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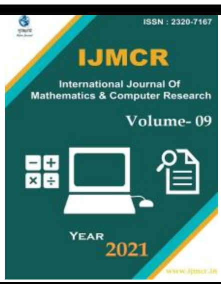

Unique Common Fixed Point Result for Rational Contraction in Complex valued Metric Space

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ABSTRACT

In this paper, we prove some unique common fixed point theorem for two pairs of weakly compatible mappings, satisfying the rational contraction conditions in complex valued metric space. The proved result, generalize and extend some known results in the literature. Finally, The main result is the application of the Urysohn integral equations to derive the existence theorem for a general solution.
AMS(MOS) Subject Classification Codes: 47H10, 54H25.

KEYWORDS:
Fixed point, common fixed point, complex valued metric space, weakly compatible metric, Urysohn integral equation.

I. INTRODUCTION

The theory of complex valued metric space (CV MS) is developed by A. Azam et al.[1] and F.Rouzkard and M. Imdad [3], which is a generalization of metric space. He created unique common fixed point results for pair of compatible mappings satisfying a rational inequality. Very much inventor have established several results of fixed points for various mappings satisfying a rational contraction in the context of the complex valued metric spaces, see for [2-18]. The purpose of this paper is to build up the common fixed point result for two sets of compatible mappings satisfying rational contraction in a complex valued metric space and we acquire the presence and uniqueness of a common solution for the arrangement of Urysohn integral equation.

II. PRELIMINARY

The following definitions and results will be needed in the sequel

Definition 2.1 *[1] Presume $\Delta_1, \Delta_2 \in \mathbb{C}$, where $\mathbb{C}$ be the set of complex number ,define a partial order $\precsim$ on $\mathbb{C}$ such that:*
$\Delta_1 \precsim \Delta_2$ if and only if $Re(\Delta_1) \leq Re(\Delta_2)$ and $Im(\Delta_1) \leq Im(\Delta_2)$ that is $\Delta_1 \precsim \Delta_2$.
Conesequently, if one of the following hold;.

(C_1) $Re(\Delta_1) = Re(\Delta_2)$ and $Im(\Delta_1) = Im(\Delta_2)$,
(C_2) $Re(\Delta_1) < Re(\Delta_2)$ and $Im(\Delta_1) = Im(\Delta_2)$,
(C_3) $Re(\Delta_1) = Re(\Delta_2)$ and $Im(\Delta_1) < Im(\Delta_2)$,
(C_4) $Re(\Delta_1) < Re(\Delta_2)$ and $Im(\Delta_1) < Im(\Delta_2)$.

Definition 2.2 [1] Let $d : X \times X \to \mathbb{C}$ be a self mapping on the set X, satisfies the following conditions:

(d) $0 - d(\alpha,\beta)$ *for all* $\alpha,\beta \in X$;

(d) $d(\alpha,\beta) = 0$ *iff* $\alpha = \beta$, $\forall\ \alpha,\beta \in X$;

(d) $d(\alpha,\beta) = d(\beta,\alpha)$ *for all* $\alpha,\beta \in X$;

(d) $d(\alpha,\beta) - d(\alpha,\gamma) + d(\gamma,\beta)$ $\forall\ \alpha,\beta,\gamma \in X$.

Then d is called a complex valued metric on X *and the pair* (X, d) *is called a complex valued metric space.*

Example 2.3 *Suppose* $\Delta_1 = \alpha_1 + i\beta_1$ *and* $\Delta_2 = \alpha_2 + i\beta_2$ *and let* $X = \mathbb{C}$. *Define the mapping* $d : X \times X \to \mathbb{C}$ *such that*

$$d(\Delta_1, \Delta_2) = |\alpha_1 - \alpha_2| + i|\beta_1 - \beta_2|,$$

Then (X, d) *is a complex valued metric space.*

Example 2.4 *Let* $X = \mathbb{C}$ *and define the mapping* $d : X \times X \to \mathbb{C}$ *such that*

$$d(\Delta_1, \Delta_2) = |\alpha_1 - \alpha_2|e^{i\theta},$$

where $\theta \in]0, \frac{\pi}{2}[$.

Definition 2.5 *[1]*

(1) Let a point $x \in X$ *is called an interior point of a set* $A \subseteq X$ *on complex valued metric space*(X, d), *whenever there exist* $0 \prec r \in \mathbb{C}$ *such that*

$$B(\alpha, r) = \{\beta \in X : d(\alpha, \beta) < r\} \subseteq A.$$

(2) A point $\alpha \in X$ *is called a limit point of* A *in complex valued metric space*(X, d), *whenever for all* $0 \prec r \in \mathbb{C}$,

$$B(\alpha, r) \cap (A - \{\alpha\}) \neq \emptyset.$$

(3) A set $A \subseteq X$ *is called an open set in complex valued metric space*(X, d), *whenever each element of* A *is an interior point of* A.

(4) A set $A \subseteq X$ *is called closed set whenever each limit point of* A *belongs to* A.

(5) A subbasis for a Hausdorff topology τ *on* X *is the family*

$$F\{B(\alpha, r) : \alpha \in X \ \ and \ \ 0 \prec r\}.$$

Definition 2.6 *[1] Presume that,* $\{x_n\}$ *be a sequence on complex valued metric space*(X, d) *and let* $x \in X$.

(1) Then $\{x_n\}$ *is said to be convergent to a point* $x \in X$ *and* x *is the limit point of* $\{x_n\}$.*If for any* $c \in \mathbb{C}$ *with* $0 \prec c$, *there exists* $N \in \mathbb{N}$ *such that for all* $n > N$, $d(x_n, x) \prec c$. *We denote this by*

$$\lim_{n \to \infty} x_n = x \ \ or \ \ x_n \to x \ \ as \ \ n \to \infty.$$

(2) If for any $c \in \mathbb{C}$ *with* $0 \prec c$, *there exists* $N \in \mathbb{N}$ *such that for all* $n > N$, $d(x_n, x_{n+m}) \prec c$, *where* $m \in \mathbb{N}$, *then* $\{x_n\}$ *is called Cauchy sequence in* X.

(3) If for every Cauchy sequence in X is convergent, then (X, d) is said to be complete complex valued metric space.

Lemma 2.7 *[1] Let $\{x_n\}$ be a sequence on complex valued metric space(X, d) and. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.8 *[1] Suppose that, $\{x_n\}$ be a sequence on complex valued metric space$X, d)$. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$ where $m \in \mathbb{N}$.*

Definition 2.9 *[17] let S and I be self -mappings of a set X. If$w = Sx = Ix$ for some $x \in X$, then x is called a point of coincidence of $S$and I, and w is called a point of coincidenceod S and I.*

Definition 2.10 *[18] let S and T be two self -mappings defined on a set X.Then S and T are called a weakly compatible if they commute at their coincidence points.*

3 MAIN RESULTS

Theorem 3.1 *: Let $R, S, T, U : X \to X$ be self mappings on complete complex valued metric space(X, d), satisfying the conditions*

(i) $d(Ra, Sb) \precsim \alpha Q(a, b), where, \alpha \in [0, 1)$
and $Q(a, b) \precsim max\{d(T_a, U_b), d(T_a, R_a), d(U_b, R_a), d(T_a, S_b), d(U_b, R_a), \frac{1}{2}[d(U_b, R_a) + d(T_a, S_b)] \frac{d(T_a, R_a)d(U_b, S_b)}{1+d(T_a, U_b)}\}.$

(ii) $R(X) \subset U(X) and S(X) \subset T(X)$.

(iii) Then R, S, T and U have a unique common fixed point in X and The pair (R, T) and (S, U) are weakly compatible.

(iv) either T and R is continuous.

Proof. Let $a_0 \in X$.From the condition (ii), there exists a_1, a_2 such that

$$b_0 = Ua_1 = Ra_0$$
$$and$$
$$b_1 = Ta_2 = Sa_1.$$

We cdefine the successive sequence $\{a_{n_i}\}$ and $\{b_{n_i}\}$ in X as follows:

$$b_{2n_i} = Ua_{2n_i+1} = Ra_{2n_i},$$
$$and$$
$$b_{2n_i+1} = Ta_{2n_i+2} = Sa_{2n_i+2}, ...(3.1)$$

Using (3.1) in (ii), we get

$$
\begin{aligned}
d(b_{2n_i}, b_{2n_i+1}) &= d(Ra_{2n_i}, Sa_{2n_i+1}) \\
&\precsim max\{d(Ta_{2n_i}, Ua_{2n_i+1}), d(Ta_{2n_i}, Ra_{2n_i}), d(Ua_{2n_i+1}, Sa_{2n_i+1}), d(Ua_{2n_i+1}, Ra_{2n_i}), \\
&d(Ta_{2n_i}, Sa_{2n_i+1}), \frac{1}{2}[d(Ua_{2n_i+1}, Ra_{2n_i}) + d(Ta_{2n_i}, Sa_{2n_i+1})], \\
&\frac{d(Ta_{2n_i}, Ra_{2n_i}), (Uan_{2n_i+1}, Sa_{2n_i+1})}{1 + d(Ta_{2n_i}, Ua_{2n_i+1})}\} \\
&= max\{d(b_{2n_i-1}, b_{2n_i}), d(b_{2n_i-1}, b_{2n_i}), d(b_{2n_i}, b_{2n_i+1}), d(b_{2n_i}, b_{2n_i}), d(b_{2n_i-1}, b_{2n_i+1}), \\
&\frac{1}{2}[d(b_{2n_i}, b_{2n_i}) + d(b_{2n_i-1}, b_{2n_i+1})], \frac{d(b_{2n_i-1}, b_{2n_i}), d(b_{2n_i}, b_{2n_i+1})}{1 + d(b_{2n_i-1}, b_{2n_i+1})}\} \\
&\precsim max\{d(b_{2n_i-1}, b_{2n_i}), d(b_{2n_i-1}, b_{2n_i}), d(b_{2n_i}, b_{2n_i+1}), d(b_{2n_i-1}, b_{2n_i+1}), \\
&\frac{1}{2}[d(b_{2n_i-1}, b_{2n_i+1})], \frac{d(b_{2n_i-1}, b_{2n_i}), d(b_{2n_i}, b_{2n_i+1})}{1 + d(b_{2n_i-1}, b_{2n_i+1})}\},
\end{aligned}
$$

we have

$$
\frac{1}{2}d(b_{2n_i-1}, b_{2n_i+1}) \precsim \frac{1}{2}[d(b_{2n_i-1}, b_2n_i) + d(b_2n_i, b_{2n_i+1})] ....(3.2.)
$$

and also we have,

$$
d(b_{2n_i-1}, b_2n_i) \precsim 1 + d(b_{2n_i-1}, b_2n_i)
$$

which implies that

$$
\frac{d(b_{2n_i-1}, b_{2n_i}), d(b_{2n_i}, b_{2n_i+1})}{1 + d(b_{2n_i-1}, b_{2n_i+1})} \precsim d(b_{2n_i}, b_{2n_i-1}) ....(3.3)
$$

from (3.2) and (3.3), we get

$$
Q(a_{2n_i}, a_{2n_i+1}) = max\{d(b_{2n_i-1}, b_{2n_i}), d(b_{2n_i}, b_{2n_i+1})\}
$$

with

$$
\begin{aligned}
d(b_{2n_i}, b_{2n_i+1}) &= d(Ra_{2n_i}, Sa_{2n_i+1}) \\
&\precsim \lambda Q(a_{2n_i}, a_{2n_i+1}).
\end{aligned}
$$

If

$$Q(a_{2n_i}, a_{2n_i+1}) = d(b_{2n_i}, b_{2n_i+1})$$
$$Then$$
$$d(b_{2n_i}, b_{2n_i+1}) \precsim \alpha d(b_{2n_i}, b_{2n_i+1})$$

therefore,

$$1 - \alpha d(b_2 n_i, b_{2n_i+1}) \precsim 0$$

which is contrdictiin, since $\alpha \in (0,1)$
we conclude that

$$d(b_2 n_i, b_{2n_i+1}) \precsim \alpha d(b_{2n_i-1}, b_2 n_i)$$

similarly,

$$d(b_{2n_i+1}, b_{2n_i+2}) \precsim \alpha d(b_2 n_i, b_{2n_i+1}).$$

It follow that,

$$\begin{aligned} d(b_{n_i}, b_{n_i+1}) &\precsim \alpha d(b_{n_i-1}, b_{n_i}) \\ &\precsim ... \precsim \\ &\precsim \alpha^{n_i} d(b_0, b_1) \end{aligned}$$

which implies taht,

$$\begin{aligned} |d(b_{n_i}, b_{n_i+1})| &\leq \alpha |d(b_{n_i-1}, b_n)| \\ &\leq ... \leq \\ &\leq \alpha^{n_i} |d(b_0, b_1)|. \end{aligned}$$

Now, for $m_i < n_i$, we have

$$\begin{aligned} d(b_{n_i}, b_{m_i}) &= d(bn_i, b_{n_i+1} + d(b_{n_i+1}, b_{n_i+2}) + ... + d(b_{m_i-1}, bm_i) \\ &\precsim \alpha^{n_i} d(b_0, b_1) + \alpha^{n_i+1} d(b_0, b_1) + ... + \alpha^{m_i-1} d(b_0, b_1) \\ &\precsim (\alpha^{n_i} + \alpha^{n_i+1} + \alpha^{n_i+2} + ... + \alpha^{m_i-1}) d(b_0, b_1) \end{aligned}$$

Therefore,

$$d(b_{n_i}, b_{m_i}) \precsim \frac{\alpha^n}{1-\alpha} d(b_0, b_1)$$

Hence

$$|d(b_{n_i}, b_{m_i})| \leq \frac{\alpha^n}{1-\alpha} |d(b_0, b_1)| \rightarrow 0, as n_i \rightarrow \infty$$

.

Thus,$\{b_{n_i}\}$ be a cauchy sequence in X. Since X is complete. So, $\exists\, x_0 \in X$ such that $b_{n_i} \rightarrow x_0$ as $n_i \rightarrow \infty$. and sub sequence, we know that
$Ua_{2n_i+1} \rightarrow x_0$, $Ra_{2n_i} \rightarrow x_0$,$Ta_{2n_i+1} \rightarrow x_0$ and $Sa_{2n_i} \rightarrow x_0$.
Now from (iv), T is continuous,then
$TTa_{2n_i} \rightarrow x_0$ and $TRa_{2n_i} \rightarrow Tx_0$, as $n \rightarrow \infty$. Because (R, T) is compatible. this implies that $RTa_{2n} \rightarrow Tx_0$. Indeed,

$$d(RTa_{2n_i}, Tx_0) \precsim d(RTa_{2n_i}, TRa_{2n_i}) + d(TRa_{2n_i}, Tx_0)$$

So,

$$|d(RTa_{2n_i}, Tx_0)| \precsim |d(RTa_{2n_i}, TRa_{2n_i})| + |d(TRa_{2n_i}, Tx_0)| \rightarrow 0 as n \infty.$$

we prove that,$Tx_0 = x_0$. On the contran, we suppose that, $Tx_0 \neq x_0$.

$$d(Tx_0, x_0) \precsim d(Tx_0, RTa_{2n_i}) + d(RTa_{2n_i}, Sa_{2n_i+1}) + d(Sa_{2n_i+1}, x_0)$$

Using (ii) with $a = Ta_{2n_i}$, $b = a_{2n_i+1}$, we get

$$d(RTa_{2n_i}, Sa_{2n_i+1}) \precsim \alpha Q d(Ta_{2n_i}, a_{2n_i+1})$$

where

$$\begin{aligned} Q(Ta_{2n_i}, a_{2n_i+1}) = max\{&d(TTa_{2n_i}, Ua_{2n_i+1}), d(TTa_{2n_i}, RTa_{2n_i}), d(Ua_{2n_i+1}, Sa_{2n_i+1}), d(Ua_{2n_i+1}, RTa_{2n_i}), \\ &d(TTa_{2n}, Sa_{2n+1}), \frac{1}{2}[d(Ta_{2n_i+1}, RTa_{2n_i}) + d(UTa_{2n_i}, Sa_{2n_i+1})], \\ &\frac{d(TTa_{2n_i}, RTa_{2n_i}), (Ua_{2n_i+1}, Sa_{2n_i+1})}{1 + d(TTa_{2n_i}, Ua_{2n_i+1})}\} \end{aligned}$$

Let $n \rightarrow \infty$, we get

$$\begin{aligned} Q(Tx_0, x_0) = max\{&d(Tx_0, x_0), d(Tx_0, x_0), d(x_0, Tx_0), d(x_0, Tx_0), d(Tx_0, x_0), \\ &\frac{1}{2}[d(Tx_0, Tx_0) + d(Tx_0, x_0)], \frac{d(Tx_0, Tx_0), (x_0, x_0)}{1 + d(Tx_0, x_0)}\} \\ &= d(Tx_0, x_0) \end{aligned}$$

Further

$$|d(Tx_0, x_0)| \leq \alpha |d(Tx_0, x_0)|.$$

So, $(1-\alpha)|d(Tx_0, x_0)| \leq 0$,which is a contradiction , that is, $|d(Tx_0, x_0) = 0$. Then $Tx_0 = x_0$. Next ,we prove that $Rx_0 = x_0$. On the contrary, we suppose that $Rx_0 \neq x_0$ such that

$$d(Rx_0, x_0)| \precsim d(Rx_0, Sa_{2n+1}) + d(Sa_{2n+1}, x_0).$$

Now using (ii) with $a = x_0$ and $b = a_{2n+1}$, we get

$$d(Rx_0, Sa_{2n+1}) \precsim \alpha Q(x_0, a_{2n+1})$$

where

$$Q(x_0, a_{2n_i+1} = max\{d(Tx_0, Ua_{2n_i+1}), d(Tx_0, Rx_0), d(Ua_{2n_i+1}, Sa_{2n_i+1}), d(Ua_{2n_i+1}, Rx_0), d(Tx_0, Sa_{2n_i+1}),$$

$$\frac{1}{2}[d(Ua_{2n_i+1}, Rx_0) + d(Tx_0, Sa_{2n_i+1})], \frac{d(Tx_0, Rx_0), d(Ua_{2n_i+1}, Sa_{2n_i+1})}{1 + d(Tx_0, Ua_{2n_i+1})}\}$$

Let $n \to \infty$

$$\begin{aligned} Q(x_0, x_0) &= max\{d(x_0, x_0), d(x_0, Rx_0), d(x_0, x_0), d(x_0, Rx_0), d(x_0, x_0), \\ &\frac{1}{2}[d(x_0, Rx_0) + d(x_0, x_0)], \frac{d(x_0, Rx_0), (x_0, x_0)}{1 + d(x_0, x_0)}\} \\ &= d(Rx_0, x_0) \end{aligned}$$

Then, $(Rx_0, x_0) \precsim \alpha d(Rx_0, x_0)$, further, $|(Rx_0, x_0) \precsim \alpha d(Rx_0, x_0)|$, which is contradiction $i.e$ $|(Rx_0, x_0)| = 0 \Rightarrow Rx_0 = x_0$. Now , we prove that, $Ux_0 = Sx_0$, as $R(x) \subset U(x)$.So, there exist $y_0 \in X$ such that $x_0 = Rx_0 = Uy_0$.

First, we shall show that $Uy_0 = SY_0$ for this we get:

$$\begin{aligned} d(Uy_0, Sy) &= d(Rx_0, Sy_0) \\ &\precsim \alpha Q(x_0, y_0) \end{aligned}$$

where

$$\begin{aligned} Q(x_0, y_0) = max\{&d(Tx_0, Uy_0), d(Tx_0, Rx_0), d(Uy_0, Sy_0), d(Uy_0, Rx_0), d(Tx_0, Sy_0), \\ &\frac{1}{2}[d(Uy_0, Rx_0) + d(Tx_0, Sy_0)], \frac{d(Tx_0, Rx_0), (Uy_0, Sy_0)}{1 + d(Tx_0, Uy_0)}\} \end{aligned}$$

Then,

$$Q(x_0, y_0) = max\{d(Uy_0, Uy_0), d(x_0, x_0), d(Uy_0, Sy_0), d(Uy_0, Uy_0), d(Uy_0, Sy_0), \frac{1}{2}[d(Uy_0, Uy_0) + d(Uy_0, Sy_0)], \frac{d(x_0, x_0), (Uy_0, Sy_0)}{1 + d(Uy_0, Uy_0)}\}$$

Then, $d(Uy_0, Sy_0) \precsim \alpha d(Uy_0, Sy_0)$, further $|d(Uy_0, Sy_0) \leq \alpha d(Uy_0, Sy_0)|$,which is contradiction,*i.e.* $|d(Uy_0, Sy_0)| = 0 \Rightarrow Uy_0 = Sy_0 = x_0$. As the pair (S, U) is weakly compatible ,so, we have $SUy_0 = USy_0$, therefore, $Ux_0 = Sx_0$.
Now, we prove $x_0 = Sx_0$, on the contrarywe suppose that $Sx_0 \neq x_0$,

$$\begin{aligned} d(x_0, Sx_0) &= d(Rx_0, Sx_0) \\ &\precsim \alpha Q(x_0, x_0) \end{aligned}$$

where

$$Q(x_0, x_0) = max\{d(Tx_0, Ux_0), d(Tx_0, Rx_0), d(Ux_0, Sx_0), d(Ux_0, Rx_0), d(Tx_0, Ux_0), \frac{1}{2}[d(Ux_0, Rx_0) + d(Tx_0, Sx_0)], \frac{d(Tx_0, Rx_0), (Ux_0, Sx_0)}{1 + d(Tx_0, Ux_0)}\}.$$

Then,

$$Q(x_0, y_0) = max\{d(x_0, Sx_0), d(x_0, x_0), d(Sx_0, Sx_0), d(Sx_0, x_0), d(Sx_0, Sx_0), \frac{1}{2}[d(Sx_0, Ux_0) + d(Sx_0, Sx_0)], \frac{d(x_0, x_0), (Sx_0, Sx_0)}{1 + d(x_0, Sx_0)}\}.$$

Then, $(x_0, x_0) \precsim d(x_0, x_0)$,further, $|d(x_0, Sx_0)| \leq \alpha |d(x_0, Sx_0)|$, which is contradiction. *i.e.* $|d(x_0, Sx_0)| = 0 \Rightarrow x_0 = Sx_0$.
Now ,we prove that $Ux_0 = x_0$. On the contrary, we suppose that $Qx_0 \neq x_0$, we have

$$\begin{aligned} d(x_0, Ux_0) &= d(Rx_0, USx_0) \\ &= d(Rx_0, SUx_0) \end{aligned}$$

from (ii) we get,

$$\begin{aligned} d(x_0, Ux_0) &= d(Rx_0, USx_0) \\ &\precsim d(Rx_0, SUx_0) \end{aligned}$$

where

$$Q(x_0, Ux_0) = max\{d(Tx_0, UUx_0), d(Tx_0, Rx_0), d(UUx_0, SUx_0), d(UUx_0, Rx_0), d(Tx_0, SUx_0),$$
$$\frac{1}{2}[d(UUx_0, Rx_0) + d(Tx_0, SUx_0)], \frac{d(Tx_0, Rx_0), (UUx_0, SUx_0)}{1 + d(Tx_0, UUx_0)}\}.$$

$$= max\{d(x_0, Ux_0), d(x_0, x_0), d(Ux_0, Ux_0), d(Ux_0, x_0), d(x_0, Ux_0),$$
$$\frac{1}{2}[d(Ux_0, x_0) + d(x_0, Ux_0)], \frac{d(x_0, x_0), (Ux_0, Ux_0)}{1 + d(x_0, Ux_0)}\}.$$
$$= d(x_0, Ux_0).$$

Further, $|d(x_0, Ux_0)| \leq \alpha|d(x_0, Ux_0)|$, which is contradiction, that is $|d(x_0, Ux_0)| = 0$.Then, $x_0 = Ux_0$. on conclude $Rx_0 = Sx_0 = Tx_0 = x_0$, when T is continuous, we get the same results , R is continuous. Now we prove the uniqueness, letx^* be another common fixed point of R, S, T and U, then

$$Rx^* = Sx^* = Tx^* = Ux^* = x^*.$$

Putting $a = x_0$ and $b = x^*$ in (ii),we get

$$d(x_0, x^*) = d(Rx_0, Sx_0)$$
$$\precsim \alpha Q(x_0, x^*),$$

where

$$Q(x_0, x^*) = max\{d(Tx_0, Ux^*), d(Tx_0, Rx_0), d(Ux^*, Sx^*), d(Ux^*, Rx_0), d(Tx_0, Sx^*),$$
$$\frac{1}{2}[d(Ux^*, Rx_0) + d(Tx_0, Sx^*], \frac{d(Tx_0, Rx_0), (Ux^*, Sx^*)}{1 + d(Tx_0, Ux^*)}\}.$$
$$= max\{d(x_0, x^*), d(x_0, x_0), d(x^*, x^*), d(x^*, x_0), d(x_0, x^*),$$
$$\frac{1}{2}[d(x^*, x_0) + d(x_0, x^*)], \frac{d(x_0, x_0), (x^*, x^*)}{1 + d(x_0, x^*)}\}.$$

Hence, $|d(x_0, x^*)| \leq \alpha d(x_0, x^*)$,which is contradiction, $i.e.$ $|d(x_0, x^*)| = 0 \Rightarrow x_0 = x^*$.
Thus ,x_0 is the unique common fixed point of $R, S, T$and U in X. This completes the proof of the theorem.

■

Corollary 3.2

Let (X, d) be a complete valued metric space, if we put$R = SandT = U = I$ with exceeding the max of the rest of term, we confirm the inequality of contraction of S in the complex valued metric space,So, we get
,

$$d(Sa, Sb) \precsim \alpha d(a, b)$$

where $\alpha \in [0, 1)$, for all $a, b \in X$.

Example 3.3 *Let $X = B(0, r)$, $r > 1$,for all $a, b \in X$. Define $d : X \times X \to \mathbb{C}$*

$$d(a(x_0), b(x_0) = \frac{i}{2\pi}|\int_\Gamma \frac{a(x_0)}{x_0} - \int_\Gamma \frac{b(x_0)}{x_0}|$$

a complete complex valued metric Γ is a closed path in X containing a zero. We prove that d is a complex valued metric.
Now

$$\begin{aligned} d(a(x_0), b(x_0) &= \frac{i}{2\pi}|\int_\Gamma \frac{a(x_0)}{x_0} - \int_\Gamma \frac{b(x_0)}{x_0}| \\ &= (\frac{i}{2\pi}|\int_\gamma \frac{a(x_0)}{x_0} - \int_\Gamma \frac{z(x_0)}{x_0} + \int_\Gamma \frac{z(x_0)}{x_0} - \int_\Gamma \frac{b(x_0)}{x_0}|) \\ &\precsim \frac{i}{2\pi}|\int_\gamma \frac{a(x_0)}{x_0} - \int_\Gamma \frac{z(x_0)}{x_0}| + \frac{i}{2\pi}|\int_\Gamma \frac{z(x_0)}{x_0} - \int_\Gamma \frac{b(x_0)}{x_0}| \\ &\precsim d(a(x_0), z(x_0)) + d(z(x_0), b(x_0)) \end{aligned}$$

Now we define the mapping $R, S, T, U : X \to X$ by $Ra(x_0) = x_0$,$Sa(x_0) = e^{\frac{x_0}{2}}$, $Ta(x_0) = e^{x_0} - 1$ and $Ua(x_0) = x^0 + \frac{1}{2}x_0$. Using the Cauchy formula,when the mapping R, S, T andU are analytic, we get:

$$\begin{aligned} d(Ra(x_0), Sb(x_0) &= \frac{i}{2\pi}|\int_\Gamma \frac{(x_0)}{x_0} - \int_\Gamma \frac{e^(x_0) - 1}{x_0}| \\ &= 0. \\ d(Ta(x_0), Ub(x_0) &= \frac{i}{2\pi}|\int_\Gamma \frac{e^(x_0)}{x_0} - \int_\Gamma \frac{e^(x_0) + \frac{1}{2}(x_0)}{x_0}| \\ &= i. \\ d(Ta(x_0), Ra(x_0) &= \frac{i}{2\pi}|\int_\Gamma \frac{e^(x_0)/2}{x_0} - \int_\Gamma \frac{(x_0)}{x_0}|, \\ &= 0. \\ d(Ub(x_0), Sb(x_0) &= \frac{i}{2\pi}|\int_\Gamma \frac{x_0 + \frac{x_0}{2}}{x_0} - \int_\Gamma \frac{e^{x_0} - 1}{x_0}|, \\ &= 0. \\ d(Ub(x_0), Ra(x_0) &= \frac{i}{2\pi}|\int_\Gamma \frac{x_0 + \frac{x_0}{2}}{x_0} - \int_\Gamma \frac{x_0}{x_0}|, \\ &= 0. \\ d(Ta(x_0), Sb(x_0) &= \frac{i}{2\pi}|\int_\Gamma \frac{e^{x_0}2}{x_0} - \int_\Gamma \frac{e^{x_0} - 1}{x_0}|, \\ &= 0. \\ Q(a(x_0), b(x_0)) &= max(i, 0) \\ &= \alpha i. \end{aligned}$$

Further,

$$\begin{aligned} 0 &= d(Ra(x_0), Sb(x_0)) \\ &\precsim \alpha i. \end{aligned}$$

Thus all the conditions of theorem 3.1 are satisfied, then the mappings R, S, T and U have a unique common fixed point in X.

4 Application

We utilize ,the Urysohn integral equation in theorem 3.1 and carry out of common solution. let us consider the two Urysohn integral equation

$$a(x_0) = \int_x^y K_1(t, s, a(x_0))ds + g(x_0)...(4.1)$$
$$a(x_0) = \int_x^y K_2(t, s, a(x_0)ds + h(h_0)...(4.2)$$

where $x_0 \in [x, y] \subset R$ and $a, g, h \in X$.

Theorem 4.1 *Let $d : X \times X \to \mathbb{C}$ be a mapping and Let $X = C([x, y], \mathbb{R}^n)$, $x > 0$ such that*

$$d(a, b) = max||a(x_0) - b(x_0)||_\infty \sqrt{1 + a^2}e^{itan^{-1}x}$$

Now, Asuumethat $K_1, K_2 : [x, y] \times [x, y] X \mathbb{R}^n \to \mathbb{R}^n$ such that $Fa, Ga \in X$, for each $a \in X$, where

$$F_a(x_0) = \int_x^y K_1(t, s, a(x_0)ds$$
$$G_a(x_0) = \int_x^y K_2(t, s, a(x_0)ds, \forall x_0 \in [x, y].$$

If there exists $\alpha \in [0, 1]$ such that

$$A(a, b)(x_0) \precsim \alpha Q(a, b)(x_0)(4.3)$$

where

$$Q(a, b) = max\{D(a, b)(x_0), B(a, b)(x_0), C(a, b)(x_0), E(a, b)(x_0), F(a, b)(x_0),$$
$$\frac{1}{2}[E(a, b)(x_0) + F(a, b)(x_0)], \frac{E(a, b)(x_0).F(a, b)(x_0)}{1 + D(a, b)(x_0)}\}$$

now, let

$$A(a, b)(x_0) = ||G_a(x_0) - H_b(x_0) + g(x_0) - h(x_0)||\sqrt{1 + a^2}e^{itan^{-1}x},$$
$$= E(a, b)x_0$$
$$C(a, b)(x_0) = ||b(x_0) - H_b(x_0) - h(x_0)||\sqrt{1 + a^2}e^{itan^{-1}x},$$
$$= F(a, b)x_0$$
$$D(a, b)(x_0) = ||a(x_0) - b(x_0)||\sqrt{1 + a^2}e^{itan^{-1}x},$$

Holds for all $a, b \in X$,then the system of Urysohn integral equations has a unique common solution in X.

Proof. Let the mapping $R, S : X \to X$ define by

$$R_a = G_a + \lambda, S_b = H_a + \phi.$$

where,

$$\begin{aligned} d(Ra, Sb) &= \max_{x_0 \in [x,y]} ||G_a(x_0) - H_b(x_0) + \lambda(x_0) - \phi(x_0)||_\infty . \sqrt{1+a^2} e^{itan^{-1}x}, \\ d(a, Ra) &= \max_{x_0 \in [x,y]} ||a(x_0) - G_a(x_0) - \lambda(x_0)||_\infty . \sqrt{1+a^2} e^{itan^{-1}x}, \\ d(b, Sb) &= \max_{x_0 \in [x,y]} ||b(x_0) - H_b(x_0) - \phi(x_0)||_\infty . \sqrt{1+a^2} e^{itan^{-1}x}, \\ d(a, b) &= \max_{x_0 \in [x,y]} ||a(x_0) - b(x_0)||_\infty . \sqrt{1+a^2} e^{itan^{-1}x}. \end{aligned}$$

From assumption 4.3, for each $x_0 \in [x, y]$, we have

$$\begin{aligned} A(a,b)(x_0) &\precsim \alpha Q(a,b)(x_0) \\ &\precsim \alpha max\{D(a,b)x_0, B(a,b)x_0, C(a,b)x_0, E(a,b)x_0, F(a,b)x_0, \\ &\frac{1}{2}[E(a,b)x_0 + F(a,b)x_0], \frac{E(a,b)x_0.F(a,b)x_0}{1+D(a,b)x_0}, \} \end{aligned}$$

which implies that

$$\begin{aligned} \max_{x_0 \in [x,y]} A(a,b)(x_0) = \alpha \max_{x_0 \in [x,y]} .max\{&D(a,b)x_0, B(a,b)x_0, C(a,b)x_0, E(a,b)x_0, F(a,b)x_0, \\ &\frac{1}{2}[E(a,b)x_0 + F(a,b)x_0], \frac{E(a,b)x_0.F(a,b)x_0}{1+D(a,b)x_0}, \} \\ &\precsim \alpha max\{ \max_{x_0 \in [x,y]} D(a,b)(x_0), \max_{x_0 \in [x,y]} B(a,b)(x_0), \max_{x_0 \in [x,y]} C(a,b)(x_0), \max_{x_0 \in [x,y]} E(a,b)(x_0), \\ &\max_{x_0 \in [x,y]} F(a,b)(x_0), \frac{1}{2}[\max_{x_0 \in [x,y]} E(a,b)x_0 + \max_{x_0 \in [x,y]} F(a,b)x_0], \\ &\frac{\max_{x_0 \in [x,y]} E(a,b)(x_0), \max_{x_0 \in [x,y]} F(a,b)(x_0)}{1 + \max_{x_0 \in [x,y]} D(a,b)(x_0)} \}. \end{aligned}$$

Therefore,

$$d(Ra, Sb) \precsim \alpha max\{d(a,b), d(a,Ra), d(b,Sb), d(b,Ra), d(a,Sb), \frac{1}{2}[d(b,Ra) + d(a,Sb)], \frac{d(a,Ra).d(b,Sb)}{1+d(a,b)}\}.$$

Thus, satisfied the all conditions of theorem 3.1 with $R = S = I_x$ .Therefore,we get the system of Urysohn integral equations have a unique common solution in X. ■

5 CONCLUSION

The intention of this article, we studied unique common fixed point results for two pairs compatible mappings in the setting of complex valued metric spaces. Our result complement several significant fixed point theorem of complex valued metric spaces. These results genealizes and improves the recent results of V.H. Badshah et al. [13] and F. Rouzkard[16]. Also, we give examples as an application for the usability of the main result. We confidence that, the described result here in will be a source of motivational for other investigators.

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