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Investigation of Some Subgroups in Determining the Frattini Subgroup of Non-Abelian Finite Groups

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ABSTRACT

Frattini subgroup, $\Phi(G)$, of a group G is the intersection of all the maximal subgroups of G, or else G itself if G has no maximal subgroups. If G is a p −group, then $\Phi(G)$ is the smallest normal subgroup N such the quotient group G/N is an elementary abelian group. It is against this background that the concept of p −subgroup and fitting subgroup play a significant role in determining Frattini subgroup (especially its order) of dihedral groups. A lot of scholars have written on Frattini subgroup, but no substantial relationship has so far been identified between the parent group G and its Frattini subgroup $\Phi(G)$ which this tries to establish using the approach of Jelten B. Napthali who determined some internal properties of non abelian groups where the centre Z(G) takes its maximum size.

KEYWORDS: Fitting subgroup, p-group, Frattini subgroup, non-abelian finite groups.

1. INTRODUCTION

Giovani Frattini (1852–1925), introduced another type of subgroup known as Frattini subgroup denoted by $\Phi(G)$. He described the elements of a finite group into two classes; generators and non-generators and noticed that the non-generators form normal subgroups, called the Frattini subgroup, which equals the intersection of all maximal subgroups of the given group. The theory of $\Phi(G)$ of a group has been well developed since that time and has proved useful in the study of various problems in the group theory. According to him, If G has no maximal subgroups, then $\Phi(G) = G$. He also showed that when G is finite, $\Phi(G)$is nilpotent and this nilpotency of $\Phi(G)$ has also been proved for certain classes of infinite groups. This subgroup, which is evidently characteristic, nilpotent if G is finite, it is the smallest normal subgroup N such that G/N is elementary Abelian. This work, intends to investigate the role of p −subgroups in determining $\Phi(G)$. The notation used is standard. For a group G, $\Phi(G)$ denotes its Frattini subgroup, $F(G)$ denotes its Fitting subgroup, $Z(G)$ denotes its center, M denote a maximal subgroup and G' its derived subgroup.

One of the main goals in finite group theory was to obtain a good understanding of the subgroup structure of groups. Researchers classified the elements of a finite group into two, generators and non-generators. They noticed that the non-generators form a normal subgroup which equals the intersection of all maximal subgroup of the given group.

James and Paul (2017) who studied the relations between nilpotency and Frattini subgroups, remarked that if a maximal subgroup M of a group G is normal, then G/M has prime order and G' is a subgroup of M if and only if G' is a subgroup of $\Phi(G)$. Over the years, the interpretation of $\Phi(G)$ has been largely on the intersection of maximal subgroups. But a product of two normal nilpotent subgroups is again nilpotent subgroup. It means that in every group there is a unique maximal normal nilpotent subgroup $F(G)$ called the Fitting subgroup. This subgroup has a great influence on the structure of a solvable group.

Bhattacharya and Mukherjee (1987), investigated the relationship between the structure of a finite group G and the properties of the maximal subgroup of group G. They considered the family of maximal subgroups whose indices are composite and co-prime to a given prime. At the end, they proved that S_p(G) is a characteristic subgroup containing the Frattini subgroup of G.

Lange (1975), studied the relationship between a finite p-group and its Frattini subgroup. In particular, attention was given to Frattini subgroups that are either cyclic or are non Abelian.

For instance, Hobby (1960), showed that some groups that cannot be derived subgroups of p-groups also cannot be Frattini subgroup of p-groups.s

Eric (1997), is a particular nice paper in which the converse of Satz is proven, in some sense completely classifying finite groups that occur as Frattini subgroups. In particular, he shows that {Φ(H)|H, Φ(H) is finite} = {Φ(H)|H is finite}

Javier (2001), reviewed the behaviour of the Frattini subgroup Φ(G) and the Frattini factor group G/Φ(G) of an infinite group.

James and Paul (2017): who studied the relations between nilpotency and Frattini subgroups, devoted their work to the structure of finite nilpotent algebras. They remarked that if a maximal subgroup M of a group G is normal, then G/M has prime order and G' is a subgroup of M. Thus M < G if and only if G' is a subgroup of M. All maximal subgroups of G are normal if and only if $G' \in$ M. All maximal subgroups of G are normal if and only if G' is a subgroup of Φ(G). So G is nilpotent.

Neuman (2011), showed that Frattini subgroup of a group coincide with the set of non generators of the group. On the Frattini subgroup of a finite group,

Besides these key points, this work intends to investigate the contribution of p-subgroups and fitting subgroup in determining Frattini subgroups of finite groups. It is against this background that this work intends to investigate the influence of fitting subgroups of finite groups in determining $\Phi(G)$.

Definition: The group of symmetries of an n − sided regular polygon for n > 1 with rotations and reflections is termed dihedral group denoted by D_n.

Definition: The centre of a finite group G denoted by Z(G), is the set

$\{x \in$ G: $xg = gx$ for all $g \in G\}$

Definition: Let G be a group and N ⊆ G, N is called normal subgroup of G if and only if $Ng = gN$; for all $g \in$ G . or, equivalently, $gNg^{-1} = N$; for all $g \in$ G. We write $N \trianglelefteq G$ for N to be a normal subgroup of G.

Definition: A group G is proper $NF -$ group if $\Phi(G) \neq \{1\}$, but $\Phi(N) = \{1\}$ for all proper normal subgroups N of G.

Definition: A group G is solvable if it has a chain of subgroups

$1 = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_1 =$ G. Where each $G_i \trianglelefteq G_{i+1}$ with Abelian quotient G_{i+1}/G_i

Definition: We say that a subgroup $H \subseteq G$ is characteristic in G if α(H) = H for every α ∈ Aut (G).

Definition: Frattini subgroup is the intersection of maximal subgroups.

Definition: The maximum possible order of Frattini subgroup $\Phi(D_n)$ is $\frac{1}{4}|D_n|$

Definition: The order of Frattini subgroup of any dihedral group whose order is a power of two is exactly $\frac{1}{4}(2^k)$, $k \geq 3$

2.0 METHODOLOGY

Some propositions, corollaries and theorems that display our investigation of the influence of some subgroups in determining Frattini subgroup of finite non abelian groups. We will be using some of these contributions in determining the relationship between G and $\Phi(G)$ (if it exist). The existing relationship between the reflections and rotations of dihedral group are to be applied to form a general formula (whose validity to be verified using tabular form) for finding Frattini subgroup of any dihedral group.

2.1 Proposition

For every $n = 3^k$; k = 1, 2, 3,.., $|\Phi(D_n)| = 3^{k-1} < \frac{1}{4}|D_n|$.

2.2 Proposition

Every dihedral group D_n ; n is odd and not raised to the power of $\frac{1}{k}$; $k = 2,3,4, \ldots$

is $|\Phi(D_{2n})| = 1 < \frac{1}{4}|D_n|$. $k = 2,3,4, \ldots$

2.3 Proposition

For n $\neq b^k$, the Frattini subgroup of any dihedral group D_n is 1. where b is any odd number and $k = 2,3,4, \ldots$

2.4 Proposition

Proposition: $\Phi(D_n) \geq 2$ for all even $n = 2^k; k \geq 2$

2.5 Theorem

Let G be a finite nilpotent dihedral group of order p. Then, G has a normal p-subgroup N such that $Z(\mathrm{G}) \leq N \leq G$

Proof.

Let G be a finite p-group with $|\mathrm{G}| = p^\alpha$ for some $\alpha \geq 1$, Z(G) is a nontrivial p-subgroup. let g ∈ Z(G) with $g \neq$ e. The order of g is p^r for some r ≥ 1. Therefore $g^{p^{r-1}}$has order p, so Z(G) contains a subgroup of order p, which must be normal in G since every subgroup of Z(G) is a normal subgroup of G. By the proof of proposition 3.5, G has a maximal subgroup M of G with $M \supseteq \Phi$(G). Taking $N = M$, implies $Z(\mathrm{G}) \leq \Phi(G) \leq N \leq G$

RESULT

The dihedral group D_n is generated by two elements. If these elements are r and s, then we have $D_n = \langle r, s \ni r^n = s^2 = (rs)^2 = 1; srs = srs^{-1} = r^{-1}\rangle$.

Since $\langle r\rangle$ is always the maximal subgroup of D_n, so we must have

$\Phi(D_n) = \langle r^i\rangle$ for some $i \in \mathbb{Z}$. But no maximal subgroup for D_n contains only element of the form sr^i as $|\langle sr^i\rangle| = 2$ and $\langle sr^i, sr^j\rangle = \langle sr^i, r^{i-j}\rangle$ which shows that it is a proper subgroup when $i - j$ and n are not relatively prime and is all of D_n otherwise. Since the general non cyclic-group of a finite dihedral group has the form $\langle r^m, r^i s\rangle$, where m/n and $0 \leq i \leq m$, a subgroup of this form is maximal if $m = n/p$, p

some prime. It is a proven fact that no element of $\langle r^k \rangle$ can be written as r^is. Therefore, we can claim that

$$\Phi(D_n) = (\cap_m \langle r^p \rangle) \quad \text{-----------} \quad 3.1$$

which can be used to find the Frattini subgroup of any dihedral group as in the table below.

3.1.0 Table

S/No.	n	|G|	|Z(G)|	$\lvert C_{D_n}(\mathbf{r}) \rvert$	|Φ(G)|
1	4	8	2	4	2
2	8	16	2	8	4
3	16	32	2	16	8
4	32	64	2	32	16
5	64	128	2	64	32

The above table shows that if $|D_n| = 2^k; k \geq 3$, then $|\Phi(D_{2n})| = \frac{1}{4}(2^k)$.

For instance,

The above table shows if $|D_n| = 2^3 = 8$, then $|\Phi(D_n)| = \frac{1}{4}(2^3) = 2$,

If $|D_n| = 2^4 = 16$, then $|\Phi(D_n)| = \frac{1}{4}(2^4) = 4$,

for $|D_n| = 2^5 = 32$, then $|\Phi(D_n)| = \frac{1}{4}(2^5) = 8$,

if we take $|D_n| = 2^6 = 64$, then $|\Phi(D_n)| = \frac{1}{4}(2^6) = 16$,

take $|D_n| = 2^7 = 128$, then $|\Phi(D_n)| = \frac{1}{4}(2^7) = 32$.

Therefore, if $|D_n| = 2^k$, then $|\Phi(D_n)| = \frac{1}{4}(2^k)$ and it can be checked that $|\Phi(D_n)| = \frac{1}{4}(2^k) = \frac{1}{4}|D_n|$.

3.1.1 Theorem

If $|D_n|$ is a perfect square root, then for all even ≥ 4, 4 divides the order of D_n.

Proof:

The identity element of D_n must be the product of two element of $\Phi(D_n)$, say. As inverse pairs always do not commute, and distinct element of $\Phi(D_n)$, we must have $h^2 = 1$ for some $h \in H$. Since $1 \in \Phi(D_n)$, there does not exist $1 = h \in H$. So h must have an order 2 and so G has an even order. It follows that the order of G with G having a perfect square root is divisible by 4.

3.1.1 Table

S/No.	n	|G|	|Z(G)|	$\lvert C_{D_n}(\mathbf{r}) \rvert$	|Φ(G)|
1	9	18	1	9	3
2	27	54	1	27	9
3	81	162	1	81	27
4	243	483	1	243	81
5	729	966	1	729	243

From table 3.1.1

If $n = 3^1$, $|D_n| = 6$ and $|\Phi(D_n)| = 1 = 3^0$,

If $n = 3^2$, $|D_n| = 18$ and $|\Phi(D_{2n})| = 3 = 3^1$,

If $n = 3^3$, $|D_n| = 54$ and $|\Phi(D_{2n})| = 9 = 3^2$,

If $n = 3^4$, $|D_n| = 162$ and $|\Phi(D_{2n})| = 27 = 3^3$.

Therefore, for every $n = 3^k; k = 1,2,3, \ldots$

$|\Phi(D_n)| = 3^{k-1}$ but $3^{k-1} < \frac{1}{4}|D_n|$. So, $|\Phi(D_n)| = 3^{k-1} < \frac{1}{4}|D_n|$.

3.1.2 Table

S/No.	n	|G|	|Z(G)|	$\lvert C_{D_n}(\mathbf{r}) \rvert$	|Φ(G)|
1	3	6	1	3	1
2	5	10	1	5	1

3	7	14	1	7	1
4	11	22	1	11	1
5	13	26	1	13	1
6	14	28	2	14	1
7	15	30	1	15	1
8	17	34	1	17	1
9	19	38	1	19	1
10	21	42	1	21	1

By the table above, for all odd $n \neq b^k$ where $k = 2,3,4, \ldots,$ $|\mathbf{\Phi(G)}| = \mathbf{1}$

3.1.3 Table

S/No.	n	|G|	|Z(G)|	$\lvert C_{D_n}(\mathbf{r})\rvert$	|Φ(G)|
1	9	18	1	9	3
2	27	54	1	27	9
3	32	64	2	32	16
4	36	72	2	36	6
5	45	90	1	45	3
6	49	98	1	49	7
7	81	162	1	81	27

The above shows that for $n = b^k$; $k = 2,3,4 \ldots,$ $|\mathbf{\Phi(G)}| \neq 1$

3.1.2 Theorem

Let D_n be a finite dihedral group and $\Phi(D_n)$ be its Frattini subgroup, then for any element $r \in D_n$:

i. $|\Phi(D_n)| = |\mathrm{Z(G)}||\Phi(D_n):\mathrm{Z(G)}|$ and $|\mathrm{G}| = |\Phi(D_n)|\ |\mathrm{G}:\Phi(D_n)|$

ii. $|\mathrm{Z(G)}| = |\Phi(D_n)|/|\Phi(D_n):\mathrm{Z(G)}|$ and $|\Phi(D_n)| = |\mathrm{G}|/|\mathrm{G}:\Phi(D_n)|$

iii. $|C_{D_n}(\mathrm{r})| = |\Phi(D_n)||C_{D_n}(\mathrm{r}):\Phi(D_n)|$

Where $|\mathrm{G}:C_{D_n}(\mathrm{r})| \geq 2$, $|C_{D_n}(\mathrm{r}):\Phi(D_n)| \geq 2$ for all even $n \geq 3$

3.1.3 Theorem

If $G = D_n$ is a finite non abelian group, then the maximum possible order of the Frattini subgroup of D_n is $\frac{1}{4}|D_n|$. That is $|\Phi(D_n)| \leq \frac{1}{4}|D_n|$.

Proof:

We assumed that $G = D_n$. Where $D_n = \{\mathrm{r}^n = \mathrm{s}^2 = \mathrm{e} \ni \mathrm{sr} = \mathrm{sr}^{-1}\}$.

But the centralizer of r in D_n denoted by $C_{D_n}(r)$, contains all powers of r, so we have $\langle r\rangle \subseteq C_{D_n}(r)$. ------------3. 2

This shows that $C_{D_n}(r)$ has at least n elements. On the other hand, $C_{D_n}(r) \neq D_n$, since it is non abelian group, there exist an element in $s \in D_n$ such that s does not belong to $C_{D_n}(r)$. $\langle r\rangle$ contains exactly half of the elements in D_n. Lagrange's theorem shows that there is no subgroup that lies strictly between $\langle r\rangle$ and D_n, it follows that any other subgroup H of G, $H \leq \langle r\rangle$. If $H = \Phi(D_n)$, then $\Phi(D_n) \subseteq \langle r\rangle$.

So, $\Phi(D_n) \subseteq \langle r\rangle \subseteq C_{D_n}(r) \subseteq D_n$ ------------3. 3

Since $C_{D_n}(r) \neq D_n$ it implies that $C_{D_n}(r) = \langle r\rangle$.

By theorem 3.1.2, $|\Phi(D_n)| \leq \frac{1}{2}|C_{D_n}(r)|$. Since $C_{D_n}(r) \neq D_n$, it imples that $C_{D_n}(r)$ is a proper subset of D_n. But the centralizer of any dihedral group is also a subgroup, and so we can write $|C_{D_n}(r)| \leq \frac{1}{2}|D_n|$.

Therefore, by (3.3), $|\Phi(D_n)| \leq \frac{1}{2}|C_{D_n}(r)| \leq \frac{1}{2}\left(\frac{1}{2}|D_n|\right) = \frac{1}{4}|D_n|$.

Hence, $|\Phi(D_n)| \leq \frac{1}{4}|D_n|$. --------------3. 4

3.1.0 Examples

Also If $n = 5^1$, $|\mathrm{G}| = |D_n| = 5$, and $|\Phi(D_n)| = 1 = 5^0$,

If $n = 5^2$, $|\mathrm{G}| = |D_n| = 25$, and $|\Phi(D_n)| = 5 = 5^1$,

If $n = 5^3$, $|\mathrm{G}| = |D_n| = 125$, and $|\Phi(D_n)| = 25 = 5^2$.

This implies that for every $n = 5^k; k = 1,2,3, \dots n$ $|\Phi(D_n)| = 5^{k-1}$. Since $5^{k-1} < \frac{1}{4}|D_{2n}|$, it implies that $|\Phi(D_n)| = 5^{k-1} < \frac{1}{4}|D_n|$.

CONCLUSION

Our investigation has shown that, fitting subgroup plays a significant role in determining Frattini subgroup of finite nilpotent dihedral group which led to the establishment of relationship between $|\Phi(\mathrm{D_n})|$ and $|\mathrm{D_n}|$.

We conclude that:

1. if $|D_n| = 2^k; k \geq 3$, then $|\Phi(D_{2n})| = \frac{1}{4}(2^k)$.

i.e $|D_n| = 2^k$, then $|\Phi(D_n)| = \frac{1}{4}(2^k)$ and it can be checked that $|\Phi(D_n)| = \frac{1}{4}(2^k) = \frac{1}{4}|D_n|$.

2. for every $n = 3^k; k = 1,2,3, \dots$

i.e $|\Phi(D_n)| = 3^{k-1}$ but $3^{k-1} < \frac{1}{4}|D_n|$. So, $|\Phi(D_n)| = 3^{k-1} < \frac{1}{4}|D_n|$.

3. for all odd $n \neq b^k$ where $k = 2,3,4, \dots$, $|\Phi(\mathrm{G})| = 1 < \frac{1}{4}|D_n|$
4. for $n = b^k; k = 2,3,4 \dots$, $|\Phi(\mathrm{G})| \neq 1 \leq \frac{1}{4}|D_n|$

   i.e $|\Phi(D_n)| = 5^{k-1}$. Since $5^{k-1} < \frac{1}{4}|D_{2n}|$, it implies that $|\Phi(D_n)| = 5^{k-1} < \frac{1}{4}|D_n|$.

Therefore, $|\Phi(\mathrm{G})| \leq \frac{1}{4}|D_n|$.

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