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GENERALIZED HYERS-ULAM-RASSIAS TYPE STABILITY OF FUNCTIONAL EQUATION DERIVING FROM QUADRATIC MAPPING IN NON-ARCHIMDEAN (l,β)-NORMED SPACE

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Abstract. In this paper, we study to solve the Hyers $-Ulam-Rassias$ *stability of functional equation deriving from mapping quadratic in* $Non-Archimdean$ (l,β)-*normed space. Then I will show that the solutions of equation are quadratic mapping. These are the main results of this paper.*

Keywords: *non-Archimdean* (n,β)*-normed space Hyers-Ulam-Rassias stability, functional equation deriving from quadratic mapping.*
Mathematics Subject Classification: 39B82, 39B72

1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ be a normed spaces on the same field $\mathbb{K}$, and $f:\mathbf{X}\to\mathbf{Y}$ be a mapping. We use the notation $\|\cdot\|_{\beta_1}$ $(\|\cdot\|_{\beta})$ for corresponding the norms on $\mathbf{X}$ and $\mathbf{Y}$. In this paper, we investigate the stability of functional equation deriving quadratic from mappimng non-Archimedean (n,β)-normed space. In fact, when $\mathbf{X}$ is a non-Archimedean (n,β)-normed space with norm $\|\cdot\|_{\beta_1}$ and that $\mathbf{Y}$ is a Banach non-Archimedean (n,β)-normed space with norm with norm $\|\cdot\|_{\beta}$

We solve and prove the $Hyers-Ulam-Rassias$ type stability of functional equation deriving from mapping quadratic in non-Archimdean (l,β)-normed space, associated to the quadratic functional equation.

$$f\Big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\Big)+f\Big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\Big)-2\sum_{j=1}^{k}f\Big(\frac{x_{k+1}}{k}\Big)-2\sum_{j=1}^{k}f\Big(x_j\Big)=0, \tag{1.1}$$

The study of the functional equation stability originated from a question of S.M. Ulam [24], concerning the stability of group homomorphisms. Let $\Big(\mathbf{G},*\Big)$ be a group and let $\Big(\mathbf{G}',\circ,d\Big)$ be a metric group with metric $d\Big(\cdot,\cdot\Big)$. Geven $\epsilon>0$, does there exist a $\delta>0$

such that if $f : \mathbf{G} \to \mathbf{G}'$ satisfies

$$d\left(f\left(x * y\right), f\left(x\right) \circ f\left(y\right)\right) < \delta$$

for all $x, y \in \mathbf{G}$ then there is a homomorphism $h : \mathbf{G} \to \mathbf{G}'$ with

$$d\left(f\left(x\right), h\left(x\right)\right) < \epsilon$$

for all $x \in \mathbf{G}$?, if the answer, is affirmative, we would say that equation of homomophism $h\left(x * y\right) = h\left(y\right) \circ h\left(y\right)$ is stable. The concept of stability for a functional equation arises when we replace functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given function equation?
The Hyers [8] gave firts affirmative partial answer to the equation of *Ulam* in *Banach* spaces.
Next The stability of quadratic functional equation was proved by Skof [23] for mappings $f : E_1 \to E_2$ where E_1 is a normed space and E_2 is a Banach space . Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The functional equation:

$$f\left(x + y\right) + f\left(x + y\right) - 2f\left(x\right) - 2f(y)$$

is called the quadratic functional equation.
The functional equation:

$$f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) - \frac{1}{2}f\left(x\right) - \frac{1}{2}f\left(y\right) = 0$$

is called a Jensen type the quadratic functional equation.

The first work on the stability problem for functional equations in non-Archimedean spaces was started by Moslehian and Rassias [11]. Moslehian and Sadeghi [10] investigated the stability of cubi functional equations in non-Archimedean normed space. Next the mathematicians Xiuzhog Yang, Yachai lui, and Nazek Alessa set up the quadratic equation in non-Archimedean (n,β)-normed space

Recently, in [3, 9, 25] the authors studied the Hyers-Ulam-Rassia stability for the following quadratic functional equation in non-Archimedean (n,β)-normed space:

$$f\left(x + 2y\right) + f\left(x - 2y\right) = 4f\left(x + y\right) + 4f\left(x - y\right) - 6f\left(x\right) + f\left(2y\right) + f\left(-2y\right) - 4f\left(y\right) - 4f\left(-y\right) \tag{1.2}$$

$$\sum_{1\le i\le j\le l} \phi\left(v_i + v_j\right) + \sum_{1\le i\le j\le l} \phi\left(v_i - v_j\right) = 2\left(l - 1\right) \sum_{1\le i\le l} \phi\left(v_i\right) \tag{1.3}$$

In this paper, we solve and proved the Hyers-Ulam-Rassias type stability for quadratic functional equation (1.1), ie the quadratic functional equation with $2k-variables$. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, we will prove that the mappings satisfying the quadratic functional equation (1.1). Thus, the results in this paper are generalization of

those in [3, 9, 25] for quadratic functional equation with $2k - variables$.
The paper is organized as followns:
In section preliminaries we remind some basic some basic properties about in non-Archimedean (n,β)-normed space in [10, 11, 25] such as We only redefine the solution definition of the quadratic equation function.
Section 3: is devoted to prove the Hyers-Ulam stability of the quadratic functional equation (1.1) when when $\mathbf{X}$ is a non-Archimedean (n,β)-normed space with norm $\left\|\cdot\right\|_{\beta_1}$ and that $\mathbf{Y}$ is a Banach non-Archimedean (n,β)-normed space with norm with norm $\left\|\cdot\right\|_{\beta}$

2. PRELIMINARIES

2.1. (n,β)-normed spaces.

Definition 2.1.

Let $\{x_n\}$ be a sequence in a normed space $\mathbf{X}$.

(1) A sequence $\{x_n\}_{n=1}^{\infty}$ in a space $\mathbf{X}$ is a Cauchy sequence iff the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero.
(2) The sequence $\{x_n\}_{n=1}^{\infty}$ is said to be convergent if, for any $\epsilon > 0$, there are a positive integer N and $x \in \mathbf{X}$ such that

$$\left\|x_n - x\right\| \leq \epsilon . \forall n \geq N,$$

for all n,m $\geq N$. Then the point x$\in$ $\mathbf{X}$ is called the limit of sequence x_n and denote $\lim_{n\to\infty} x_n = x$.

(3) If every sequence Cauchy in $\mathbf{X}$ converges, then the normed space $\mathbf{X}$ is called a Banach space.

Definition 2.2.

Let $\mathbf{X}$ be a linear space over $\mathbb{R}$ with dim$\mathbf{X} \geq n$, $n \in \mathbb{N}$ and $0 < \beta \leq 1$ let $\left\|\cdot, ..., \cdot\right\| : \mathbf{X^n} \to \mathbb{R}$. be a function satisfying the following properties:

(1) $\left\|x_1, ..., x_n\right\|_{\beta} = 0$ if and only if $x_1, ..., x_n$ are linearly dependent,
(2) $\left\|x_1, ..., x_n\right\|_{\beta}$ is invariant under permutations of $x_1, ..., x_n$
(3) $\left\|\alpha x_1, ..., x_n\right\|_{\beta} = \left|\alpha\right|^{\beta}\left\|x_1, ..., x_n\right\|$
(4) $\left\|x_1, ..., x_n, y + z\right\|_{\beta} \leq \left\|x_1, ..., x_n, y\right\|_{\beta} + \left\|x_1, ..., x_n, z\right\|_{\beta}, \forall x_1, ..., x_n, y, z \in \mathbf{X}$ and $\alpha \in \mathbb{R}$. Then the function $\left\|\cdot, ..., \cdot\right\|$ is called an (n,β)-norm on $\mathbf{X}$ and the pair $\left(\mathbf{X}, \left\|\cdot, ..., \cdot\right\|\right)$ is called a linear (n,β)-normed space or an (n,β)-normed space.
* Note that the concept of a linear (n,β)-normed space is a generalization of a linear n-normed space $(\beta = 1)$ and of a linear n-normed space $(n = 1)$

Definition 2.3.

A sequence $\{x_n\}$ in a linear (n,β)-normed space $\mathbf{X}$ is called a convergent sequence if there is $x \in \mathbf{X}$ such that $\lim_{n\to\infty} \|x_n - x, z_1, z_2, ..., z_{n-1}\|_\beta = 0$ for all $z_1, z_2, ..., z_{n-1} \in \mathbf{X}$. *Note we call that $\{x_n\}$ convergent to xor that x is the limit of $\{x_n\}$, witer $\{x_n\} \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

Definition 2.4.

A sequence $\{x_n\}$ in a linear (n,β)-normed space $\mathbf{X}$ is called a Cauchy sequence if $\lim_{n,m\to\infty} \|x_n - x_m, z_1, z_2, ..., z_{n-1}\|_\beta = 0$ for all $z_1, z_2, ..., z_{n-1} \in \mathbf{X}$.

Definition 2.5.

A linear (n,β)-normed space in which every Cauchy sequence is convergent is called a complete (n,β)-normed space.

2.2. The properties of (n,β)-normed spaces.

Lemma 2.6.

Let $\left(\mathbf{X}, \|\cdot, ..., \cdot\|_\beta\right)$ be a linear (n,β)-normed space, $k \geq 1$, $0 < \beta \leq 1$. If $x_1 \in \mathbf{X}$ and $\|x_1, z_1, z_2, ..., z_{n-1}\|_\beta = 0$ for all $z_1, z_2, ..., z_{n-1} \in \mathbf{X}$, then $x_1 = 0$.

Lemma 2.7.

For a convergent sequence $\{x_n\}$ in a linear (n,β)-normed space $\mathbf{X}$,

$$\lim_{n\to\infty} \|x_n, z_1, z_2, ..., z_{n-1}\|_\beta = \|\lim_{n\to\infty} x_m, z_1, z_2, ..., z_{n-1}\|_\beta = 0$$

for all $z_1, z_2, ..., z_{n-1} \in \mathbf{X}$.

2.3. non-Archimedean (n,β)-normed spaces.

In this subsection we recall some basic notations from [9,10] such as non-Archimedean fields, non-Archimedean normed spaces and non-Archimedean Banach spaces.

A valuation is a function $|\cdot|$ from a field $\mathbb{K}$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation,

$$|r| = 0 \Leftrightarrow r = 0$$

$$|r \cdot s| := |r|\ |s|, \forall r, s \in \mathbb{K}$$

and the triangle inequality holds, i.e.,

$$|r + s| \leq |r| + |s|, \forall r, s \in \mathbb{K}.$$

A field $\mathbb{K}$ is called a valued field if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuation. Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the strong triangle inequality is replaced by

$$|r + s| \leq max\{|r|, |s|\}, \forall r, s \in \mathbb{K},$$

then the function $|\cdot|$ is called a non-Archimedean valuation. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1, \forall n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function

$\big|\cdot\big|$ talking everything except for 0 into 1 and $\big|0\big| = 0$. In this paper, we assume that the base field is a non-Archimedean field with $\big|2\big| \neq 1$, hence call it simply a field.

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Definition 2.8. Let be a vecter space over a filed K with a non -Archimedean $\Big|\cdot\Big|$. A function $\Big\|\cdot\Big\| : X \to \Big[0,\infty\Big)$ is said a non -Archimedean norm if it satisfies the follwing conditions:

(1) $\Big\|x\Big\| = 0$ if and only if $x = 0$;

(2) $\Big\|rx\Big\| = \Big|r\Big|\Big\|x\Big\|(r \in \mathbb{K}, x \in X)$;

(3) $\Big\|x+y\Big\| \leq max\Bigg\{\Big\|x\Big\|, \Big\|y\Big\|\Bigg\}x, y \in X$ hold.

Then $\Bigg(X, \Big\|\cdot\Big\|\Bigg)$ is called a norm -Archimedean norm space.

Definition 2.9.

A sequence $\Big\{x_n\Big\}$ in a norm -Archimedean $\big(n,\beta\big)$-normed space $\mathbf{X}$ is a Cauchy sequence if and only if $\Big\{x_n - x_m\Big\} \to 0$.

Definition 2.10. Let $\Big\{x_n\Big\}$ be a sequence in a norm -Archimedean normed space X.

(1) A sequence $\Big\{x_n\Big\}_{n=1}^{\infty}$ in a non -Archimedean space is a Cauchy sequence iff the sequence $\Big\{x_{n+1} - x_n\Big\}_{n=1}^{\infty}$ converges to zero.

(2) The sequence $\Big\{x_n\Big\}$ is said to be convergent if, for any $\epsilon > 0$, there are a positive integer N and $x \in X$ such that

$$\Big\|x_n - x\Big\| \leq \epsilon . \forall n \geq N,$$

for all n,m $\geq N$. The we call x$\in X$ a limit of sequence x_n and denote $\lim_{n\to\infty} x_n = x$.

(3) If every sequence Cauchy in X converger, then the norm -Archimedean normed space X is called a norm -Archimedean Bnanch space.

Definition 2.11.

Let $\mathbf{X}$ be a real space with dim$\mathbf{X} \geq n$ over a scalar filed $\mathbb{K}$ with a non -Archimedean nontrivial valuation $\big|\cdot\big|$, , where n is a positive integer and β is a constant with $0 < \beta \leq 1$. A real-valued function let $\Big\|\cdot, ..., \cdot\Big\| : \mathbf{X^n} \to \mathbb{R}$. is called an $\big(n,\beta\big)$-norn on $\mathbf{X}$ satisfying the following properties:

(1) $\Big\|x_1, ..., x_n\Big\|_\beta = 0$ if and only if $x_1, ..., x_n$ are linearly dependent,

(2) $\left\|x_1, ..., x_n\right\|_\beta$ is invariant under permutations of $x_1, ..., x_n$

(3) $\left\|\alpha x_1, ..., x_n\right\|_\beta = \left|\alpha\right|^\beta \left\|x_1, ..., x_n\right\|$

(4) $\left\|x_0 + x_1, ..., x_n\right\|_\beta \leq max\left\{\left\|x_0, ..., x_n\right\|_\beta, \left\|x_1, ..., x_n\right\|_\beta\right\}, \forall x_0, x_1, ..., x_n \in \mathbf{X}$ and $\alpha \in \mathbb{K}$. Then the function $\left\|\cdot, ..., \cdot\right\|$ is called an (n, β)-norm on $\mathbf{X}$ and the pair $\left(\mathbf{X}, \left\|\cdot, ..., \cdot\right\|\right)$ is called a non -Archimedean (n, β)-normed space or an (n, β)-normed space.

* Note that the concept of a non -Archimedean (n, β)-normed space is a non -Archimedean n-normed space if $(\beta = 1)$ and a a non -Archimedean β-normed space if n=1 respectively.

2.4. **Solutions of the equation.** The functional equation

$$f\left(x+y\right) + f\left(x-y\right) = 2f\left(x\right) + 2f\left(y\right)$$

is called the quadratic equation. In particular, every solution of the quadratic equation is said to be an *quadratic mapping*.

3. Stability Results.

Here, we consider $\left|k\right| \neq 1$ and examine the Hyers-Ulam stability of the functional equation (1.1) when when $\mathbf{X}$ is a non-Archimedean (n,β)-normed space with norm $\| \cdot \|_{\beta_1}$ and that $\mathbf{Y}$ is a Banach non-Archimedean (n,β)-normed space with norm with norm $\| \cdot \|_\beta$ Under this setting, we can show that the mapping satisfying (1.1) is quadratic. These results are give in the following.

Theorem 3.1.

Suppose That $\mathbf{X}$ is a non-Archimedean β_1-normed space and that $\mathbf{Y}$ is a complete non-Archimedean $(l, \beta)-$normed space, where $l \geq 2$, $0 < \beta, \beta_1 \leq 1$. Let $\epsilon \in [0, \infty)$, $p \in \left(0, \infty\right)$ with $\beta_1 p > \beta$ and let

$$\varphi : \mathbf{Y^{l-1}} \to [0, \infty)$$

be a function. Suppose that a mapping

$$\mathbf{f}\text{: } \mathbf{X^{2k}} \to \mathbf{Y}$$

satisfying $f(0)=0$ and the inequality

$$\Big\|f\Big(\sum_{j=1}^{k}x_j+\frac{1}{k}\sum_{j=1}^{k}x_{k+j}\Big)+f\Big(\sum_{j=1}^{k}x_j-\frac{1}{k}\sum_{j=1}^{k}x_{k+j}\Big)-2\sum_{j=1}^{k}f\Big(x_j\Big)$$
$$-2\sum_{j=1}^{k}f\Big(\frac{x_{k+j}}{k}\Big),z_1,z_2,...,z_{l-1}\Big\|_{\beta}$$
$$\le\epsilon\Big(\sum_{j=1}^{k}\Big\|x_j\Big\|_{\beta_1}^{p}+\sum_{j=1}^{k}\Big\|x_{k+j}\Big\|_{\beta_1}^{p}\Big)\varphi\Big(z_1,z_2,...,z_{l-1}\Big) \tag{3.1}$$

for all $x_j,x_{k+j}\in\mathbf{X}$ for all $j=1\to k$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Then there exists a unique quadratic mapping

$$H:\mathbf{X}\to\mathbf{Y}$$

satisfying

$$\Big\|f\Big(x\Big)-H\Big(x\Big),z_1,z_2,...,z_{l-1}\Big\|_{\beta}\le\epsilon k\Big|(2k)^{-\beta}\Big|\Big\|x\Big\|_{\beta_1}^{p}\varphi\Big(z_1,z_2,...,z_{l-1}\Big) \tag{3.2}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{n-1}\in\mathbf{Y}$.

Proof. Replace $\Big(x_1,x_2,...,x_k,x_{k+1},x_{k+2},,...,x_{2k}\Big)$ by $\Big(x,x,...,x,0,0,...,0\Big)$ we have

$$\Big\|2f\Big(kx\Big)-2kf\Big(x\Big),z_1,z_2,...,z_{l-1}\Big\|_{\beta}\le\epsilon\cdot k\Big\|x\Big\|_{\beta_1}^{p}\varphi\Big(z_1,z_2,...,z_{l-1}\Big) \tag{3.3}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$

and dividing both sides by $\Big|(2k)^{\beta}\Big|$, we get

$$\Big\|\frac{f\Big(kx\Big)}{k}-f\Big(x\Big),z_1,z_2,...,z_{l-1}\Big\|_{\beta}\le\epsilon\cdot k\Big|(2k)^{-\beta}\Big|\Big\|x\Big\|_{\beta_1}^{p}\varphi\Big(z_1,z_2,...,z_{l-1}\Big) \tag{3.4}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Replacing x by k^nx in (3.4)

and dividing both sides by $\Big|k^{n\beta}\Big|$, we get

$$\Big\|\frac{f\Big(k^{n+1}x\Big)}{k^{n+1}}-\frac{f\Big(k^nx\Big)}{k^n},z_1,z_2,...,z_{l-1}\Big\|_{\beta}$$
$$\le\epsilon\cdot k\Big|\frac{1}{k^{n\beta}}\Big|\Big|\frac{1}{(2k)^{\beta}}\Big||k^{n\beta_1p}|\Big\|x\Big\|_{\beta_1}^{p}\varphi\Big(z_1,z_2,...,z_{l-1}\Big)$$
$$=\epsilon\cdot k\Big|\frac{1}{(2k)^{\beta}}\Big|\Big|k^{(\beta_1p-\beta)}\Big|^{n}\Big\|x\Big\|_{\beta_1}^{p}\varphi\Big(z_1,z_2,...,z_{l-1}\Big) \tag{3.5}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Since $p\beta_1 > \beta$ and $\left|k\right| \neq 1$, we obtain that

$$\lim_{n\to\infty} \left\| \frac{f\left(k^{n+1}x\right)}{k^{n+1}} - \frac{f\left(k^n x\right)}{k^n}, z_1, z_2, ..., z_{l-1} \right\|_\beta = 0 \tag{3.6}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$

. It follows from (3.6) that the sequence $\left\{ \frac{f\left(k^n x\right)}{k^n} \right\}$ is Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is completes space, the sequence $\left\{ \frac{f\left(k^n x\right)}{k^n} \right\}$ coverges. So one can define the mapping H: $\mathbf{X} \to \mathbf{Y}$ by

$$H\left(x\right) := \lim_{n\to\infty} \frac{f\left(k^n x\right)}{k^n} \tag{3.7}$$

for all $x \in \mathbf{X}$.

It follows from (3.1) and (3.7) and lemma 2.7 that

$$\begin{aligned}
&\Bigg\| H\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) + H\left(\sum_{j=1}^{k} x_j - \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) - 2\sum_{j=1}^{k} H\left(x_j\right) \\
&- 2\sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1} \Bigg\|_\beta \\
&= \lim_{n\to\infty} \left|k^{-n\beta}\right| \Bigg\| f\left[k^n\left(\sum_{j=1}^{n} x_j + \frac{1}{k}\sum_{j=1}^{n} x_{k+j}\right) + H\left(\sum_{j=1}^{k} x_j - \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right)\right] \\
&- 2\sum_{j=1}^{k} f\left(k^n x_j\right) - 2\sum_{j=1}^{k} f\left(k^n \frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1} \Bigg\|_\beta \\
&\leq \lim_{n\to\infty} \epsilon \left|k^{-n\beta}\right| \left(\sum_{j=1}^{k} \left\|k^n x_j\right\|_{\beta_1}^p + \sum_{j=1}^{k} \left\|k^n x_{k+j}\right\|_{\beta_1}^p\right) \varphi\left(z_1, z_2, ..., z_{l-1}\right) \\
&= \lim_{n\to\infty} \epsilon \left|k^{n(\beta_1 p - \beta)}\right|^n \left(\sum_{j=1}^{k} \left\|k^n x_j\right\|_{\beta_1}^p\right) \varphi\left(z_1, z_2, ..., z_{l-1}\right)
\end{aligned}$$

and so for all x_j, $x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Since $p\beta_1 > \beta$ and $\left|k\right| \neq 1$, we get

$$\begin{aligned}
&\Bigg\| H\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) + H\left(\sum_{j=1}^{k} x_j - \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) - 2\sum_{j=1}^{k} H\left(x_j\right) \\
&\qquad - 2\sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1} \Bigg\|_\beta = 0
\end{aligned}$$

for all x_j, $x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. By lemma 2.6, we get

$$H\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) + H\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) - 2\sum_{j=1}^{k} H\left(x_j\right) - 2\sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right) = 0$$

for all $x_j,\ x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. So mapping H is quadratic.

replace x by kx in (3.3) and dividing both sides by $\left|(2k^2)^{\beta}\right|$, we get

$$\begin{aligned}&\left\|\frac{f\left(k^2x\right)}{k^2} - \frac{f\left(kx\right)}{k}, z_1, z_2, ..., z_{l-1}\right\|_{\beta} \\ &\qquad \leq \epsilon \cdot k\left|(2k^2)^{-\beta}\right|\left\|kx\right\|_{\beta_1}^{p} \varphi\left(z_1, z_2, ..., z_{l-1}\right)\end{aligned} \tag{3.8}$$

Thus, by(3.3) and (3.8)

$$\begin{aligned}&\left\|f\left(x\right) - \frac{f\left(k^2x\right)}{k^2}, z_1, z_2, ..., z_{l-1}\right\|_{\beta} \\ &\quad \leq max\left\{\left\|\frac{f\left(kx\right)}{k} - f\left(x\right), z_1, z_2, ..., z_{l-1}\right\|_{\beta}, \left\|\frac{f\left(k^2x\right)}{k^2} - \frac{f\left(kx\right)}{k}, z_1, z_2, ..., z_{l-1}\right\|_{\beta}\right\} \\ &\quad \leq max\left\{\epsilon \cdot k\left|(2k)^{-\beta}\right|\left\|x\right\|_{\beta_1}^{p} \varphi\left(z_1, z_2, ..., z_{l-1}\right), \epsilon \cdot k\left|(2k^2)^{-\beta}\right|\left\|x\right\|_{\beta_1}^{p} \varphi\left(z_1, z_2, ..., z_{l-1}\right)\right\}\end{aligned} \tag{3.9}$$

Since $p\beta_1 > \beta$ and $\left|k\right| \neq 1$, we obtain

$$\left\|f\left(x\right) - \frac{f\left(k^2x\right)}{k^2}, z_1, z_2, ..., z_{l-1}\right\|_{\beta} \leq \epsilon \cdot k\left|k^{-2\beta}\right|\left\|x\right\|_{\beta_1}^{p} \varphi\left(z_1, z_2, ..., z_{l-1}\right) \tag{3.10}$$

for all $x \in \mathbf{X}$. and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. By induction on n, we can conclude that

$$\left\|f\left(x\right) - \frac{f\left(k^nx\right)}{k^n}, z_1, z_2, ..., z_{l-1}\right\|_{\beta} \leq \epsilon \cdot k\left|(2k)^{-\beta}\right|\left\|x\right\|_{\beta_1}^{p} \varphi\left(z_1, z_2, ..., z_{l-1}\right) \tag{3.11}$$

for all $x \in \mathbf{X}, n \in \mathbb{N}$. and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Replacing x by kx in (3.11)

and dividing both sides by $\left|k^{\beta}\right|$, we get

$$\left\|\frac{f\left(kx\right)}{k} - \frac{f\left(k^{n+1}x\right)}{k^{n+1}}, z_1, z_2, ..., z_{l-1}\right\|_{\beta} \leq \epsilon \cdot k\left|(2k)^{-\beta}\right|\left|k^{-\beta}\right|\left\|kx\right\|_{\beta_1}^{p} \varphi\left(z_1, z_2, ..., z_{l-1}\right) \tag{3.12}$$

for all $x \in \mathbf{X}, n \in \mathbb{N}$. and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. It follow from (3.3) and (3.12)

$$\left\|f(x)-\frac{f(k^n x)}{k^n}, z_1, z_2, ..., z_{l-1}\right\|_\beta \leq \epsilon\cdot k\left|k^{-2\beta}\right|\left\|x\right\|_{\beta_1}^p \varphi(z_1, z_2, ..., z_{l-1}) \qquad (3.13)$$

for all $x \in \mathbf{X}, n \in \mathbb{N}.$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Passing the limit as $n \to \infty$ (3.11), we can get (3.2) Next, we prove the uniqueness of H. Assume that $H_1 : \mathbf{X} \to \mathbf{Y}$ is an additive mapping satisfing (3.2). Then we have

$$\begin{aligned}
&\left\|H(x)-H_1(x), z_1, z_2, ..., z_{l-1}\right\|_\beta \\
&= \left|k^{-n\beta}\right|\left\|H(k^n x)-H_1(k^n x), z_1, z_2, ..., z_{l-1}\right\|_\beta \\
&\leq \left|k^{-n\beta}\right| max\Big\{\left\|H(k^n x)-f(k^n x), z_1, z_2, ..., z_{l-1}\right\|_\beta, \\
&\left\|f(k^n x)-H_1(k^n x), z_1, z_2, ..., z_{l-1}\right\|_\beta\Big\} \\
&\leq \epsilon\cdot k\left|k^{-n\beta}\right|\left|(2k)^{-\beta}\right|\left\|k^n x\right\|_{\beta_1}^p \varphi(z_1, z_2, ..., z_{l-1}) \\
&= \epsilon\cdot k^{kq\beta_1}\left|k^{p\beta_1-\beta}\right|^n\left|(2k)^{-\beta}\right|\left\|x\right\|_{\beta_1}^{p)} \varphi(z_1, z_2, ..., z_{l-1}) \qquad (3.14)
\end{aligned}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Taking the limit as $n \to \infty$, we have

$$\left\|H(x)-H_1(x), z_1, z_2, ..., z_{l-1}\right\|_\beta = 0$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. By lemma 2.6, we get $H(x) = H_1(x)$ for all $x \in \mathbf{X}$. So H is the unique additive mapping satisfying (3.2)Let

□

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Theorem 3.2.

Suppose That $\mathbf{X}$ is a non-Archimedean β_1-normed space and that $\mathbf{Y}$ is a complete non-Archimedean $(l, \beta)-$normed space, where $l \geq 2$, $0 < \beta, \beta_1 \leq 1$. Let $\epsilon \in [0, \infty)$, $p \in (0, \infty)$ with $\beta_1 p < \beta$ and let

$$\varphi : \mathbf{Y^{l-1}} \to [0, \infty)$$

be a function. Suppose that a mapping

$$\mathbf{f}\colon \mathbf{X^{2k}} \to \mathbf{Y}$$

satisfying $f(0)=0$ and the inequality

$$\left\|f\Big(\sum_{j=1}^{k}x_j+\frac{1}{k}\sum_{j=1}^{k}x_{k+j}\Big)+f\Big(\sum_{j=1}^{k}x_j-\frac{1}{k}\sum_{j=1}^{k}x_{k+j}\Big)-2\sum_{j=1}^{k}f\Big(x_j\Big)\right.$$
$$\left.-2\sum_{j=1}^{k}f\Big(\frac{x_{k+j}}{k}\Big),z_1,z_2,...,z_{l-1}\right\|_{\beta}$$
$$\leq\epsilon\Big(\sum_{j=1}^{k}\Big\|x_j\Big\|_{\beta_1}^{p}+\sum_{j=1}^{k}\Big\|x_{k+j}\Big\|_{\beta_1}^{p}\Big)\varphi\Big(z_1,z_2,...,z_{l-1}\Big) \tag{3.15}$$

for all $x_j,x_{k+j}\in\mathbf{X}$ for all $j=1\rightarrow k$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Then there exists a unique additive mapping

$$H:\mathbf{X}\rightarrow\mathbf{Y}$$

satisfying

$$\Big\|f\big(x\big)-H\big(x\big),z_1,z_2,...,z_{l-1}\Big\|_{\beta}\leq\epsilon k\Big|k^{-\beta}\Big|\Big\|x\Big\|_{\beta_1}^{p}\varphi\big(z_1,z_2,...,z_{l-1}\big) \tag{3.16}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$.

Proof. Put $x_j=x$ and $x_{k+j}=0$ for all $j=1\rightarrow k$ in (3.15),we get

$$\Big\|2f\big(kx\big)-2kf\big(x\big),z_1,z_2,...,z_{l-1}\Big\|_{\beta}\leq\epsilon\cdot k\Big\|x\Big\|_{\beta_1}^{p}\varphi\big(z_1,z_2,...,z_{l-1}\big) \tag{3.17}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$ replace x by $\frac{x}{k}$ and dividing both sides by $\big|2^{\beta}\big|$ in (3.17) we get we get

$$\Big\|f\big(x\big)-kf\big(\frac{x}{k}\big),z_1,z_2,...,z_{l-1}\Big\|_{\beta}\leq\epsilon\cdot\big|2^{-\beta}\big|k\Big|k^{-p\beta_1}\Big|\Big\|x\Big\|_{\beta_1}^{p}\varphi\big(z_1,z_2,...,z_{l-1}\big) \tag{3.18}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Replacing x by $\frac{x}{k^n}$ in (3.18)

$$\Big\|k^nf\big(\frac{x}{k^n}\big)-k^{n+1}f\big(\frac{x}{k^{n+1}}\big),z_1,z_2,...,z_{n-1}\Big\|_{\beta}$$
$$\leq\epsilon\big|2^{-\beta}\big|\cdot k\Big|k^{n\beta}\Big|\Big|k^{-p\beta_1}\Big|\Big|k^{-n\beta_1p}\Big|\Big\|x\Big\|_{\beta_1}^{p}\varphi\Big(z_1,z_2,...,z_{l-1}\big)$$
$$=\epsilon\big|2^{-\beta}\big|\cdot k\Big|k^{n\beta}\Big|\big|k^{-\beta_1p(n+1)}\Big|\Big\|x\Big\|_{\beta_1}^{p}\varphi\big(z_1,z_2,...,z_{l-1}\Big) \tag{3.19}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Since $p\beta_1<\beta$ and $\big|k\big|\neq1$, we obtain that

$$\lim_{n\to\infty}\Big\|k^{n+1}f\big(\frac{x}{k^{n+1}}\big)-k^nf\big(\frac{x}{k^n}\big),z_1,z_2,...,z_{l-1}\Big\|_{\beta}=0 \tag{3.20}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$

. It follows from (3.20) that the sequence $\left\{k^n f\left(\frac{x}{k^n}\right)\right\}$ is Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is completes space, the sequence $\left\{k^n f\left(\frac{x}{k^n}\right)\right\}$ coverges. So one can define the mapping $H\colon \mathbf{X} \to \mathbf{Y}$ by

$$H\left(x\right) := \lim_{n\to\infty} k^n f\left(\frac{x}{k^n}\right) \tag{3.21}$$

for all $x \in \mathbf{X}$.

It follows from (3.15) and (3.21) and lemma 2.7 that

$$\begin{aligned}
&\left\| H\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) + H\left(\sum_{j=1}^{k} x_j - \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) - 2\sum_{j=1}^{k} H\left(x_j\right)\right.\\
&\left. - 2\sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1}\right\|_{\beta}\\
&= \lim_{n\to\infty} \left|k^{n\beta}\right| \left\| f\left[k^{-n}\left(\sum_{j=1}^{n} x_j + \frac{1}{k}\sum_{j=1}^{n} x_{k+j}\right) + H\left(\sum_{j=1}^{k} x_j - \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right)\right]\right.\\
&\left. - 2\sum_{j=1}^{k} f\left(k^{-n} x_j\right) - 2\sum_{j=1}^{k} f\left(k^{-n}\frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1}\right\|_{\beta}\\
&\le \lim_{n\to\infty} \epsilon \left|k^{n\beta}\right| \left(\sum_{j=1}^{k} \left\|k^{-n} x_j\right\|_{\beta_1}^{p} + \sum_{j=1}^{k} \left\|k^{-n} x_{k+j}\right\|_{\beta_1}^{p}\right) \varphi\left(z_1, z_2, ..., z_{l-1}\right)\\
&= \lim_{n\to\infty} \epsilon \left|k^{(\beta-\beta_1 p)}\right|^{n} \left(\sum_{j=1}^{k} \left\|x_j\right\|_{\beta_1}^{p}\right) \varphi\left(z_1, z_2, ..., z_{l-1}\right)
\end{aligned}$$

and so for all $x_j,\, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k.$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Since $p\beta_1 < \beta$ and $\left|k\right| \neq 1$, we get

$$\begin{aligned}
&\left\| H\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) + H\left(\sum_{j=1}^{k} x_j - \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) - 2\sum_{j=1}^{k} H\left(x_j\right)\right.\\
&\qquad\qquad \left. - 2\sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1}\right\|_{\beta} = 0
\end{aligned}$$

for all $x_j,\, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k.$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. By lemma 2.6, we get

$$H\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) + H\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) - 2\sum_{j=1}^{k} H\left(x_j\right) - 2\sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right) = 0$$

for all $x_j,\, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. So mapping H is quadratic.

replace x by $\frac{x}{k}$ in (3.18) and multiplying both sides by $\left|k^{\beta}\right|$, we get

$$\left\|k^2 f\Big(\frac{x}{k^2}\Big) - kf\Big(\frac{x}{k}\Big), z_1, z_2, ..., z_{l-1}\right\|_\beta$$
$$\leq \epsilon k\left|2^{-\beta}\right| \cdot \left|k^\beta\right|\left\|\frac{x}{k}\right\|_{\beta_1}^p \varphi\big(z_1, z_2, ..., z_{l-1}\big) \quad (3.22)$$

Thus, by(3.17) and (3.22)

$$\left\|f(x) - k^2 f(\frac{x}{k^2}), z_1, z_2, ..., z_{l-1}\right\|_\beta$$
$$\leq max\left\{\left\|kf(\frac{x}{k}) - f(x), z_1, z_2, ..., z_{l-1}\right\|_\beta, \left\|k^2 f(\frac{x}{k^2}) - kf(\frac{x}{k}), z_1, z_2, ..., z_{l-1}\right\|_\beta\right\}$$
$$\leq max\left\{\epsilon k \cdot \left|2^{-\beta}\right|\left\|\frac{x}{k}\right\|_{\beta_1}^p \varphi\big(z_1, z_2, ..., z_{l-1}\big), \epsilon k \cdot \left|2^{-\beta}\right|\left|k^{2\beta}\right|\left\|\frac{x}{k^2}\right\|_{\beta_1}^p \varphi\big(z_1, z_2, ..., z_{l-1}\big)\right\} \quad (3.23)$$

Since $p\beta_1 < \beta$ and $\left|k\right| \neq 1$, we obtain

$$\left\|f(x) - k^2 f(\frac{x}{k^2}), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq \epsilon \cdot k\left|k^{-p\beta_1}\right|\left\|x\right\|_{\beta_1}^p \varphi\big(z_1, z_2, ..., z_{l-1}\big) \quad (3.24)$$

for all $x \in \mathbf{X}$. and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. By induction on n, we can conclude that

$$\left\|f(x) - k^n f(\frac{x}{k^n}), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq \epsilon \cdot k\left|k^{-2\beta}\right|\left\|x\right\|_{\beta_1}^p \varphi\big(z_1, z_2, ..., z_{l-1}\big) \quad (3.25)$$

for all $x \in \mathbf{X}, n \in \mathbb{N}$. and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Replacing x by $\frac{x}{k}$ in (3.11)
and multiplying both sides by $\left|k^\beta\right|$, we get

$$\left\|kf(\frac{x}{k}) - k^{n+1} f(\frac{x}{k^{n+1}}), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq \epsilon \cdot k\left|k^\beta\right|\left|k^{-p\beta_1}\right|\left\|x\right\|_{\beta_1}^p \varphi\big(z_1, z_2, ..., z_{l-1}\big) \quad (3.26)$$

for all $x \in \mathbf{X}, n \in \mathbb{N}$. and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. It follow from (3.17) and (3.43)

$$\left\|f(x) - k^{n+1} f(\frac{x}{k^{n+1}}), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq \epsilon \cdot k\left|k^{-p\beta_1}\right|\left\|x\right\|_{\beta_1}^p \varphi\big(z_1, z_2, ..., z_{l-1}\big) \quad (3.27)$$

for all $x \in \mathbf{X}, n \in \mathbb{N}$. and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Passing the limit as $n \to \infty$ (3.59), we can get (3.16) Next, we prove the uniqueness of H. Assume that $H_1 : \mathbf{X} \to \mathbf{Y}$ is an additive mapping satisfing (3.16). Then we have

$$\Big\|H(x)-H_1(x),z_1,z_2,...,z_{l-1}\Big\|_\beta$$
$$=\Big|k^{n\beta}\Big|\Big\|H(\frac{x}{k^n})-H_1(\frac{x}{k^n}),z_1,z_2,...,z_{l-1}\Big\|_\beta$$
$$\leq\Big|k^{n\beta}\Big|max\Big\{\Big\|H(\frac{x}{k^n})-f(\frac{x}{k^n}),z_1,z_2,...,z_{l-1}\Big\|_\beta,$$
$$\Big\|f(\frac{x}{k^n})-H_1(\frac{x}{k^n}),z_1,z_2,...,z_{l-1}\Big\|_\beta\Big\}$$
$$\leq\epsilon\cdot k\Big|k^{n\beta}\Big|\Big|k^{-p\beta_1}\Big|\Big\|\frac{x}{k^n}\Big\|_{\beta_1}^p\varphi(z_1,z_2,...,z_{l-1})$$
$$=\epsilon\cdot k\Big|k^{\beta-p\beta_1}\Big|^n\Big|k^{-p\beta_1}\Big|\Big\|x\Big\|_{\beta_1}^p\varphi(z_1,z_2,...,z_{l-1}) \tag{3.28}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Taking the limit as $n\to\infty$, we have

$$\Big\|H(x)-H_1(x),z_1,z_2,...,z_{l-1}\Big\|_\beta=0$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. By lemma (1.4), we get $H(x)=H_1(x)$ for all $x\in\mathbf{X}$. So H is the unique additive mapping satisfying (3.16) Let

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Theorem 3.3.

Suppose That $\mathbf{X}$ be a vector space and that $\mathbf{Y}$ is a complete non-Archimedean $(l,\beta)-$normed space, where $l\geq 2$, $0<\beta\leq 1$. Let

$$\varphi:\mathbf{X^{2k}}\to[0,\infty)$$

be a function such that

$$\lim_{n\to\infty}\Big|\frac{1}{k^{n\beta}}\Big|\varphi\Big(k^nx_1,k^nx_2,...,k^nx_k,k^nx_{k+1},k^nx_{k+2},...,k^nx_{2k}\Big)=0 \tag{3.29}$$

for all $x_j,x_{k+j}\in\mathbf{X}$ for all $j=1\to k$, and

suppose that a mapping

$$\psi\colon\mathbf{Y^{l-1}}\to[0,\infty)$$

be a function. The limit

$$\lim_{n\to\infty}max\Big\{\Big|\frac{1}{k^{i\beta}}\Big|\varphi\big(k^{i-1}x_1,k^{i-1}x_2,...,k^{i-1}x_k,0,0,...,0\big),1\leq i\leq n\Big\} \tag{3.30}$$

exists for $x \in \mathbf{X}$, and it is denoted by $\widetilde{\varphi}(x)$. Suppose that a mapping

$$f : \mathbf{X} \to \mathbf{Y}$$

satisfying $f(0) = 0$ the inequality

$$\begin{aligned}\Bigg\| f\Bigg(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} \frac{x_{k+j}}{k}\Bigg) + f\Bigg(\sum_{j=1}^{k} x_j - \sum_{j=1}^{k} \frac{x_{k+j}}{k}\Bigg) - 2\sum_{j=1}^{k} f\Big(x_j\Big) \\ - 2\sum_{j=1}^{k} f\Bigg(\frac{x_{k+j}}{k}\Bigg), z_1, z_2, ..., z_{l-1}\Bigg\|_{\beta} \\ \le \varphi\big(x_1, x_2, ..., x_k, x_{k+1}, x_{k+2}, ..., x_{2k}\big) \cdot \psi\Big(z_1, z_2, ..., z_{l-1}\Big) \quad (3.31)\end{aligned}$$

for all $x_j, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Then there exists a unique additive mapping

$$H : \mathbf{X} \to \mathbf{Y}$$

satisfying

$$\Big\| f\big(x\big) - H\big(x\big), z_1, z_2, ..., z_{l-1} \Big\|_{\beta} \le \widetilde{\varphi}(x)\varphi\big(z_1, z_2, ..., z_{l-1}\big) \quad (3.32)$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Moreover, if

$$\lim_{h\to\infty} \lim_{n\to\infty} max\Bigg\{\Bigg|\frac{1}{k^{i\beta}}\Bigg|\varphi\big(k^{i-1}x_1, k^{i-1}x_2, ..., k^{i-1}x_k, 0, 0, ..., 0\big), 1 + h \le i \le n + h\Bigg\} = 0 \quad (3.33)$$

for all $x \in \mathbf{X}$, then H is a unique additive mapping satisfying (3.32).

Proof. Put $x_j = x, x_{k+j} = 0$ for all $j = 1 \to k$ in (3.31) and dividing both sides by $\Big|(2k)^{\beta}\Big|$, we get

$$\begin{aligned}\Bigg\| \frac{f\big(kx\big)}{k} - f\big(x\big), z_1, z_2, ..., z_{l-1} \Bigg\|_{\beta} \\ \le \Big|(2k)^{-\beta}\Big| \varphi\big(x, x, ..., x, 0, 0, ..., 0\big)\psi\big(z_1, z_2, ..., z_{l-1}\big) \quad (3.34)\end{aligned}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Replacing x by $k^i x$ in (3.34)

and dividing both sides by $\Big|k^{i\beta}\Big|$, we get

$$\begin{aligned}\Bigg\| \frac{f\big(k^{i+1}x\big)}{k^{i+1}} - \frac{f\big(k^i x\big)}{k^i}, z_1, z_2, ..., z_{l-1} \Bigg\|_{\beta} \\ \le \Big|(2k)^{-\beta}\Big|\Big|k^{-i\beta}\Big| \varphi\big(k^i x, k^i x, ..., k^i x, 0, 0, ..., 0\big)\psi\big(z_1, z_2, ..., z_{l-1}\big) \quad (3.35)\end{aligned}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$, $i \in \mathbb{N}$. Taking the limit as $i \to \infty$ and considering (3.29)

$$\lim_{i\to\infty} \left\| \frac{f\left(k^{i+1}x\right)}{k^{i+1}} - \frac{f\left(k^{i}x\right)}{k^{i}}, z_1, z_2, ..., z_{l-1} \right\|_\beta = 0 \tag{3.36}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$

. It follows from (3.36) that the sequence $\left\{\frac{f\left(k^n x\right)}{k^n}\right\}$ is Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is completes space, the sequence $\left\{\frac{f\left(k^n x\right)}{k^n}\right\}$ coverges. So one can define the mapping $H\colon \mathbf{X} \to \mathbf{Y}$ by

$$H\left(x\right) := \lim_{n\to\infty} \frac{f\left(k^n x\right)}{k^n} \tag{3.37}$$

for all $x \in \mathbf{X}$.

It follows from (3.31), (3.54) and lemma 1.4 that

$$\Bigg\| H\Big(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\Big) + H\Big(\sum_{j=1}^{k} x_j - \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\Big) - 2\sum_{j=1}^{k} H\Big(x_j\Big)$$

$$- 2\sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1} \Bigg\|_\beta$$

$$= \lim_{n\to\infty} \left|k^{-n\beta}\right| \Bigg\| f\left[k^n\left(\sum_{j=1}^{n} x_j + \frac{1}{k}\sum_{j=1}^{n} x_{k+j}\right)\right] + f\left[k^n\left(\sum_{j=1}^{n} x_j - \frac{1}{k}\sum_{j=1}^{n} x_{k+j}\right)\right] - 2\sum_{j=1}^{k} f\left(k^n x_j\right)$$

$$- 2\sum_{j=1}^{k} f\left(k^n \frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1} \Bigg\|_\beta$$

$$\leq \lim_{n\to\infty} \left|k^{-n\beta}\right| \varphi\big(k^n x, k^n x, ..., k^n x, 0, 0, ..., 0\big)\psi\big(z_1, z_2, ..., z_{l-1}\big)$$

and so for all x_j, $x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$.Taking the limit as $i \to \infty$ and considering (3.29)we get

$$\Bigg\| H\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) + H\left(\sum_{j=1}^{k} x_j - \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) - 2\sum_{j=1}^{k} H\Big(x_j\Big) - 2\sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right)$$

$$, z_1, z_2, ..., z_{l-1} \Bigg\| = 0$$

for all x_j, $x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. By lemma 1.4, we get

$$H\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) + H\left(\sum_{j=1}^{k} x_j - \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) - 2\sum_{j=1}^{k} H\Big(x_j\Big) - 2\sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right) = 0$$

for all $x_j,\ x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. So mapping H is quadratic. Replace x by kx in (3.34) and dividing both sides by $\left|k^{\beta}\right|$, we get

$$\left\|\frac{f\left(k^2x\right)}{k^2} - \frac{f\left(kx\right)}{k}, z_1, z_2, ..., z_{l-1}\right\|_{\beta} \leq \left|\left|k^{-2\beta}\right|\varphi\left(k^nx, k^nx, ..., k^nx, 0, 0, ..., 0\right)\psi\left(z_1, z_2, ..., z_{l-1}\right) \tag{3.38}$$

for all $x \in \mathbf{X}$, $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Considering (3.34), we get

$$\left\|f\left(x\right) - \frac{f\left(k^2x\right)}{k^2}, z_1, z_2, ..., z_{l-1}\right\|_{\beta} \leq max\left\{\left|\left|k^{-\beta}\right|\varphi\left(x, x, ..., x, 0, 0, ..., 0\right), \left|k^{-2\beta}\right|\varphi\left(kx, kx, ..., kx, 0, 0, ..., 0\right)\right\}\psi\left(z_1, z_2, ..., z_{l-1}\right) \tag{3.39}$$

for all $x \in \mathbf{X}$, $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. By induction on n, we get

$$\left\|f\left(x\right) - \frac{f\left(k^nx\right)}{k^n}, z_1, z_2, ..., z_{l-1}\right\|_{\beta} \leq max\left\{\frac{\varphi\left(k^{h-1}x, k^{h-1}x, ..., k^{h-1}x, 0, 0, ..., 0\right)}{\left|k^{h\beta}\right|}, 1 \leq h \leq n\right\}\psi\left(z_1, z_2, ..., z_{l-1}\right) \tag{3.40}$$

replacing x by kx in (3.40)and dividing both sides by $\left|k^{\beta}\right|$, we get

$$\left\|\frac{f\left(kx\right)}{k} - \frac{f\left(k^{n+1}x\right)}{k^{n+1}}, z_1, z_2, ..., z_{l-1}\right\|_{\beta} \leq max\left\{\frac{\varphi\left(k^hx, k^hx, ..., k^hx, k^hx, k^hx, ..., k^hx\right)}{\left|k^{(h+1)\beta}\right|}, 1 \leq h \leq n\right\}\psi\left(z_1, z_2, ..., z_{l-1}\right) \tag{3.41}$$

for all $x \in \mathbf{X}$, $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$ and $n \in \mathbb{N}$, which together with (3.34) implies .

$$\left\|f\left(kx\right)-\frac{f\left(k^{n+1}x\right)}{k^{n+1}},z_1,z_2,...,z_{l-1}\right\|_{\beta}$$

$$\leq max\left\{\frac{\varphi\left(x,x,...,x,kx,kx,...,kx\right)}{\left|k^{\beta}\right|},\frac{\varphi\left(k^hx,k^hx,...,k^hx,k^hx,k^hx,...,k^hx\right)}{\left|k^{(h+1)\beta}\right|}\right.$$

$$\left.,1\leq h\leq n\right\}\psi\left(z_1,z_2,...,z_{l-1}\right)$$

$$=max\left\{\frac{\varphi\left(k^hx,k^hx,...,k^hx,k^hx,k^hx,...,k^hx\right)}{\left|k^{(h+1)\beta}\right|},1\leq h\leq n\right\}\psi\left(z_1,z_2,...,z_{l-1}\right)$$

$$=max\left\{\frac{\varphi\left(k^hx,k^hx,...,k^hx,k^hx,k^hx,...,k^hx\right)}{\left|k^{h\beta}\right|},1\leq h\leq n+1\right\}\psi\left(z_1,z_2,...,z_{l-1}\right)\quad(3.42)$$

for all $x\in\mathbf{X}$, $z_1,z_2,...,z_{2k-1}\in\mathbf{Y}$ and $n\in\mathbb{N}$. This is completes the proof of (3.57) Taking the limit as $n\to\infty$ in (3.57). Now we need to prove the uniqueness of H. Let H' be another additive mapping satisfying (3.48). Sence

$$\lim_{h\to\infty}\left|\frac{1}{k^{h\beta}}\right|\widetilde{\varphi}(k^hx)=\lim_{h\to\infty}\lim_{n\to\infty}max\left\{\left|\frac{1}{k^{i\beta}}\right|\varphi\left(k^{i+h-1}x_1,k^{i+h-1}x_2,...,k^{h+i-1}x_k,\right.\right.$$

$$\left.\left.k^{h+i-1}kx_{k+1},k^{h+i-1}x_{k+2},...,k^{h+i-1}x_{2k}\right),1\leq i\leq n\right\}$$

$$=\lim_{h\to\infty}\lim_{n\to\infty}max\left\{\left|\frac{1}{k^{i\beta}}\right|\varphi\left(k^{i-1}x_1,k^{i-1}x_2,...,k^{i-1}x_k,\right.\right.$$

$$\left.\left.k^{i-1}kx_{k+1},k^{i-1}x_{k+2},...,k^{i-1}x_{2k}\right),1+h\leq i\leq n+h\right\}\quad(3.43)$$

for all $x\in\mathbf{X}$, $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$, it follows from then H is a unique additive mapping satisfying (3.49) that.

$$\left\|H\left(x\right)-H'\left(x\right),z_1,z_2,...,z_{l-1}\right\|_\beta$$

$$=\lim_{h\to\infty}\left|\frac{1}{k^{h\beta}}\right|\left\|H\left(k^hx\right)-H'\left(k^hx\right),z_1,z_2,...,z_{l-1}\right\|_\beta$$

$$\leq\lim_{h\to\infty}\left|\frac{1}{k^{h\beta}}\right|max\left\{\left\|H\left(k^hx\right)-f\left(k^hx\right),z_1,z_2,...,z_{l-1}\right\|_\beta,\right.$$

$$\left.\left\|f\left(k^hx\right)-H'\left(k^hx\right),z_1,z_2,...,z_{l-1}\right\|_\beta\right.$$

$$\leq\lim_{h\to\infty}\left|\frac{1}{k^{h\beta}}\right|\widetilde{\varphi}(k^{h\beta}x)\psi\left(z_1,z_2,...,z_{l-1}\right)=0 \tag{3.44}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. □

. From lemma 6, we conclude that the quadratic function H is unique.

Theorem 3.4.

Suppose That $\mathbf{X}$ be a vector space and that $\mathbf{Y}$ is a complete non-Archimedean $(l,\beta)-$normed space, where $k\geq 1$, $0<\beta\leq 1$. Let

$$\varphi:\mathbf{X^{2k}}\to[0,\infty)$$

be a function such that

$$\lim_{n\to\infty}\left|k^{n\beta}\right|\varphi\left(\frac{x_1}{k^n},\frac{x_2}{k^n},...,\frac{x_k}{k^n},\frac{x_{k+1}}{k^n},\frac{x_{k+2}}{k^n},...,\frac{x_{2k}}{k^n}\right)=0 \tag{3.45}$$

for all $x_j,x_{k+j}\in\mathbf{X}$ for all $j=1\to k$, and

suppose that a mapping

$$\psi\colon\mathbf{Y^{l-1}}\to[0,\infty)$$

be a function. The limit

$$\widetilde{\varphi}(x)=\lim_{n\to\infty}max\left\{\left|k^{(i-1)\beta}\right|\varphi\left(k^{-i}x,k^{-i}x,...,k^{-i}x,0,0,...,0\right),1\leq i\leq n\right\} \tag{3.46}$$

exists for $x\in\mathbf{X}$, and it is denoted by $\widetilde{\varphi}(x)$. Suppose that a mapping

$$f:\mathbf{X}\to\mathbf{Y}$$

satisfying $f(0)=0$ the inequality

$$\Bigg\|f\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}\frac{x_{k+j}}{k}\right)+f\left(\sum_{j=1}^{k}x_j-\sum_{j=1}^{k}\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\Big(x_j\Big)$$
$$-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right),z_1,z_2,...,z_{l-1}\Bigg\|_\beta$$
$$\leq\varphi\Big(x_1,x_2,...,x_k,x_{k+1},x_{k+2},...,x_{2k}\Big)\cdot\psi\Big(z_1,z_2,...,z_{l-1}\Big)\quad(3.47)$$

for all $x_j,x_{k+j}\in\mathbf{X}$ for all $j=1\to k$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Then there exists a unique quadratic mapping

$$H:\mathbf{X}\to\mathbf{Y}$$

satisfying

$$\Big\|f\big(x\big)-H\big(x\big),z_1,z_2,...,z_{l-1}\Big\|_\beta\leq\widetilde{\varphi}(x)\varphi\big(z_1,z_2,...,z_{l-1}\big)\qquad(3.48)$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Moreover, if

$$\lim_{h\to\infty}\lim_{n\to\infty}max\left\{\left|k^{(i-1)\beta}\right|\varphi\big(k^{-i}x,k^{-i}x,...,k^{-i}x,0,0,...,0\big),1+h\leq i\leq n+h\right\}=0\quad(3.49)$$

for all $x\in\mathbf{X}$, then H is a unique additive mapping satisfying (3.48).

Proof. Put $x_j=x,x_{k+j}=0$ for all $j=1\to k$ in (3.47) we get

$$\Big\|2f\big(kx\big)-2kf\big(x\big),z_1,z_2,...,z_{l-1}\Big\|_\beta\leq\varphi\big(x,x,...,x,0,0,...,0\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\qquad(3.50)$$

Dividing both sides by $\left|2^{-\beta}\right|$ and replace x by $\frac{x}{k}$ in (3.50) we get

$$\Big\|f\big(x\big)-kf\left(\frac{x}{k}\right),z_1,z_2,...,z_{l-1}\Big\|_\beta$$
$$\leq\left|2^{-\beta}\right|\varphi\big(\frac{x}{k},\frac{x}{k},...,\frac{x}{k},0,0,...,0\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\qquad(3.51)$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Replacing x by $\frac{x}{k^{i-1}}$ in (3.51)

and multipling both sides by $\left|k^{(i-1)\beta}\right|$, we get

$$\Big\|k^if\big(\frac{x}{k^i}\big)-k^{(i-1)}f\big(\frac{x}{k^{i-1}}\big),z_1,z_2,...,z_{l-1}\Big\|_\beta$$
$$\leq\left|2^{-\beta}\right|\left|k^{(i-1)\beta}\right|\varphi\big(\frac{x}{k^i},\frac{x}{k^i},...,\frac{x}{k^i},0,0,...,0\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\qquad(3.52)$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$, $i \in \mathbb{N}$. Taking the limit as $i \to \infty$ and considering (3.45)

$$\lim_{i\to\infty} \left\| k^i f\big(\frac{x}{k^i}\big) - k^{(i-1)} f\big(\frac{x}{k^{i-1}}\big), z_1, z_2, ..., z_{l-1} \right\|_\beta = 0 \tag{3.53}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$

. It follows from (3.53) that the sequence $\left\{k^i f\big(\frac{x}{k^i}\big)\right\}$ is Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is completes space, the sequence $\left\{k^i f\big(\frac{x}{k^i}\big)\right\}$. coverges. So one can define the mapping H: $\mathbf{X} \to \mathbf{Y}$ by

$$H\big(x\big) := \lim_{i\to\infty} k^i f\big(\frac{x}{k^i}\big) \tag{3.54}$$

for all $x \in \mathbf{X}$. It follows from (3.47), (3.53) and lemma 1.4 that

$$\begin{aligned}
&\Bigg\| H\Bigg(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\Bigg) + H\Bigg(\sum_{j=1}^{k} x_j - \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\Bigg) - 2\sum_{j=1}^{k} H\Big(x_j\Big) - 2\sum_{j=1}^{k} H\Bigg(\frac{x_{k+j}}{k}\Bigg) \\
&, z_1, z_2, ..., z_{l-1}\Bigg\|_\beta \\
&= \lim_{n\to\infty} \left|k^{n\beta}\right| \Bigg\| f\Bigg[k^{-n}\Bigg(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\Bigg)\Bigg] + f\Bigg[k^{-n}\Bigg(\sum_{j=1}^{k} x_j - \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\Bigg)\Bigg] \\
&- 2\sum_{j=1}^{k} f\Bigg(\frac{1}{k^n} x_j\Bigg) - 2\sum_{j=1}^{k} f\Bigg(\frac{x_{k+j}}{k^{n+1}}\Bigg), z_1, z_2, ..., z_{l-1}\Bigg\|_\beta \\
&\le \lim_{n\to\infty} \left|k^{n\beta}\right| \varphi\big(k^{-n}x, k^{-n}x, ..., k^{-n}x, 0, 0, ..., 0\big)\psi\big(z_1, z_2, ..., z_{l-1}\big)
\end{aligned}$$

and so for all x_j, $x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$.Taking the limit as $n \to \infty$ and considering (3.45)we get

$$\begin{aligned}
&\Bigg\| H\Bigg(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\Bigg) + H\Bigg(\sum_{j=1}^{k} x_j - \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\Bigg) - 2\sum_{j=1}^{k} H\Big(x_j\Big) - 2\sum_{j=1}^{k} H\Bigg(\frac{x_{k+j}}{k}\Bigg) \\
&\qquad\qquad , z_1, z_2, ..., z_{l-1}\Bigg\| = 0
\end{aligned}$$

for all x_j, $x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. and $z_1, z_2, ..., z_{2k-1} \in \mathbf{Y}$. By lemma 1.4, we get

$$H\Bigg(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\Bigg) + H\Bigg(\sum_{j=1}^{k} x_j - \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\Bigg) - 2\sum_{j=1}^{k} H\Big(x_j\Big) - 2\sum_{j=1}^{k} H\Bigg(\frac{x_{k+j}}{k}\Bigg) = 0$$

for all x_j, $x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. So mapping H is quadratic.

Replace x by $\frac{x}{k}$ in (3.51) and multiplying both sides by $\left|k^\beta\right|$, we get

$$\left\|k^2 f\big(\frac{x}{k^2}\big) - k f\big(\frac{x}{k}\big), z_1, z_2, ..., z_{l-1}\right\|_\beta$$
$$\leq \left|2^{-\beta}\right|\left|k^{2\beta}\right|\varphi\big(\frac{x}{k^2}, \frac{x}{k^2}, ..., \frac{x}{k^2}, 0, 0, ..., 0\big)\psi\big(z_1, z_2, ..., z_{l-1}\big) \qquad (3.55)$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Considering (3.51), we receive

$$\left\|f\big(x\big) - k^2 f\big(\frac{x}{k^2}\big), z_1, z_2, ..., z_{l-1}\right\|_\beta$$
$$\leq max\Big\{\left|2^{-\beta}\right|\left|k^{\beta}\right|\varphi\big(\frac{x}{k}, \frac{x}{k}, ..., \frac{x}{k}, 0, 0, ..., 0\big), \left|2^{-\beta}\right|\left|k^{2\beta}\right|\varphi\big(\frac{x}{k^2}, \frac{x}{k^2}$$
$$, ..., \frac{x}{k^2}, 0, 0, ..., 0\big)\Big\}\psi\big(z_1, z_2, ..., z_{l-1}\big) \qquad (3.56)$$

for all $x \in \mathbf{X}$, $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. By induction on n, we get

$$\left\|f\big(x\big) - k^n f\big(\frac{x}{k^n}\big), z_1, z_2, ..., z_{l-1}\right\|_\beta$$
$$\leq max\Big\{\left|k^{(i-1)\beta}\right|\varphi\big(k^{-i}x, k^{-i}x, ..., k^{-i}x, 0, 0, ..., 0\big)\left|2^{-\beta}\right|, 1 \leq i \leq n\Big\}\psi\big(z_1, z_2, ..., z_{l-1}\big) \qquad (3.57)$$

replacing x by $\frac{x}{k}$ in (3.57)and multiping both sides by $\left|k^\beta\right|$, we get

$$\left\|kf\left(\frac{x}{k}\right) - k^{n+1} f\big(\frac{x}{k^{n+1}}x\big), z_1, z_2, ..., z_{l-1}\right\|_\beta$$
$$\leq max\Big\{\left|k^{(h+1)\beta}\right|\varphi\big(\frac{x}{k^h}, \frac{x}{k^h}, ..., \frac{x}{k^h}, \frac{x}{k^h}, \frac{x}{k^h}, ..., \frac{x}{k^h}\big),$$
$$1 \leq h \leq n\Big\}\psi\big(z_1, z_2, ..., z_{l-1}\big) \qquad (3.58)$$

for all $x \in \mathbf{X}$, $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$ and $n \in \mathbb{N}$, which together with (3.50) implies .

$$\left\|f(x)-f\left(\frac{x}{k^{n+1}}\right),z_1,z_2,...,z_{l-1}\right\|_\beta$$
$$\leq max\left\{\varphi(\frac{x}{k},\frac{x}{k},...,\frac{x}{k},0,0,...,0),\left|k^{(h-1)\beta}\right|\varphi(\frac{x}{k^h},\frac{x}{k^h},...,\frac{x}{k^h},0,0,...,0),1\leq h\leq n\right\}$$
$$\psi(z_1,z_2,...,z_{2k-1})$$
$$\leq max\left\{\left|k^{(h-1)\beta}\right|\varphi(\frac{x}{k^h},\frac{x}{k^h},...,\frac{x}{k^h},0,0,...,0),1\leq h\leq n\right\}\psi(z_1,z_2,...,z_{2k-1}) \tag{3.59}$$

for all $x\in\mathbf{X}$, $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$ and $n\in\mathbb{N}$. passing the limit as $n\to\infty$ in (3.57) , we get (3.48). Now we need to prove the quadratic function uniqueness of H. Let H' be another quadratic mapping satisfying (3.48). Sence

$$\lim_{h\to\infty}\left|k^{h\beta}\right|\widetilde{\varphi}\left(\frac{x}{k^h}\right)=\lim_{h\to\infty}\left|k^{h\beta}\right|\lim_{n\to\infty}max\left\{\left|k^{(h-1)\beta}\right|\varphi(\frac{x}{k^h},\frac{x}{k^h},...,\frac{x}{k^h},0,0,...,0),1\leq h\leq n\right\}$$
$$=\lim_{h\to\infty}\lim_{n\to\infty}max\left\{\left|k^{(h-1)\beta}\right|\varphi(\frac{x}{k^h},\frac{x}{k^h},...,\frac{x}{k^h},0,0,...,0),1+h\leq i\leq n+h\right\} \tag{3.60}$$

for all $x\in\mathbf{X}$. It follows from (3.49) that.

$$\left\|H(x)-H'(x),z_1,z_2,...,z_{2k-1}\right\|_\beta$$
$$=\lim_{h\to\infty}\left|k^{h\beta}\right|\left\|H\left(\frac{x}{k^h}\right)-H'\left(\frac{x}{k^h}\right),z_1,z_2,...,z_{l-1}\right\|_\beta$$
$$\leq\lim_{h\to\infty}\left|k^{h\beta}\right|max\left\{\left\|H\left(\frac{x}{k^h}\right)-f\left(\frac{x}{k^h}\right),z_1,z_2,...,z_{l-1}\right\|_\beta,\right.$$
$$\left.\left\|f\left(\frac{x}{k^h}\right)-H'\left(\frac{x}{k^h}\right),z_1,z_2,...,z_{l-1}\right\|_\beta\right.$$
$$\leq\lim_{h\to\infty}\left|k^{h\beta}\right|\widetilde{\varphi}\left(\frac{x}{k^h}\right)\psi(z_1,z_2,...,z_{l-1})=0 \tag{3.61}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. □

. From lemma 6, we conclude that the quadratic function H is unique.

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