International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 09 Issue 12 December 2021, Page no. – 2526-2529

Index Copernicus ICV: 57.55, Impact Factor: 7.184

DOI: 10.47191/ijmcr/v9i12.03

Modification of Varignon's Theorem

Ariska[1], Mashadi[2], Leli Deswita[3]

[1,2,3]Department of Mathematics, University of Riau, Pekanbaru, Indonesia

ARTICLE INFO	ABSTRACT
Published Online: 14 December 2021 Corresponding Author: **Mashadi**	This paper discusses the modification of Varignon's theorem on quadrilaterals by dividing the sides of the quadrilateral into three, four, five to become n parts. The process of proving it is done in a very simple way, namely by using congruence. The result obtained is that the area of the plane formed from the modification of Varignon's theorem has a relationship with the area of the basic quadrilateral.

KEYWORDS: Varignon theorem, quadrilateral, congruence, sequence

I. INTRODUCTION

Mathematics is the science of logic about shapes, arrangements, quantities and concepts that relate to one another in large numbers, which are divided into three fields, namely algebra, analysis and geometry [5]. Furthermore, in [10] said that Geometry is a branch of mathematics that deals with statements of shape, size, image and the nature of space.

One of the many theorems in the field of geometry, especially those that discuss plane figures, is the Varignon Parallelogram theorem. The Varignon theorem was introduced by Pierre Varignon's (1654-1722) which is known as the Varignon's Parallelogram. The French academics of the Newton and Leibniz era provided strong evidence for the Varignon theorem that was published in 1731 [11]. In 2001 an area formed from Varignon was described [12]. The Varignon's Parallelogram theorem states The figure formed when the midpoints of the sides of a quadrilateral are joined in order is a parallelogram, and its area is half that of the quadrilateral [6].

Several evidences and cases of Varignon concave and cross-parallelograms have also been expanded at other points by dividing by three, four, or generally dividing n sides of each quadrilateral and investigated using the GeoGebra application [7] and the Sketpatch program [9]. Varignon theorem in triangles and its modifications have been discussed by Arisa et al. [13].

Based on the above conditions, the author will modify [10] Varignon theorem by dividing by three, four, five and in general dividing each side [9 and 7], then take the starting point of each side and connect it to form a modified quadrilateral.

II. VARIGNON'S THEOREM

The Varignon theorem was discovered by Pierre Varignon (1654-1722). Nine years after Varignon died, this theorem was published [11]. In 2001 in [12] explained the area formed from the Varignon theorem.

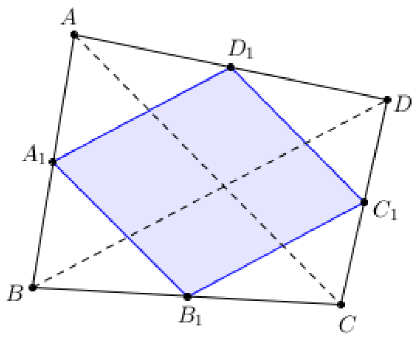

Fig. 1: Varignon Parallelogram

The Varignon Parallelogram theorem and its proofs have been discussed in [2, 6, 7, 8, 11, 12 and 15] stating that the quadrilateral formed is a parallelogram and the area is half of the basic quadrilateral, as described in Theorem 1.

Theorem 1. (Theorem Varignon Paralellogram) Given any $\square ABCD$, if the sides AB, BC, CD and AD is divided into two parts of equal length are formed points A_1, B_1, C_1 and D_1. If the four points are connected, then formed a parallelogram and $A_1B_1C_1D_1$ and $L\square A_1B_1C_1D_1 = \frac{1}{2} L\square ABCD$.

Proof. Notice $\triangle ABC$ applies $A_1B_1 \,//\, AC$ and $A_1B_1 = \frac{1}{2} AC$, to $\triangle CDA$ apply $C_1D_1 \,//\, AC$ and $C_1D_1 = \frac{1}{2}AC$ then A_1B_1 //

C_1D_1 and $A_1B_1 = C_1D_1$. Next, $\triangle ABD$ apply $A_1D_1 \,//\, BD$ and $A_1D_1 = \frac{1}{2}BD$, to $\triangle BCA$ apply $B_1C_1 = \frac{1}{2}BD$ then $A_1D_1 \,//\, B_1C_1$ and $A_1D_1 = B_1C_1$. Because, $A_1B_1 = C_1D_1$, $A_1D_1 = B_1C_1$, $A_1B_1 \,//\, C_1D_1$ and $A_1D_1 \,//\, B_1C_1$ then $A_1B_1C_1D_1$ is a parallelogram.

Then, using the midpoint theorem [3] and congruence [1 and 14] on $\triangle AA_1D_1$, $\triangle A_1BB_1$, $\triangle B_1CC_1$, and $\triangle C_1DD_1$ we get:

$$L \triangle AA_1D_1 = \frac{1}{4} L \triangle ABD, \quad (1)$$

$$L \triangle A_1BB_1 = \frac{1}{4} L \triangle ABC, \quad (2)$$

$$L \triangle B_1CC_1 = \frac{1}{4} L \triangle BCD, \quad (3)$$

$$L \triangle C_1DD_1 = \frac{1}{4} L \triangle CDA. \quad (4)$$

Furthermore, equation (1), (2), (3) and (4) is substituted into the equation spacious $\square A_1B_1C_1D_1$ obtained

$$\begin{aligned} L\square A_1B_1C_1D_1 &= L\square ABCD - L \triangle A_1BB_1 - L \triangle B_1CC_1 \\ &\quad - L \triangle C_1DD_1 - L \triangle D\_1\ AA\_1, \\ &= L\square ABCD - \frac{1}{4} L \triangle ABD - \frac{1}{4} L \triangle ABC \\ &\quad - \frac{1}{4} L \triangle BCD - \frac{1}{4} L \triangle CDA, \\ &= L\square ABCD - \left(\frac{1}{4} L \triangle ABD + \frac{1}{4} L \triangle BCD\right) \\ &\quad + \left(\frac{1}{4}\ L \triangle ABC + \frac{1}{4} L \triangle CDA\right) \\ &= L\square ABCD - \frac{1}{4} L\square ABCD \\ &\quad + \frac{1}{4} L\square ABCD, \\ &= L\square ABCD - \frac{2}{4} L\square ABCD, \end{aligned}$$

$$L\square A_1B_1C_1D_1 = \frac{1}{2} L\square ABCD. \quad \blacksquare$$

III. MODIFICATION

Modification of Varignon's theorem is to divide three, four, five to n parts of each side of the quadrilateral then connect each point on the side of the quadrilateral so that another quadrilateral is formed in the initial quadrilateral.

A. Quadrilateral with sides dividing by three

In Figure 2, for each side of the quadrilateral is divided into three equal parts from side AB to form a points A_1 and A_2, from side BC to form a points B_1 and B_2, from side CD to form a points C_1 and C_2 from side DA to form a points D_1 and D_2.

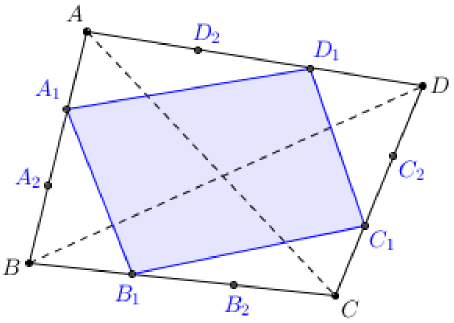

Fig. 2: Quadrilateral divided by three

Theorem 2. Given any $\square ABCD$, if the sides AB, BC, CD and DA each divided into three equal parts will form a points $A_1, A_2, B_1, B_2, C_1, C_2, D_1$ and D_2. If the points A_1, B_1, C_1 and D_1 connected, then apply $L\square A_1B_1C_1D_1 = \frac{5}{9} L\square ABCD$.

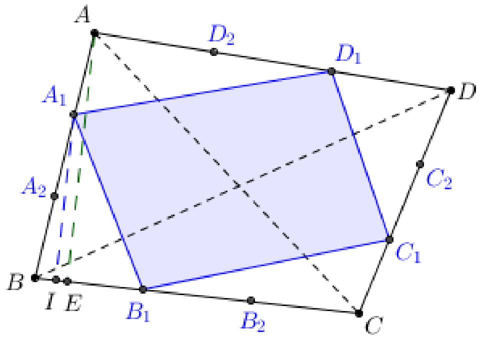

Fig. 3: Illustration of proof of Varignon modification divided three

Proof. To prove $L\square A_1B_1C_1D_1$ first shown $L \triangle A_1BB_1, L \triangle B_1CC_1, L \triangle C_1DD_1$ and $L \triangle D_1AA_1$. Pay attention to Figure 3 then it is obtined

$$L \triangle A_1BB_1 = \frac{1}{2} BB_1 \cdot A_1I. \quad (5)$$

Since $A_1B = \frac{2}{3}AB$ and by using the congruence we get

$$A_1I = \frac{2}{3} AE. \quad (6)$$

Furthermore, the value $BB_1 = \frac{1}{3}BC$ and equation (6) are substituted into equation (5) so that we get

$$L \triangle A_1BB_1 = \frac{2}{9} L \triangle ABC. \quad (7)$$

In the same way, we get:

$$L \triangle B_1CC_1 = \frac{2}{9} L \triangle BCD, \quad (8)$$

$$L \triangle C_1DD_1 = \frac{2}{9} L \triangle CDA, \quad (9)$$

$$L \triangle AA_1D_1 = \frac{2}{9} L \triangle ABD. \quad (10)$$

Furthermore, equation (7), (8), (9) and (10) is substituted into the equation $L\square A_1B_1C_1D_1$ in Figure 2 is obtained

$$\begin{aligned} L\square A_1B_1C_1D_1 &= L\square ABCD - L\ \ \triangle A_1BB_1 - L \triangle B_1CC_1 \\ &\quad - L \triangle C_1DD_1 - L \triangle AA_1D_1 \\ &= L\square ABCD - \frac{2}{9} L \triangle ABC - \frac{2}{9} L \triangle BCD \\ &\quad - \frac{2}{9} L \triangle CDA - \frac{2}{9} L \triangle ABD \\ &= L\square ABCD - \left(\frac{2}{9} L \triangle ABC + \frac{2}{9} L \triangle CDA\right) \\ &\quad - \left(\frac{2}{9} L \triangle BCD + \frac{2}{9} L \triangle ABD\right) \\ &= L\square ABCD - \frac{2}{9} L\square ABCD \\ &\quad - \frac{2}{9} L\square ABCD \\ &= L\square ABCD - \frac{4}{9} L\square ABCD \end{aligned}$$

$$L\square A_1B_1C_1D_1 = \frac{5}{9} L\square ABCD. \quad \blacksquare$$

B. Quadrilateral with sides dividing by four

In Figure 4, each side of the quadrilateral is divided into four equal parts from the side AB to form points A_1, A_2 and A_3, from the side BC to form points B_1, B_2 and B_3, from the side CD to form points C_1, C_2 and C_3 and from the side DA to form points D_1, D_2 and D_3.

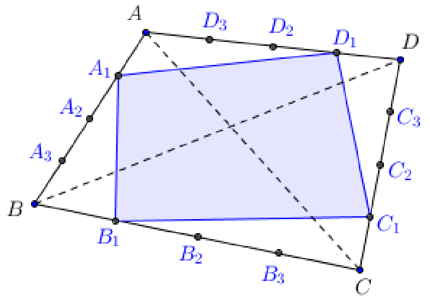

Fig. 4: Quadrilateral divided by four

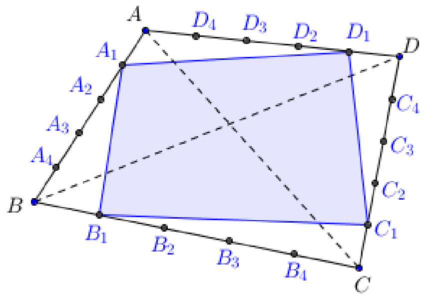

Fig. 5: Quadrilateral divided by five

Theorem 3. Given any $\square ABCD$, if the sides AB, BC, CD and DA is divided into four equal parts will form a point A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3, D_1, D_2 and D_3. If the points A_1, B_1, C_1 and D_1 are connected, then apply $L\square A_1B_1C_1D_1 = \frac{10}{16} L\square ABCD$.

Proof. In the same way using the similarity in equations (7), (8), (9) and (10), it is obtained:

$$L \triangle A_1BB_1 = \frac{3}{16} L \triangle ABC, \quad (11)$$

$$L \triangle B_1CC_1 = \frac{3}{16} L \triangle BCD, \quad (12)$$

$$L \triangle C_1DD_1 = \frac{3}{16} L \triangle CDA, \quad (13)$$

$$L \triangle AA_1D_1 = \frac{3}{16} L \triangle DAB. \quad (14)$$

Next, use the equation (11), (12), (13) and (14) into the equation subtitusikan $L\square A_1B_1C_1D_1$ is obtained

$$\begin{aligned} L\square A_1B_1C_1D_1 &= L\square ABCD - L \ \triangle A_1BB_1 - L \triangle B_1CC_1 \\ &\quad - L \triangle C_1DD_1 - L \triangle AA_1D_1 \\ &= L\square ABCD - \frac{3}{16} L \triangle ABC - \frac{3}{16} L \triangle BCD \\ &\quad - \frac{3}{16} L \triangle CDA - \frac{3}{16} L \triangle ABD \\ &= L\square ABCD - \left(\frac{3}{16} L \triangle ABC + \frac{3}{16} L \triangle CDA\right) \\ &\quad - \left(\frac{3}{16} L \triangle BCD + \frac{3}{16} L \triangle ABD\right) \\ &= L\square ABCD - \frac{3}{16} L\square ABCD \\ &\quad - \frac{3}{16} L\square ABCD \\ &= L\square ABCD - \frac{6}{16} L\square ABCD \end{aligned}$$

$$L\square A_1B_1C_1D_1 = \frac{10}{16} L\square ABCD. \quad \blacksquare$$

C. Quadrilateral with sides dividing by five

In Figure 5, each side of the quadrilateral is divided into five equal parts from the sides AB forming points A_1,A_2, A_3 and A_4, from the side BC forming points B_1,B_2, B_3 and B_4 from the side CD forming points C_1, C_2, C_3 and C_4 then from the side DA forming points D_1,D_2, D_3 and D_4.

Theorem 4. Given any $\square ABCD$, if the sides AB, BC, CD and DA is divided into five equal parts will form a points A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3 C_4, D_1, D_2, D_3 and D_4. If the points A_1, B_1, C_1 and D_1 connected, then apply $L\square A_1B_1C_1D_1 = \frac{17}{25} L\square ABCD$.

Proof. In the same way using the similarity in equation (7), (8), (9) and (10), it is obtained:

$$L \triangle A\_1\ BB\_1 = \frac{4}{25}\ L \triangle ABC, \quad (15)$$

$$L \triangle B\_1\ CC\_1 = \frac{4}{25}\ L \triangle BCD, \quad (16)$$

$$L \triangle C\_1\ DD\_1 = \frac{4}{25}\ L \triangle CDA, \quad (17)$$

$$L \triangle AA\_1\ D\_1 = \frac{4}{25}\ L \triangle ABD. \quad (18)$$

Furthermore, equation (15), (16), (17) and (18) is substituted into the equation $L\square A_1B_1C_1D_1$ it is obtained

$$\begin{aligned} L\square A_1B_1C_1D_1 &= L\square ABCD - L \ \triangle A_1BB_1 - L \triangle B_1CC_1 \\ &\quad - L \triangle C_1DD_1 - L \triangle AA_1D_1 \\ &= L\square ABCD - \frac{4}{25}\ L \triangle ABC - \frac{4}{25}\ L \triangle BCD \\ &\quad - \frac{4}{25}\ L \triangle CDA - \frac{4}{25}\ L \triangle ABD \\ &= L\square ABCD - \left(\frac{4}{25} L \triangle ABC + \frac{4}{25} L \triangle CDA\right) \\ &\quad - \left(\frac{4}{25} L \triangle BCD + \frac{4}{25} L \triangle ABD\right) \\ &= L\square ABCD - \frac{4}{25} L\square ABCD \\ &\quad - \frac{4}{25} L\square ABCD \\ &= L\square ABCD - \frac{8}{25} L\square ABCD \end{aligned}$$

$$L\square A_1B_1C_1D_1 = \frac{17}{25} L\square ABCD. \quad \blacksquare$$

IV. CONCLUTIONS

In this section, it will be determined in general that if each side is divided into n parts, then there is a comparison of the modified area of the Varignon quadrilateral with the area of the original quadrilateral to the sequence pattern.

From the results for the side varignon divided by two, the side varignon modification divided by three, the side varignon modification divided by four and the side varignon modification divided by five, the rows obtained

$$L\square A_1B_1C_1D_1 = \frac{2}{4}, \frac{5}{9}, \frac{10}{16}, \frac{17}{25}, \cdots$$

From the above sequence, the modified area of the Varignon triangle can be written in the form $\frac{a_n}{b_n}$. Obviously $b_n = (n+1)^2$ so

$$L_n = \frac{a_n}{(n+1)^2}. \quad (19)$$

Using Newton's binomial [4] translation or the system of linear equations above, the sequence for $a_n = n^2 + 1$. Furthermore, if it is substituted into equation (19) then we get

$$L_n = \frac{n^2+1}{(n+1)^2}.$$

REFERENCES

1. Mashadi, 2015, Geometri, UR Press, Pekanbaru.
2. Jupri, A, 2017, Investigating primary school mathematic teachers’ deductive reasoning ability through varignon’s theorem, International Conference On Mathematics and Science Education (ICMScE), 2017.
3. Krishna, D. N. V, 2016, The New Proof of Ptolemy’s Theorem and Nine Point Circle Theorem, Mathematics and Computer Science, 1 (4), pp. 93-100
4. Andriani , D, Mashadi, S. Gemawati, 2018, Barisan Sisi Alas Piramida Heptagonal. Karismatika. 4(3) (2018)
5. Suherman, E. dkk, 2003, Strategi Pembelajaran Matematika Kontemporer, Universitas Pendidikan Indonesia Bandung
6. Coxeterand, H.S.M, S.L. Greitzer, 1967, Geometri Revisited, MAA, Washington, 52
7. Contreas, J. N, 2014, Investigating Variations of Varignon Theorem Using GeoGebra, Ball State University, 3(2014), 29-36.
8. Ndlovu, M, 2014, Definitional conflicts between euclidean geometry and dynamic geometry environments: varignon theorem as an example, Proceeding of INTED2014 Conference, Valencia, pp 6158-6166.
9. Villiers, M. D, 1998, A Sketchpad Discovery Involving Triangles and Quadrilaterals, KZN Mathematics Journal, 3(1), 11-18.
10. Syawaludin, M. R, Mashadi and Sri Gemawati, 2018, Modification Cross’ Theorem on Triangle with Congruence, SciencePG; 4(5); 40-44
11. Oliver, P. N, 2001, Pierre Varignon and The Parallelogram Theorem, Mathematics Teacher of Mathematic, 94(2001), 316-319.
12. Oliver, P. N, 2001, Consequence of The Varignon Parallelogram Theorem, Mathematics Teacher of Mathematic, 94 (2001), 406-408.
13. Arisa, Y, Mashadi, and L. Deswita, 2020, Modification of the Varignon Theorem on the Triangle, International Journal of Recent Scientific Research, 11(09) (2020), 39642-39646.
14. Mashadi, 2015, Geometri Lanjut, UR Press, Pekanbaru.
15. Palatnik, A, 2017, Proof without words: Varignon’s theorem, College Mathematic Journal, 48 (2017), 354.