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Study of S-Function in Relationship with Natural Transform

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ABSTRACT
The objective of this paper is to investigate natural transform of the S -function defined and studied by Saxena and Daiya [4] in which Riemann–Liouville integrals are replaced by more general Prabhakar integrals. We analyze and discuss its properties in terms of Mittag-Leffler functions. Further, we show some applications of these natural transform in classical equations of mathematical physics, like the heat and the free electron laser equations, and in difference-differential equations governing the dynamics of generalized renewal stochastic processes.

KEYWORDS: S-function, Natural transform, Riemann Liouville fractional Integrals, Mittag-Leffler Function.

INTRODUCTION

Partial differential equations and their applications arise frequently in many branches of physics, Engineering, and other sciences. There are many works that provided using integral transform method to solve some types of partial differential equations, for example in [06, 7] integral transform is used to solve boundary value problems and integral equations.

Laplace transform is one of the most used in the mathematical and engineering community. Definition of the Laplace transform, notations, and Laplace transforms of some elementary functions can be found in [8].

Furthermore, one dimensional Laplace transform was extended to two dimensional and called as double Laplace transform. The first introduction of double Laplace transform was in [7]. Some operation calculus of double Laplace transform can be found in [9]. Double Laplace transform was used to solve heat, wave, and Laplace's equations with convolution terms (see [10]), telegraph and partial integro differential equations. Sumudu transform was first introduced by [14] and some of its applications were given by [15]. The aims of this study are to generalize the definition of single Natural transform to double Natural transform and achieve its main properties, in order to solve telegraph, wave and partial integro-differential equations.

DEFINITIONS AND PRELIMINARIES USED IN THIS PAPER

S-Function:

The S -function defined and studied by Saxena and Daiya [10] as follows:

$$\underset{\mathbf{p},\mathbf{q}}{\overset{(\alpha,\beta,\gamma,\tau)}{\mathbf{S}}}\left[\begin{matrix} a_1, a_2, \ldots a_p \\ b_1, b_2, \ldots . b_q \end{matrix} \;,\; x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\,,(a_2)_n\,,\ldots\ldots\ldots(a_p)_n \quad (\gamma)_{n\tau,k}}{(b_1)_n\,,(b_2)_n\,,\ldots\ldots\ldots(b_q)_n \quad \Gamma(n\alpha+\beta)} \frac{x^n}{n!}$$

Where, $k \in R, \alpha, \beta, \gamma, \tau \in C.\, R(\alpha) > 0, a_1, a_2, .. a_p, b_1, b_2, \ldots . b_q, R(\alpha) > kR(\tau)$ and $p < q+1$. The Pochhammer symbol $(\tau)_\mu$ defined interms of gamma function as follows:

$$(\tau)_\mu = \frac{\Gamma(\tau+\mu)}{\Gamma(\tau)} = \begin{Bmatrix} 1, & \mu = 0, \tau, \mu \in c \\ \tau(\tau+1)(\tau+2)\ldots\ldots(\tau+\mu-1) & \end{Bmatrix}$$

Natural transform:

The Natural transform earlier called as N transform [7] was extensively surveyed by the authors in [3], where the said transform is sketched out from the Fourier integral to define the complex inverse Natural transform as well. Albeit it combines the futures of Laplace and Sumudu transforms [1, 2] and therefore the region of convergence also includes both of them. Hence the Natural transform of the function $f(t) \in R^2$ is given by the following integral equation [3, 7].

$N[f(t)] = \int_0^\infty e^{-st} f(ut)dt \ , Re(S) > 0, u(-\tau_1, \tau_2)$

Provided the function $f(t) \in R^2$ *is defined in the set*

$$A = \{ f(t) \,|\, \exists M, \tau_1, \tau_2 > 0. |f(t)| < M \ e^{\frac{|t|}{\tau_j}} \}.$$

Lemma-I:

For instance the Natural transform of the t^n, $n > -1$ is given by [3]

$$N[\,t^n] = \int_0^\infty e^{-st}\,(ut)^n dt\ , Re(S) > 0,$$

$$= u^n \int_0^\infty e^{-st}\, t^n dt$$

$$= u^n \frac{\Gamma(n+1)}{s^{n+1}}$$

MAIN RESULTS

In this section, we consider the natural transform of the generalized function of fractional calculus called S-function and making use of the given above lemma to derive following useful results.

Theorem (1.1): Let $A = \{ f(t) \,|\, \exists M, \tau_1, \tau_2 > 0. |f(t)| < M \ e^{\frac{|t|}{\tau_j}} \}$, and $N[f(t)]$ be the natural transform associated with S-function . Then there holds the following relationship

$$N\left\{ \begin{matrix} (\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\tau}) \\ \mathbf{S} \\ \mathbf{p},\mathbf{q} \end{matrix} \left[\begin{matrix} \boldsymbol{a_1}, \boldsymbol{a_2}, .. \boldsymbol{a_p} \\ \boldsymbol{b_1}, \boldsymbol{b_2}, ... \boldsymbol{b_q} \end{matrix} \ , \ \boldsymbol{x} \right] \right\} = \frac{1}{s} \begin{matrix} (\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\tau}) \\ \mathbf{S} \\ \mathbf{p+1},\mathbf{q} \end{matrix} \left[\begin{matrix} \boldsymbol{a_1}, \boldsymbol{a_2}, .. \boldsymbol{a_p}, 1 \\ \boldsymbol{b_1}, \boldsymbol{b_2}, ... \boldsymbol{b_q} \end{matrix} \ , \ \boldsymbol{x} \right]$$

Provided the function $f(t) \in R^2$.

Proof: By using the definition of the generalized S -function of fractional calculus and the natural transform, we get

$$N\left\{ \begin{matrix} (\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\tau}) \\ \mathbf{S} \\ \mathbf{p},\mathbf{q} \end{matrix} \left[\begin{matrix} \boldsymbol{a_1}, \boldsymbol{a_2}, .. \boldsymbol{a_p} \\ \boldsymbol{b_1}, \boldsymbol{b_2}, ... \boldsymbol{b_q} \end{matrix} \ , \ \boldsymbol{x} \right] \right\} = N\left\{ \sum_{n=0}^{\infty} \frac{(\boldsymbol{a_1})_n, (\boldsymbol{a_2})_n, ...(\boldsymbol{a_p})_n \quad (\boldsymbol{\gamma})_{n\tau,k} \qquad \boldsymbol{x^n}}{(\boldsymbol{b_1})_n, (\boldsymbol{b_2})_n, ...(\boldsymbol{b_q})_n \quad \boldsymbol{\Gamma(n\alpha+\beta)} \quad \boldsymbol{n!}} \right\}$$

$$N\left\{ \begin{matrix} (\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\tau}) \\ \mathbf{S} \\ \mathbf{p},\mathbf{q} \end{matrix} \left[\begin{matrix} \boldsymbol{a_1}, \boldsymbol{a_2}, .. \boldsymbol{a_p} \\ \boldsymbol{b_1}, \boldsymbol{b_2}, ... \boldsymbol{b_q} \end{matrix} \ , \ \boldsymbol{x} \right] \right\} = \sum_{n=0}^{\infty} \frac{(\boldsymbol{a_1})_n, (\boldsymbol{a_2})_n, ...(\boldsymbol{a_p})_n \quad (\boldsymbol{\gamma})_{n\tau,k}}{(\boldsymbol{b_1})_n, (\boldsymbol{b_2})_n, ...(\boldsymbol{b_q})_n \quad \boldsymbol{\Gamma(n\alpha+\beta)} \quad \boldsymbol{n!}} N\{\boldsymbol{x^n}\}$$

By making use of lemma –I in above equation, we get

$$N\left\{ \begin{matrix} (\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\tau}) \\ \mathbf{S} \\ \mathbf{p},\mathbf{q} \end{matrix} \left[\begin{matrix} \boldsymbol{a_1}, \boldsymbol{a_2}, .. \boldsymbol{a_p} \\ \boldsymbol{b_1}, \boldsymbol{b_2}, ... \boldsymbol{b_q} \end{matrix} \ , \ \boldsymbol{x} \right] \right\} =$$

$$\sum_{n=0}^{\infty} \frac{(\boldsymbol{a_1})_n, (\boldsymbol{a_2})_n, ...(\boldsymbol{a_p})_n \quad (\boldsymbol{\gamma})_{n\tau,k}}{(\boldsymbol{b_1})_n, (\boldsymbol{b_2})_n, ...(\boldsymbol{b_q})_n \quad \boldsymbol{\Gamma(n\alpha+\beta)} \quad \boldsymbol{n!}} u^n \frac{\Gamma(n+1)}{s^{n+1}}$$

Or

$$N\left\{ \begin{matrix} (\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\tau}) \\ \mathbf{S} \\ \mathbf{p},\mathbf{q} \end{matrix} \left[\begin{matrix} \boldsymbol{a_1}, \boldsymbol{a_2}, .. \boldsymbol{a_p} \\ \boldsymbol{b_1}, \boldsymbol{b_2}, ... \boldsymbol{b_q} \end{matrix} \ , \ \boldsymbol{x} \right] \right\} = \frac{1}{s} \begin{matrix} (\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\tau}) \\ \mathbf{S} \\ \mathbf{p+1},\mathbf{q} \end{matrix} \left[\begin{matrix} \boldsymbol{a_1}, \boldsymbol{a_2}, .. \boldsymbol{a_p}, 1 \\ \boldsymbol{b_1}, \boldsymbol{b_2}, ... \boldsymbol{b_q} \end{matrix} \ , \ \boldsymbol{x} \right]$$

CONCLUSION

This work deals with definition of Natural transform and S-function. Fundamental properties of Natural transform are obtained. Further, some examples and applications on Natural transform are presented. Using Natural transform to solve some types of equations with variable coefficients will be a future work.

REFERENCES

1. Belgacem, F. B. M. *Applications of Sumudu transform to indefinite periodic parabolic equations.* Proceedings of the 6th International Conference on Mathematical Problems & Aerospace Sciences, (ICNPAA 06), Chap. 6, pp 51-60, Cambridge Scientific Publishers, Cambridge, UK, 2007.

2. Belgacem, F. B. M. *Sumudu transform applications to Bessel's Functions and Equations.* Applied Mathematical Sciences. Vol. 4. No. 74. pp 3665-3686. 2010.
3. Belgacem, F. B. M and Silambarasan, R. *Theory of the Natural transform.* Mathematics in Engg Sci and Aerospace (MESA) journal. Vol. 3. No. 1. pp 99-124. 2012.
4. R. K. Saxena, J. Daiya, Integral transforms of the S -function, Le Mathematiche, 70 (2015), 147–**159.**
5. Khan, Z. H. and Khan, W. A. *N-transform properties and applications.* NUST Jour. of Engg. Sciences. Vol. 1. No. 1. pp 127-133. 2008.
6. D. G. Duffy, Transform methods for solving partial differential equations, Second edition, Chapman & Hall/CRC, BocaRaton, FL, (2004).
7. T. A. Estrin, T. J. Higgins, The solution of boundary value problems by multiple Laplace transformations, J. Franklin Inst., **252** (1951), 153–167
8. M. R. Spiegel, Schaum's outline of theory and problems of Laplace transforms, McGraw-Hill, New York, (1965).
9. Y. A. Brychkov, H.-J. Glaeske, A. P. Prudnikov, V˜u Kim Tuˆan, Multidimensional integral transformations, Gordon and Breach Science Publishers, Philadelphia, PA, (1992).
10. H. Eltayeb, A. Kılıc¸man, A note on solutions of wave, Laplace's and heat equations with convolution terms by using a double Laplace transform, Appl. Math. Lett., **21** (2008), 1324–1329.
11. G. K. Watugala, Sumudu transform: a new integral transform to solve differential equations and control engineering problems, Internat. J. Math. Ed. Sci. Tech., **24** (1993), 35–43.
12. Y. A. Brychkov, H.-J. Glaeske, A. P. Prudnikov, V˜u Kim Tuˆan, Multidimensional integral transformations, Gordon and Breach Science Publishers, Philadelphia, PA, (1992).
13. A. Kılıc¸man, H. E. Gadain, On the applications of Laplace and Sumudu transforms, J. Franklin Inst., **347** (2010), 848–862.
14. G. K. Watugala, Sumudu transform: a new integral transform to solve differential equations and control engineering problems, Internat. J. Math. Ed. Sci. Tech., **24** (1993), 35–43. 1
15. A. Kılıc¸man, H. Eltayeb, On a new integral transform and differential equations, Math. Probl. Eng., **2010** (2010), 13 pages.