0$, which implies that $x^{[r,s]} \to x$ in p_0^2 as $r,s \to \infty$. Thus, p_0^2 has AK property.

Definition 4 [Basar & Sever, 2009]: A double sequence space X is solid if, and only if $\tilde{X} = \{u = (u_{jk}) \in \omega^2 : \exists\, x = (x_{jk}) \in X \text{ such that } |u_{jk}| \le |x_{jk}| \; for\ all\ j,k \in \mathbb{N}\} \subset X$.

Definition 5 [Yesilkayagil & Basar, 2016]: The space X of double sequence spaces is monotone if $xu = (x_{jk}u_{jk}) \in X$ fore very $x = (x_{jk}) \in X$ and $u = (u_{jk}) \in \chi^2$, where χ^2 denotes the double sequence space of 0s and 1s.

Theorem 3: The double sequence spaces $p_\infty^2, p_c^2, p_{bc}^2$ and p_0^2 are not monotone.

Proof: We prove for p_0^2 and that of the rest can be done similarly. So, $x = (x_{jk})$ and $u = (u_{jk})$ by $x_{jk} = \left(\frac{1}{2}\right)^{j+k}$ and $u_{jk} = \begin{cases} 1, \text{if } j+k \text{ is even} \\ 0, \quad \text{otherwise} \end{cases}$ respectively. Then

$$z = (z_{jk}) = (x_{jk})(u_{jk})$$
$$= \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k}\left(\frac{1}{2}\right)^{j+k} u_{jk}$$
$$= \sum_{j=0}^{m} \binom{m}{m-j}\left(\frac{1}{2}\right)^{j} \sum_{k=0}^{n} \binom{n}{n-k}\left(\frac{1}{2}\right)^{k} \sum_{j,k=0}^{m,n} u_{jk}$$
$$= 2^{m+n} \sum_{j=0}^{m}\left(\frac{1}{2}\right)^{j} \sum_{k=0}^{n}\left(\frac{1}{2}\right)^{k} \sum_{j,k=0}^{m,n} u_{jk}$$
$$= 2^m\left(1 - \frac{1}{2^m}\right) 2^n \left(1 - \frac{1}{2^n}\right) \sum_{j,k=0}^{m,n} u_{jk}$$
$$= (2^m - 1)(2^n - 1) \sum_{j,k=0}^{m,n} u_{jk}$$
$$x_{jk} = \lim_{m,n\to\infty} (2^m - 1)(2^n - 1) = \infty$$

Therefore, $z_{jk} = (x_{jk})(u_{jk}) \notin p_0^2$. Hence, p_0^2 is not monotone.

Theorem 4: The sets p_∞^2 and p_{bc}^2 are linear spaces with coordinatewise addition and scalar multiplication, and are Banach spaces with the norm

$$||\widetilde{x}||_\infty = \sup_{m,n\in\mathbb{N}} |(Px)_{mn}| \qquad (15)$$

which are linearly isomorphic to the spaces l_∞^2 and c_b^2, respectively. That is, $p_\infty^2 \cong l_\infty^2$ and $p_{bc}^2 \cong c_b^2$.

Proof: To avoid repetition of same sense in different words, the proof of the theorem is only given for p_∞^2. The first part of the theorem is a routine verification, where it can be easily seen that (i) p_∞^2 is not empty; (ii) the sum of any two elements in p_∞^2 is also in p_∞^2; and (iii) the scalar multiplication $\alpha x \in p_\infty^2$ $\forall\, \alpha \in \mathbb{C}$ and $x \in p_\infty^2$. Thus, p_∞^2 is a linear space with coordinatewise addition and scalar multiplication. Now, we can show that p_∞^2 is a Banach space with the norm defined by (15). Let $(x^\alpha)_{\alpha\in\mathbb{N}}$ be any Cauchy sequence in the space p_∞^2, where $x^\alpha = \{x_{jk}{}^{(\alpha)}\}_{j,k=0}^{\infty}$ for every fixed $\alpha \in \mathbb{N}$. Then for a given $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that

$$\|x^\alpha - x^\beta\|_{p_\infty^2} = \sup_{m,n\in\mathbb{N}} \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k}(x_{jk}^\alpha - x_{jk}^\beta) \right| < \varepsilon \quad \forall\, \alpha,\beta > N$$

which yields for each $m,n \in \mathbb{N}$ that

$$\left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha - \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\beta \right| < \varepsilon.$$

This means that $\left(\sum_{j,k=0}^{m,n}\binom{m}{m-j}\binom{n}{n-k}x_{jk}^\alpha\right)_{\alpha\in\mathbb{N}}$ is a Cauchy sequence with complex terms for every fixed $m,n \in \mathbb{N}$. Since $\mathbb{C}$ is complete, it converges, i.e.

$$\sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha \to \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} \;\; as\ \alpha \to \infty \qquad (16)$$

It can now be seen by (16) that

$$\lim_{\alpha\to\infty} \left\| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha - \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} \right\|_{p_\infty^2} = 0.$$

Since $\left(\sum_{j,k=0}^{m,n}\binom{m}{m-j}\binom{n}{n-k}x_{jk}^\alpha\right)_{m,n\in\mathbb{N}} \in p_\infty^2$ for each fixed $\alpha \in \mathbb{N}$, there exists a positive real number K_α such that

$$\sup_{m,n\in\mathbb{N}} \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha \right| \le K_\alpha.$$

Therefore, taking supremum over m,n in the following relation

$$\left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} \right|$$
$$= \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} - \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha + \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha \right|$$
$$\le \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} - \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha \right| + \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha \right|$$
$$\le \varepsilon + K_\alpha.$$

This shows that $x = (x_{jk}) \in p_\infty^2$. Since $\{x^\alpha\}_{\alpha\in\mathbb{N}}$ is an arbitrary Cauchy sequence, then the space p_∞^2 is complete. Thus, p_∞^2 is a Banach space with the norm $\|x\|_{p_\infty^2} = \sup_{m,n}|(Px)_{mn}|$.

To prove the fact that p_∞^2 is linearly isomorphic to l_∞^2, we have to show the existence of a linear bijection between the spaces p_∞^2 and l_∞^2. Consider the transformation τ defined from p_∞^2 to l_∞^2 by $x \mapsto y = \tau x = \{(Px)_{mn}\}$. Clearly, τ is linear, $\tau(u) + \tau(v) = \tau(u+v)$ for all $u = (u_{jk}), v = (v_{jk}) \in p_\infty^2$; and $K \cdot \tau(x) = \tau(Kx)$ for all $K \in \mathbb{C}, x = (x_{jk}) \in p_\infty^2$. Further, we can see that $x = \theta$ whenever $\tau x = \theta$ which shows that τ is injective. Now, let $y = (y_{jk}) \in l_\infty^2$ and define a sequence $x = (x_{jk})$ via y by

$$x_{jk} = \sum_{u,v=0}^{j,k} (-1)^{(j-u)+(k-v)} \binom{j}{j-u}\binom{k}{k-v} y_{uv} \quad \forall\, u,v \in \mathbb{N}.$$

Hence, by taking into account the hypothesis $y \in l_\infty^2$, one can derive by taking supremum over $m,n \in \mathbb{N}$ on the following equality

$$|(Px)_{mn}| = \left|\sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} \sum_{u,v=0}^{j,k} (-1)^{(j-u)+(k-v)} \binom{j}{j-u}\binom{k}{k-v} y_{uv}\right| = |y_{mn}|.$$

That is, $\|Px\|_\infty = \|y\|_\infty$, which implies that $x \in p_\infty^2$. Therefore, τ is surjective. Hence, $p_\infty^2 \cong l_\infty^2$.

Theorem 5: The sets p_c^2 and p_0^2 become linear spaces with the coordinatewise addition and scalar multiplication which are linearly isomorphic to the spaces c^2 and c_0^2 respectively; and p_c^2 and p_0^2 are complete seminormed spaces with the seminorm

$$\|x\| = \lim_{k\to\infty}\left[\sup_{m,n\geq k}|(Px)_{mn}|\right].$$

Proof: The proof is similar to the proof of theorem 4.

Theorem 6: The inclusions $l_\infty^2 \subset p_\infty^2$ and $c_b^2 \subset p_c^2$ strictly holds.

Proof: Suppose we take any $x = (x_{jk}) \in l_\infty^2$, then there exists a positive real number K such that $\sup_{jk}|x_{jk}| \leq K$. Therefore, one can see that

$$\sup_{m,n\in\mathbb{N}} |(Px)_{mn}| = \sup_{m,n\in\mathbb{N}} \left|\sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}\right|$$

$$= \sup_{m,n\in\mathbb{N}} \left|2^{m+n}\sum_{j,k=0}^{m,n} x_{jk}\right|$$

$$\leq 2^{m+n}\sup_{m,n\in\mathbb{N}} \left|\sum_{j,k=0}^{m,n} x_{jk}\right| \leq K.$$

That means that $x \in p_\infty^2$. Now, consider the sequence $x = (x_{jk})$ defined by

$$x_{jk} = \begin{cases} (-1)^j 2^{-j}, k = 0, j \in \mathbb{N} \\ 0 \qquad\qquad ,\text{otherwise.}\end{cases}$$

It is obvious that $x \in p_\infty^2 \setminus l_\infty^2$ which shows that the inclusion $l_\infty^2 \subset p_\infty^2$ strictly holds. The other part can similarly be shown.

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