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Triangle of Triangular Numbers

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ARTICLE INFO	ABSTRACT
Published Online: 05 October 2021 Corresponding Author: **Dr. R. Sivaraman**	Among several interesting number triangles that exist in mathematics, Pascal's triangle is one of the best triangle possessing rich mathematical properties. In this paper, I will introduce a number triangle containing triangular numbers arranged in particular fashion. Using this number triangle, I had proved five interesting theorems which help us to generate Pythagorean triples as well as establish bijection between whole numbers and set of all integers.

KEYWORDS: Triangular Numbers, Square Pyramidal Numbers, Pythagorean Triples, One – One Correspondence.

1. INTRODUCTION

Triangular numbers are numbers obtained by summing consecutive natural numbers. In this paper, I will introduce a number triangle using triangular numbers and explore some of its mathematical properties. These results will have interesting connections with other branches of mathematics and help us to connect between two different concepts. The ideas presented in the paper would provide more insights in understanding of triangular numbers.

2. DEFINITION

The sum of consecutive natural numbers is called a triangular number. The nth triangular number is given by

$$T_n = 1+2+3+\cdots+n = \frac{n(n+1)}{2} \quad (2.1).$$

3. CONSTRUCTION OF NUMBER TRIANGLE

In this section, I will formally introduce the number triangle consisting of triangular numbers as shown in Figure 1. If we assume that $t_{n,m}$ is the entry located in nth row and mth position of such triangle, we notice that $t_{1,1} = 1$ and for $n \geq 2, 1 \leq m \leq 2n-1$ we have

$$t_{n,m} = \begin{cases} T_m = \dfrac{m(m+1)}{2}, & 1 \leq m \leq n \\ T_{2n-m} = \dfrac{(2n-m)(2n-m+1)}{2}, & n+1 \leq m \leq 2n-1 \end{cases} \quad (3.1)$$

Row Number

```
1                     1
2                   1 3 1
3                 1 3 6 3 1
4               1 3 6 10 6 3 1
5            1 3 6 10 15 10 6 3 1
6          1 3 6 10 15 21 15 10 6 3 1
7       1 3 6 10 15 21 28 21 15 10 6 3 1
8     1 3 6 10 15 21 28 36 28 21 15 10 6 3 1
      - - - - - - - - - - - - - - - - - - -
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Figure 1: Triangle of Triangular Numbers

We further notice that for any natural number *n*, row *n* contains exactly 2*n* – 1 entries where each entry is a particular triangular number giving its name Triangle of Triangular Numbers. In upcoming sections, I will prove interesting mathematical properties related to triangle shown in Figure 1.

4. THEOREM 1 (ROW SUM PROPERTY)

For all $n \geq 1$, the sum of all entries in *n*th row of triangle of triangular numbers is the *n*th square pyramidal number. That is, for all $n \geq 1$,

$$\sum_{m=1}^{2n-1} t_{n,m} = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 \quad (4.1)$$

Proof: First, using (2.1), we compute the sum $\sum_{m=1}^{r} T_m$

$$\sum_{m=1}^{r} T_m = \sum_{m=1}^{r} \frac{m(m+1)}{2} = \frac{1}{2}\left(\sum_{m=1}^{r} m^2 + \sum_{m=1}^{r} m\right) = \frac{1}{2}\left(\frac{r(r+1)(2r+1)}{6} + \frac{r(r+1)}{2}\right) = \frac{r(r+1)(r+2)}{6} \quad (4.2)$$

Now using (3.1) and (4.2), we see that

$$\sum_{m=1}^{2n-1} t_{n,m} = \sum_{m=1}^{n} t_{n,m} + \sum_{m=n+1}^{2n-1} t_{n,m} = \sum_{m=1}^{n} T_m + \sum_{m=n+1}^{2n-1} T_{2n-m} = \sum_{m=1}^{n} T_m + \sum_{m=1}^{n-1} T_m$$

$$= \frac{n(n+1)(n+2)}{6} + \frac{(n-1)n(n+1)}{6} = \frac{n(n+1)(2n+1)}{6}$$

$$= 1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2$$

This completes the proof.

5. THEOREM 2

The centred numbers of the triangle of triangular numbers in Figure 1 are given by $t_{n,n} = T_n = \frac{n(n+1)}{2}$ (5.1)

For $n \geq 2$, we have $t_{n,n-1} + t_{n,n} = n^2$ (5.2) and for $1 \leq k \leq n-1$, we have $t_{n,n-k} + t_{n,n+k} = 2T_{n-k}$ (5.3)

Proof: In Figure 1, we notice that each row contains 2*n* – 1 triangular numbers. Hence the middle term of each row namely $t_{n,n}$ are called as centred numbers of triangle of triangular numbers. Now according to (3.1), we have $t_{n,n} = T_n = \frac{n(n+1)}{2}$ proving (5.1).

These numbers are shown in red color in Figure 1.

Similarly using (3.1), we have

$$t_{n,n-1} + t_{n,n} = T_{n-1} + T_n = \frac{(n-1)n}{2} + \frac{n(n+1)}{2} = \frac{n}{2} \times \left[(n-1) + (n+1)\right] = n^2$$

This proves (5.2).

Similarly considering both possible values of *m* from (3.1), we have

$$t_{n,n-k} + t_{n,n+k} = T_{n-k} + T_{2n-(n+k)} = T_{n-k} + T_{n-k} = 2T_{n-k}$$

This proves (5.3) and completes the proof.

6. THEOREM 3 (ALTERNATE SUM PROPERTY)

The alternating sum of entries of *n*th row is given by

$$\sum_{m=1}^{2n-1} (-1)^{m-1} t_{n,m} = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is } odd \\ -\frac{n}{2}, & \text{if } n \text{ is } even \end{cases} \quad (6.1)$$

Proof: Using (3.1), we have

$$\sum_{m=1}^{2n-1}(-1)^{m-1}t_{n,m}=\sum_{m=1}^{n}(-1)^{m-1}\frac{m(m+1)}{2}+\sum_{m=n+1}^{2n-1}(-1)^{m-1}\frac{(2n-m)(2n-m+1)}{2}$$

$$=\left[\frac{1\times 2}{2}-\frac{2\times 3}{2}+\frac{3\times 4}{2}-\cdots+(-1)^{n-2}\frac{(n-1)n}{2}+(-1)^{n-1}\frac{n(n+1)}{2}\right]+$$

$$\left[(-1)^{n}\frac{(n-1)n}{2}+(-1)^{n+1}\frac{(n-2)(n-1)}{2}+\cdots+(-1)^{2n-3}\frac{2\times 3}{2}+(-1)^{2n-2}\frac{1\times 2}{2}\right]$$

$$=(1\times 2)-(2\times 3)+(3\times 4)-(4\times 5)+\cdots+(-1)^{n}(n-1)\times n-(-1)^{n}\frac{n(n+1)}{2} \quad (6.2)$$

Now, if n is odd, then $(-1)^n=-1$ and so we get

$$(1\times 2)-(2\times 3)+(3\times 4)-(4\times 5)+\cdots+(-1)^{n}(n-1)\times n$$

$$=(1\times 2)-(2\times 3)+(3\times 4)-(4\times 5)+\cdots+\big((n-2)(n-1)-(n-1)\times n\big)$$

$$=2(-2)+4(-2)+6(-2)+\cdots+(n-1)(-2)$$

$$=-4\left(1+2+3+\cdots+\frac{n-1}{2}\right)=-2\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)=-\frac{(n-1)(n+1)}{2}$$

Hence if n is odd, then from (6.2), we get

$$\sum_{m=1}^{2n-1}(-1)^{m-1}t_{n,m}=-\frac{(n-1)(n+1)}{2}+\frac{n(n+1)}{2}=\frac{n+1}{2}\times\big(n-n+1\big)=\frac{n+1}{2}$$

If n is even, then $(-1)^n=1$ and so we get

$$\sum_{m=1}^{2n-1}(-1)^{m-1}t_{n,m}=(1\times 2)-(2\times 3)+(3\times 4)-(4\times 5)+\cdots+(-1)^{n}(n-1)\times n-(-1)^{n}\frac{n(n+1)}{2}$$

$$=(1\times 2)-(2\times 3)+(3\times 4)-(4\times 5)+\cdots+(n-1)\times n-\frac{n(n+1)}{2}$$

$$=(1\times 2)-(2\times 3)+(3\times 4)-(4\times 5)+\cdots+(n-1)\times n-n\times(n+1)+\frac{n(n+1)}{2}$$

$$=2(-2)+4(-2)+6(-2)+\cdots+(n)(-2)+\frac{n(n+1)}{2}$$

$$=-4\left(1+2+3+\cdots+\frac{n}{2}\right)+\frac{n(n+1)}{2}=-\left(2\times\frac{n}{2}\times\frac{n+2}{2}\right)+\frac{n(n+1)}{2}$$

$$=-\frac{n}{2}\big(n+2-n-1\big)=-\frac{n}{2}$$

This completes the proof.

7. THEOREM 4

If n is any positive integer, and if $t_{n,n}$ are centred numbers of triangle of triangular numbers then $\left(2n+1,4t_{n,n},4t_{n,n}+1\right)$ (7.1) will be a primitive Pythagorean triple.

Proof: Using (5.1), we note that the centred numbers of number triangle of triangular numbers in Figure 1 are numbers given by $t_{n,n}=T_n=\frac{n(n+1)}{2}$. Let $a=2n+1, b=4t_{n,n}, c=4t_{n,n}+1$.

Squaring and adding a, b we get

$$a^2+b^2=(2n+1)^2+4n^2(n+1)^2=4n^2+4n+1+4n^4+8n^3+4n^2$$
$$=4n^4+8n^3+8n^2+4n+1=(2n^2+2n+1)^2$$
$$=\left(4\times\frac{n(n+1)}{2}+1\right)^2=(4t_{n,n}+1)^2=c^2$$

It is also clear that the greatest common divisor of *a*, *b*, *c* is always 1. Hence $(2n+1,4t_{n,n},4t_{n,n}+1)$ will always be a primitive Pythagorean triple. This completes the proof.

8. THEOREM 5 (RHOMBUS PROPERTIES)

$$\text{If } m\le n-r \text{ then } (t_{n-r,m-r}\times t_{n+r,m+r})-(t_{n,m-r}\times t_{n,m+r})=0 \quad (8.1)$$

$$\text{If } 1\le m\le n \text{ then } (t_{n-m,n-m}\times t_{n+m,n+m})-(t_{n,n-m}\times t_{n,n+m})=(2n+1)mT_{n-m} \quad (8.2)$$

Proof: Using (3.1), we have $(t_{n-r,m-r}\times t_{n+r,m+r})-(t_{n,m-r}\times t_{n,m+r})=(T_{m-r}\times T_{m+r})-(T_{m-r}\times T_{m+r})=0$

This proves (8.1). Similarly using (3.1) we have

$$(t_{n-m,n-m}\times t_{n+m,n+m})-(t_{n,n-m}\times t_{n,n+m})=(T_{n-m}\times T_{n+m})-(T_{n-m}\times T_{2n-(n+m)})$$
$$=T_{n-m}\times(T_{n+m}-T_{n-m})=T_{n-m}\times\left(\frac{(n+m)(n+m+1)}{2}-\frac{(n-m)(n-m+1)}{2}\right)$$
$$=(2n+1)mT_{n-m}$$

This completes the proof.

9. CONCLUSION

Upon introducing the number triangle consisting of triangular numbers as in Figure 1, using the location of its entries as given in (3.1), I had proved five theorems comprising of eight results. In particular, I have proved in (4.1) that the sum of entries in *n*th row of the triangle of triangular numbers is the *n*th square pyramidal number, which forms nice pyramid shapes resembling great structures of ancient Egypt. In theorem 2 through equation (5.1), I had proved that the centred numbers of the number triangle are precisely the *n*th triangular number. Similarly through (5.2), I had shown the well known result namely sum of two consecutive triangular numbers are square numbers. Through (5.3), I had proved that sum of two entries located at equidistant from centred numbers of the number triangle is twice a triangular number. In proving (6.1) of theorem 3, regarding obtaining alternate sum of entries of triangle of triangular numbers, I had obtained an expression which establishes the one – one correspondence between set of non – negative integers (whole numbers) and integers. In fact, through (6.1), we observe that the images of all even whole numbers are mapped on to negative integers and images of all odd whole numbers are mapped on to positive integers making such a map a bijection between whole numbers and set of all integers.

In theorem 4 through (7.1), I had determined a way to generate primitive Pythagorean triples using the centred numbers of triangle of triangular numbers. This result establishes the relationship between triangular numbers and Pythagorean triples. In theorem 8, I had proved two results calling them Rhombus properties since the four entries considered forms vertices of a Rhombus of the number triangle in Figure 1. The eight amusing results derived in this paper would not only be entertaining but they also provide more insights in understanding of the behavior of fascinating triangular numbers.

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