International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 10 Issue 01 January 2022, Page no. – 2546-2550

Index Copernicus ICV: 57.55, Impact Factor: 7.184

DOI: 10.47191/ijmcr/v10i1.04

Mathematical Analysis of a Degenerate Reaction-Diffusion Model

Salim Mesbahi[1], Khaoula Imane Saffidine[2]

[1,2] Ferhat Abbas University, Faculty of Sciences, Department of Mathematics, LMFN Laboratory, Setif, Algeria

ARTICLE INFO

Published Online: 25 January 2022

Corresponding Author: **Salim Mesbahi**

ABSTRACT

In this paper, we are interested in the study of a degenerate reaction-diffusion model, where we prove the existence of positive maximal and minimal solutions, including the uniqueness of the positive solution. The technique used is mainly based on the method of upper and lower solutions.

KEYWORDS: Reaction Diffusion System, Degenerate Elliptic System, Upper And Lower Solutions, Diffusion Phenomena, Mathematical Modelling.

I. INTRODUCTION

Mathematicians are increasingly interested, especially in recent years, in the study of reaction-diffusion systems, due to their paramount importance and frequent use in the modeling of many diffusion phenomena that we observe in nature and which also result from various natural sciences and engineering, such as (Coronavirus, hepatitis, population dynamics, migration of biological species, quenching). In Murray [13,14], we find many real models in different scientific fields.

There are many techniques that can be used to investigate this type of problems. The reader can see some of them in the works of Alaa and Mesbahi et *al*. [1,2,10-12,19] and the references therein.

Perhaps the most important class of interaction diffusion systems that has received special attention are those that are degenerated, due to their wide applications in various sciences, particularly in the theory of shells, Brownian motion and in many problems of physics, engineering, biology, ecology, and others. Among the important works on degenerate systems, we mention for example Alaa et *al*. [2], Alvarez et *al*. [3], Anderson [4], Einav et *al*. [6], Fitzgibbon et *al*. [7], Saffidine and Mesbahi [19], Wang and Zhao [20].

The content of this paper is in this context, we will study the existence of positive maximal and minimal solutions for a quasilinear elliptic degenerate system, including the uniqueness of the positive solution. The elliptic operators of the considered system can degenerate in the sense that at least one of the operators $D_i(u_i)$ is degenerate. To achieve the desired result, we will use a technique based mainly on the method of upper and lower solutions. In order to better understand this technique, see Deuel and Hess [5], Pao et *al*. [15-18].

We shall then study the following system

$$(1.1):\begin{cases}-div(D_i(u_i)\nabla u_i) = f_i(x,u) & \text{in } \Omega\\ u_i(x) = g_i(x)\ ,\ 1 \le i \le N & \text{on } \partial\Omega,\end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \ge 2)$ with boundary $\partial\Omega$. $D_i(u_i)$, f_i, $i = 1,\dots,N$ are prescribed functions satisfying the conditions in hypotheses (H_1) and (H_3) which we will mention later.

The system (1.1) can model the circulation of an ideal gas in a homogeneous porous medium with an isentropic flow. He can also model the steady state of phenomena such as the heat propagation in a two-component combustible mixture, chemical processes, the interaction of two non-self-limiting biological groups, etc. We send the reader to see many models and applications in Deuel and Hess [5], Ladyženskaja et *al*. [8], Lei and Zheng [9], especially Pao [16,17] and the references therein.

The rest of this paper is organized as follows: In the next section, we state our main result. In the third section, we present some preliminary results that we will use later. Next, we give some results regarding the approximate problem. The fifth section is devoted to prove the main result. Finally, we conclude with a conclusion and some perspectives.

II. STATEMENT OF THE MAIN RESULT

Below, we denote $u \equiv (u_1,\dots,u_N)$, $\tilde{u}_s \equiv (\tilde{u}_1,\dots,\tilde{u}_N)$, $\hat{u}_s \equiv (\hat{u}_1,\dots,\hat{u}_N)$. The inequality $\hat{u}_s \le \tilde{u}_s$ means that $\hat{u}_i \le \tilde{u}_i$ for all $1 \le i \le N$.

A. *Assumptions*

Below we will denote $C^\alpha(\Omega)$ to the space of Hölder continuous functions in Ω. We start with the following important definition.

Definition 1. *A pair of functions* $\tilde{u}_s \equiv (\tilde{u}_1, \dots, \tilde{u}_N)$, $\hat{u}_s \equiv (\hat{u}_1, \dots, \hat{u}_N)$ *in* $C^2(\Omega) \cap C(\bar{\Omega})$ *are called ordered upper and lower solutions of* (1.1) *if* $\hat{u}_s \leq \tilde{u}_s$ *and*

$$(2.1): \begin{cases} -div(D(\hat{u}_i)\nabla \hat{u}_i) \leq f(x, \hat{u}_s) & \text{in } \Omega \\ \hat{u}_i(x) \leq g_i(x)\ ,\ 1 \leq i \leq N & \text{on } \partial\Omega, \end{cases}$$

and $\tilde{u}_i$ *satisfies* (2.1) *with inequalities reversed.*

For any given pair of ordered upper and lower solutions $\tilde{u}_s$ and $\hat{u}_s$, we define for all $1 \leq i \leq N$,

$$S_i^* = \{u_i \in C(\bar{\Omega}) \mid \hat{u}_i \leq u_i \leq \tilde{u}_i\},$$

and

$$S^* = \{u \in C(\bar{\Omega}) \mid \hat{u}_s \leq u \leq \tilde{u}_s\}$$

We will study our problem under the following hypotheses, we assume for each $1 \leq i \leq N$:

(H_1) $f_i(x, .) \in C^\alpha(\bar{\Omega})$ and $g_i \equiv g_i(x) \in C^\alpha(\partial\Omega)$.

(H_2) $D_i(u_i) \in C^2([0, M_i])$, $D_i(u_i) > 0$ for $u_i \in [0, M_i]$, and $D_i(0) \geq 0$, where $M_i = |\tilde{u}_i|_{C(\bar{\Omega})}$.

(H_3) $f_i(., u) \in C^1(S^*)$, and

$$\frac{\partial f_i}{\partial u_j}(., u) \geq 0,\ i \neq j \quad \text{for all u} \in S^*$$

(H_4) There exists a constant $\delta_0 > 0$ such that for any $x_0 \in \partial\Omega$ there exists a ball K outside of Ω with radius $r \geq \delta_0$ such that $K \cap \bar{\Omega} = \{x_0\}$.

In the above system, we further assume $D_i(0) = 0$ for some or all $1 \leq i \leq N$ and $D_i(0) \geq 0$ for the remaining i. Let $\gamma_i(x)$ be smooth positive functions satisfying for all $1 \leq i \leq N$,

$$\gamma_i(x) \geq max\left\{-\frac{\partial f_i}{\partial u_i}(x,\ u)\ ;\ u \in S^*\right\}$$

$$\gamma_i(x) \geq C_i(x) + \delta_i,$$

for some constants $\delta_i > 0$, where $C_i(x)$ are analogous to $C(x)$ defined in section 3 by the relations (3.5). We define for all $u \in S^*$

$$F_i(x, u) = \gamma_i(x) u_i + f_i(x, u), 1 \leq i \leq N.$$

If we take, for example,

$$D_1(u_1) = \lambda u_1^{\lambda-1}\ ,\ D_2(u_2) = \mu u_2^{\mu-1}$$
$$D_3(u_3) = \upsilon u_3^{\upsilon-1}\ ,\ f_1 = p(x) u_1^{j_1} u_2^{k_1} u_3^{\ell_1}$$
$$f_2 = q(x) u_1^{j_2} u_2^{k_2} u_3^{\ell_2}\ ,\ f_3 = r(x) u_1^{j_3} u_2^{k_3} u_3^{\ell_3},$$

we get the following typical example, where the result of this paper can be applied to it

$$\begin{cases} -\Delta u_1^\lambda = p(x) u_1^{j_1} u_2^{k_1} u_3^{\ell_1} & \text{in } \Omega \\ -\Delta u_2^\mu = q(x) u_1^{j_2} u_2^{k_2} u_3^{\ell_2} & \text{in } \Omega \\ -\Delta u_3^\upsilon = r(x) u_1^{j_3} u_2^{k_3} u_3^{\ell_3} & \text{in } \Omega \\ u_1 = u_2 = u_3 = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda,\ \mu,\ \upsilon > 1$, j_s , k_s , $\ell_s > 0$, $1 \leq s \leq 3$, and $p(x)$, $q(x)$, $r(x) > 0$ in Ω.

Lemma 1. $F_i(x, u)$ *are nondecreasing functions in* u *for all* $u \in S^*$, $1 \leq i \leq N$.

Proof. According to (H_3) and (2.3), we have for all $1 \leq i \leq N$ and $u \in S^*$,

$$\frac{\partial f_i}{\partial u_j}(., u) \geq 0,\ i \neq j.$$

By $(2.2) - (2.3)$, we get

$$\frac{\partial F_i}{\partial u_i}(x, u) = \gamma_i(x) + \frac{\partial f_i}{\partial u_i}(x, u) \geq 0,$$

and that is exactly what we want to get.

B. The main result

The main result of this paper is what the following theorem states.

Theorem 1. *Let* $\tilde{u}_s$, $\hat{u}_s$ *be ordered positive upper and lower solutions of* (1.1), *and let hypotheses* $(H_1) - (H_4)$ *hold. Then problem* (1.1) *has a minimal solution* $\underline{u}_s$ *and a maximal solution* $\overline{u}_s$ *such that* $\hat{u}_s \leq \underline{u}_s \leq \overline{u}_s \leq \tilde{u}_s$. *If* $\underline{u}_s = \overline{u}_s (\equiv u_s^*)$ *then* u_s^* *is the unique positive solution in* S^*.

III. PRELIMINARY RESULTS

All. We first consider the following scalar problem

$$(3.1): \begin{cases} -div(D(w)\nabla w) = h(x, w) & \text{in } \Omega \\ u(x) = h(x) & \text{on } \partial\Omega, \end{cases}$$

where D and h satisfy the above hypotheses $(H_1) - (H_4)$.

The following Theorem ensures the existence of positive solutions to problem (3.1) . The proof of this theorem and some important clarifications can be found in Ladyženskaja et *al*. [8], Pao and Ruan [15].

Theorem 2. *Let* $\widetilde{w}_s(x)$, $\widehat{w}_s(x)$ *be a pair of upper and lower solutions of* (3.1) *such that* $\widetilde{w}_s(x) \geq \widehat{w}_s(x) > 0$ *in* Ω *, and let hypotheses* (H_1) *and* (H_3) *hold. Then problem* (3.1) *has a classical solution* $w_s(x)$ *such that* $\widehat{w}_s(x) \leq w_s(x) \leq \widetilde{w}_s(x)$ *in* $\overline{\Omega}$ *. Furthermore, there are maximal and minimal solutions* $\overline{w}_s(x)$ *and* $\underline{w}_s(x)$ *such that every solution* $w_s \in S_0^*$ *satisfies* $\underline{w}_s(x) \leq w_s(x) \leq \overline{w}_s(x)$.

We consider the scalar problem (3.1) for w . This will lead to

$$(3.2):\ -div(D(\widehat{w})\nabla\widehat{w}) \leq h(x, \widehat{w})\ , \text{in } \Omega$$

$$(3.3):\ -div(D(\widetilde{w})\nabla\widetilde{w}) \geq h(x, \widetilde{w})\ , \text{in } \Omega,$$

and therefore

$$-div\left[D_1(\widehat{w})\nabla(\widehat{w} - \widetilde{w}) + \nabla\widetilde{w}\left(\frac{D(\widehat{w}) - D(\widetilde{w})}{\widehat{w} - \widetilde{w}}(\widehat{w} - \widetilde{w})\right)\right]$$
$$\leq \frac{h(x, \widehat{w}) - h(x, \widetilde{w})}{\widehat{w} - \widetilde{w}}(\widehat{w} - \widetilde{w}).$$

The mean value theorem states that there exist $\theta_1, \theta_2 \in [0, M]$, where $M = \|\widetilde{w}\|_{C(\bar{\Omega})}$ such that

$$-div[D(\widehat{w})\nabla z + \nabla\widetilde{w}(D'(\theta_1)z)] \leq h_w(x, \theta_2) z,$$

with $z = \widehat{w} - \widetilde{w}$, then

$$-div(\nabla z)\big(D(\widehat{w})\big) - \nabla\big(D(\widehat{w})\big)\nabla z$$
$$-div\big(\nabla\widetilde{w}(D'(\theta_1)z)\big)$$
$$-\nabla(\widetilde{w})D'(\theta_1)\nabla z - h_w(x, \theta_2) z \leq 0.$$

We obtain

$$-D(\widehat{w})\Delta z+[-\nabla D(\widehat{w})-D'(\theta_1)\nabla(\widetilde{w})]\nabla z$$
$$+[-\nabla.\nabla(\widetilde{w})D'(\theta_1)-h_w(x,\theta_2)]z\leq 0$$

We denote

(3.4): $B(x)=-\nabla D(\widehat{w})-D'(\theta_1)\nabla(\widetilde{w})$

(3.5): $C(x)=-div\nabla(\widetilde{w})D'(\theta_1)-h_w(x,\theta_2).$

We will also need the following lemma.

Lemma 2. *If* $\underline{z}$, $\overline{z}$ *are in* $C^2(\Omega)\cap C(\bar{\Omega})$ *and satisfy the relation*

$$\begin{cases}-\Gamma[\underline{z}]+\gamma\underline{z}\leq-\Gamma[\overline{z}]+\gamma\overline{z} & \text{in } \Omega\\ \underline{z}(x)\leq\overline{z}(x) & \text{on } \partial\Omega,\end{cases}$$

with $\Gamma[u]=div(D(w)\nabla w)$ *, then* $\underline{z}(x)\leq\overline{z}(x)$ *on* $\overline{\Omega}$.

Proof. Let $z(x)=\underline{z}(x)-\overline{z}(x)$, we have

$$-\Gamma[\underline{z}]+\gamma\underline{z}\leq-\Gamma[\overline{z}]+\gamma\overline{z}=\gamma\overline{z}+h(x,\bar{z})\equiv ?(x,\bar{z}),$$

then

(3.6): $-\Gamma[\underline{z}]+\gamma(\underline{z}-\bar{z})-h(x,\bar{z})\leq 0,$

and

(3.7): $\Gamma[\overline{z}]+\gamma(\underline{z}-\overline{z})+h(x,\underline{z})\leq 0.$

Adding (3.6) and (3.7), we get

$$-div\left[D(\underline{z})\nabla(\underline{z}-\overline{z})+\nabla\overline{z}\left(\frac{D(\underline{z})-D(\overline{z})}{\underline{z}-\overline{z}}(\underline{z}-\overline{z})\right)\right]$$
$$+2\gamma(\underline{z}-\bar{z})+\frac{h(x,\underline{z})-h(x,\overline{z})}{\underline{z}-\overline{z}}(\underline{z}-\overline{z})\leq 0.$$

The mean value theorem confirms the existence of θ_1,θ_2 in $[0,M]$ such that

$$-div[D(\underline{z})\nabla z+\nabla\overline{z}(D'(\theta_1)z)]+$$
$$2\gamma(\underline{z}-\bar{z})+\frac{h(x,\underline{z})-h(x,\overline{z})}{\underline{z}-\overline{z}}(\underline{z}-\overline{z})\leq 0$$

with $z=\underline{z}-\overline{z}$, then we have

$$-div(\nabla z)\left(D(\underline{z})\right)-\nabla\left(D(\underline{z})\right)\nabla z$$
$$-div(\nabla\overline{z}(D'(\theta_1)z))$$
$$-\nabla(\overline{z})D'(\theta_1)\nabla z+h_w(x,\theta_2)z\leq 0,$$

and therefore

$$-D(\underline{z})\Delta z-[\nabla D(\underline{z})-D'(\theta_1)\nabla(\overline{z})]\nabla z$$
$$-div\nabla(\overline{z})D'(\theta_1)z+2\gamma z+h_w(x,\theta_2)z\leq 0$$

$$-D(\underline{z})\Delta z+[-\nabla D(\underline{z})-D'(\theta_1)\nabla(\overline{z})]\nabla z$$
$$+[\gamma+div\nabla(\overline{z})D'(\theta_1)+h_w(x,\theta_2)]z\leq 0$$

which give

$$-D(\underline{z})\Delta z+(B(x))\nabla z+(\gamma-C(x))z\leq 0,$$

where $B(x)$ and $C(x)$ are defined in the same way as $B(x)$ and $C(x)$ of relations (3.4) and (3.5) , i.e.,

$$B(x)=-\nabla D(\underline{z})-D'(\theta_1)\nabla(\overline{z})$$
$$C(x)=-div\nabla(\overline{z})D'(\theta_1)+h_w(x,\theta_2).$$

Assume, by contradiction, that $z(x)$ has a positive maximum at some point $x_0\in\overline{\Omega}$. Then $x_0\in\Omega$, $\Delta z(x_0)\leq 0$ and $\nabla z(x_0)=0$. This leads to $(\gamma-C)z(x_0)\leq 0$, which contradicts the fact that $\gamma-C=\delta>0$.

IV. APPROXIMATING SCHEME

To prove the main Theorem, we use the method of upper and lower solutions. Using then either $\hat{u}_s$ or $\tilde{u}_s$ as the initial iteration we construct a sequence $\left\{u_s^{(m)}\right\}$ from the iteration process for

$$(4.1):\begin{cases}-\Phi_i[u_i^{(m)}]+\gamma_i u_i^{(m)}=F_i(x,u_s^{(m-1)}) & \text{in } \Omega\\ u_i^{(m)}(x)=g_i(x)\ ,\ 1\leq i\leq N & \text{on } \partial\Omega,\end{cases}$$

with

$$\Phi_i[u_i]=div(D_i(u_i)\nabla u_i).$$

We denote the sequence by $\left\{\underline{u}_s^{(m)}\right\}$ if $u_s^{(0)}=\hat{u}_s$, and by $\left\{\overline{u}_s^{(m)}\right\}$ if $u_s^{(0)}=\tilde{u}_s$. We call them as minimal and maximal sequences, respectively.

Lemma 3. *The minimal and maximal sequences* $\left\{\underline{u}_s^{(m)}\right\}$, $\left\{\overline{u}_s^{(m)}\right\}$ *exist and possess the monotone property*

$$(4.2):\ \hat{u}_s\leq\underline{u}_s^{(m)}\leq\underline{u}_s^{(m+1)}\leq\overline{u}_s^{(m+1)}\leq\overline{u}_s^{(m)}\leq\tilde{u}_s$$

for all $m\geq 1$

Proof. Firstly, we consider the following scalar problem

$$(4.3):\begin{cases}-\Phi_i[u_i^{(m)}]+\gamma_i u_i^{(m)}=F_i(x,u_s^{(m-1)}) & \text{in } \Omega\\ u_i^{(m)}(x)=g_i(x)\ ,\ 1\leq i\leq N & \text{on } \partial\Omega.\end{cases}$$

We will use the technique of proof by induction. Starting from $m=1$ and $u_s^{(0)}=\hat{u}_s$. Since by Definition 1, the components $\hat{u}_i$ of $\hat{u}_s$ satisfy the relations

$$(4.4):\begin{cases}-\Phi_i[\hat{u}_i]+\gamma_i\hat{u}_i\leq F(x,\hat{u}_s)=F(x,\underline{u}_s^{(0)}) & \text{in } \Omega\\ \hat{u}_i(x)\leq g_i(x)\ ,\ 1\leq i\leq N & \text{on } \partial\Omega,\end{cases}$$

and therefore $\tilde{u}_i$ of $\tilde{u}_s$ satisfy

$$\begin{cases}-\Phi_i[\tilde{u}_i]+\gamma_i\tilde{u}_i\geq F(x,\tilde{u}_s)\geq F(x,\underline{u}_s^{(0)}) & \text{in } \Omega\\ \tilde{u}_i(x)\geq g_i(x)\ ,\ 1\leq i\leq N & \text{on } \partial\Omega\end{cases}$$

Similarly, by considering the case $m=1$ and $u_s^{(0)}=\tilde{u}_s$, we get

$$(4.5):\begin{cases}-\Phi_i[\hat{u}_i]+\gamma_i\hat{u}_i\leq F(x,\hat{u}_s)\leq\\ F(x,\tilde{u}_s)=F\left(x,\overline{u}_s^{(0)}\right) & \text{in } \Omega\\ \hat{u}_i(x)\leq g_i(x)\ ,\ 1\leq i\leq N & \text{on } \partial\Omega,\end{cases}$$

and therefore $\tilde{u}_i$ of $\tilde{u}_s$ satisfy

$$\begin{cases}-\Phi_i[\tilde{u}]+\gamma_i\tilde{u}_i\geq F(x,\tilde{u}_s)=F\left(x,\overline{u}_s^{(0)}\right) & \text{in } \Omega\\ \tilde{u}_i(x)\geq g_i(x)\ ,\ 1\leq i\leq N & \text{on } \partial\Omega.\end{cases}$$

We see that $\tilde{u}_i$ and $\hat{u}_i$ are ordered upper and lower solutions of (4.3) for the case $m=1$. By Theorem 2, problem (4.3) has also a minimal solution $\underline{u_i}$ and a maximal solution $\overline{u_i}$ such that $\hat{u}_i\leq\underline{u}_i\leq\overline{u}_i\leq\tilde{u}_i$. We choose $\underline{u}_i$ (or $\overline{u}_i$) as $\underline{u}_i^{(1)}$

if $u_s^{(0)} = \hat{u}_s$ and $\overline{u_i}$ (or $\underline{u}_i$) as $\overline{u}_i^{(1)}$ if $u_s^{(0)} = \tilde{u}_s$. So we get $\hat{u}_i \le \underline{u}_i^{(1)} \le \overline{u}_i^{(1)} \le \tilde{u}_i$.

This proves that $\underline{u}_s^{(1)} \equiv \left(\underline{u}_1^{(1)}, \dots, \underline{u}_N^{(1)}\right)$ and $\overline{u}_s^{(1)} \equiv \left(\overline{u}_1^{(1)}, \dots, \overline{u}_N^{(1)}\right)$ are solutions of (4.1) for $m = 1$ and satisfy $\hat{u}_s \le \underline{u}_s^{(1)} \le \overline{u}_s^{(1)} \le \tilde{u}_s$.

Assume that $\underline{u}_s^{(m-1)} \le \underline{u}_s^{(m)} \le \overline{u}_s^{(m)} \le \overline{u}_s^{(m-1)}$ for $m > 1$. By Lemma 1, we have

$$\begin{cases} -\Phi_i\left[\underline{u}^{(m)}\right] + \gamma_i \underline{u}_i^{(m)} = F\left(x, \underline{u}_s^{(m-1)}\right) \le F\left(x, \underline{u}_s^{(m)}\right) \\ -\Phi_i\left[\overline{u}^{(m)}\right] + \gamma_i \overline{u}_i^{(m)} = F\left(x, \overline{u}_s^{(m-1)}\right) \ge F\left(x, \overline{u}_s^{(m)}\right) \\ \underline{u}_i^{(m)} = \overline{u}_i^{(m)} = g_i(x)\ ,\ 1 \le i \le N. \end{cases}$$

This implies that $\overline{u}_i^{(m)}$, $\underline{u}_i^{(m)}$ are ordered upper and lower solutions of (4.3) when $(m-1)$ is replaced by m and $u_s^{(m)}$ is either $\underline{u}_s^{(m)}$ or $\overline{u}_s^{(m)}$ Again by Theorem 2, problem (4.3) has a minimal solution $\underline{u}_i$ and a maximal solution $\overline{u}_i$.

We choose $\underline{u}_i$ (or $\overline{u}_i$) as $\underline{u}_i^{(m+1)}$ if $u_s^{(m)} = \underline{u}_s^{(m)}$ and $\underline{u}_i$ (or $\overline{u}_i$) as $\overline{u}_i^{(m+1)}$ if $u_s^{(m)} = \overline{u}_s^{(m)}$, which gives us $\underline{u}_i^{(m)} \le \underline{u}_i^{(m+1)} \le \overline{u}_i^{(m+1)} \le \overline{u}_i^{(m)}$.

This choice ensures that $\underline{u}_s^{(m+1)} \equiv \left(\underline{u}_1, \dots, \underline{u}_N^{(m+1)}\right)$ and $\overline{u}_s^{(m+1)} \equiv \left(\overline{u}_1^{(m+1)}, \dots, \overline{u}_N^{(m+1)}\right)$ are solutions of (4.1) and possess the monotone property (4.2).

V. PROOF OF THE MAIN RESULT

We will dedicate this paragraph to the proof of the main result**.**

Proof of Theorem 1. In view of Lemma 3, the pointwise limits

$$(5.1):\ \lim_{m\to\infty} \underline{u}_s^{(m)} = \underline{u}_s \quad , \quad \lim_{m\to\infty} \overline{u}_s^{(m)} = \overline{u}_s$$

exist and satisfy $\hat{u}_s \le \underline{u}_s \le \overline{u}_s \le \tilde{u}_s$. To prove that $\underline{u}_s$ and $\overline{u}_s$ are respectively the minimal and maximal solutions of (1.1). We first consider the minimal sequence $\left\{\underline{u}_s^{(m)}\right\} \equiv \left\{\underline{u}_1^{(m)}, \dots, \underline{u}_N^{(m)}\right\}$. Define for each m and $1 \le i \le N$:

$$\begin{cases} \underline{w}_i^{(m)}(x) = I_i\left(\underline{u}_i^{(m)}\right) = \int_0^{\underline{u}_i^{(m)}} D_1(s)ds \\ \underline{Q}_i^{(m)}(x) = -\gamma_i(x)\underline{u}_i^{(m)} + F_i\left(x, \underline{u}^{(m-1)}\right). \end{cases}$$

It is obvious that for all $1 \le i \le N$, $I_i'\left(\underline{u}\right) = D_i\left(\underline{u}\right)$, and the inverses of $I_i\left(\underline{u}\right)$ exist and are denoted by $q_i\left(\underline{w}_i\right)$. Now we can write problem (4.1) in the following scalar form

$$(5.2):\begin{cases} -\nabla^2 \underline{w}_i^{(m)} = \underline{Q}_i^{(m)}(x) & \text{in } \Omega \\ \underline{w}_i^{(m)}(x) = g_i^*(x) = I_i(g_i) \ge 0 & \text{on } \partial\Omega. \end{cases}$$

It is clear from (5.1) and (2.4) that $\underline{w}_i^{(m)} \to \underline{w}_i \equiv I_i\left(\underline{u}\right)$ and $\underline{Q}_i^{(m)} \to f_i\left(x, \underline{u}_s\right)$ as $m \to \infty$. By the argument in the proof for the scalar problem (3.1), $\underline{w}_i$ is the unique solution of the linear problem

$$\begin{cases} -\nabla^2 \underline{w}_i^{(m)}(x) = \underline{Q}_i^{(m)}(x) \\ \underline{w}_i^{(m)}(x) = g_i^*(x)\ , 1 \le i \le N \end{cases}$$

This show that $\underline{u}_s \equiv \left(\underline{u}_1, \dots, \underline{u}_N\right)$, where $\underline{u}_i = q_i\left(\underline{w}_i\right)$ for $1 \le i \le N$ is a solution of (1.1) and $\underline{u}_s \in S^*$.

Now, we show that $\overline{u}_s$ is a solution of (1.1) in S^*, for this we consider the maximal sequence $\left\{\overline{u}_s^{(m)}\right\} \equiv \left\{\overline{u}_1^{(m)}, \dots, \overline{u}_N^{(m)}\right\}$. Define for each m and for $1 \le i \le N$

$$\begin{cases} \overline{w}_i^{(m)}(x) = I_i\left(\overline{u}_i^{(m)}\right) = \int_0^{\overline{u}_i^{(m)}} D_i(s)ds \\ \overline{Q}_i^{(m)}(x) = -\gamma_i(x)\overline{u}_i^{(m)} + F_i\left(x, \overline{u}^{(m-1)}\right). \end{cases}$$

Then, the quasilinear problem (4.1) may be written as the scalar linear problem

$$\begin{cases} -\nabla^2 \overline{w}_i^{(m)} = \overline{Q}_i^{(m)}(x) & \text{in } \Omega \\ \overline{w}_i^{(m)}(x) = g_i^*(x) & \text{on } \partial\Omega. \end{cases}$$

It is clear from (5.1) and (2.4) that $\overline{w}_i^{(m)} \to \overline{w}_i \equiv I_i\left(\underline{u}_i\right)$, $\overline{Q}_i^{(m)} \to f_i(x, \overline{u}_s)$, for $1 \le i \le N$ as $m \to \infty$. As in the proof of the scalar problem, $\overline{w}_1$ is the unique solution of problem (5.2). This show that $\overline{u}_s \equiv (\overline{u}_1, \dots, \overline{u_N})$, where $\overline{u_i} = q_i(\overline{w}_i)$ is a solution of (1.1) and $\overline{u}_s \in S^*$.

To show that $\underline{u}_s$ and $\overline{u}_s$ are, respectively, minimal and maximal solutions of (1.1) in S^*. We observe that every solution $u = (u_1, \dots, u_N)$ of (1.1) in S^* satisfies for each $1 \le i \le N$

$$\begin{cases} -\Phi_i[u_i] + \gamma_i u_i = F(x, u_s) \ge F\left(x, \underline{u}_s^{(0)}\right) & \text{in } \Omega \\ u_i(x) = g_i(x) & \text{on } \partial\Omega. \end{cases}$$

By (4.1) (with $m = 1$ and $u_i^{(1)} = \underline{u}_i^{(1)}$ for $1 \le i \le N$) we have

$$F_i\left(x, \underline{u}_s^{(0)}\right) = -\Phi_i\left[\underline{u}_i^{(1)}\right] + \gamma_i \underline{u}_i^{(1)},$$

then

$$-\Phi_i[u_i] + \gamma_i u_i \ge -\Phi_i\left[\underline{u_i}^{(1)}\right] + \gamma_i \underline{u_i}^{(1)}.$$

By lemma 2 we have for each $1 \le i \le N$, $u_i \ge \underline{u}_i^{(1)}$, i.e., $u \ge \underline{u}_s^{(1)}$. This implies, by Lemma 1, that $F_i(x, u) \ge F_i\left(x, u_s^{(1)}\right)$. It follows by induction

$$F_i(x, u) \ge F_i\left(x, \underline{u}_s^{(1)}\right) \ge \cdots \ge F_i\left(x, \underline{u}_s^{(m)}\right),$$

then

$$u \ge \underline{u}_s^{(m)}\ \text{, for every } m \ge 1.$$

By following the same steps, we observe that every solution $u = (u_1, \dots, u_N)$ of (1.1) in S^* satisfies

$$\begin{cases} -\Phi_i[u_i] + \gamma_i u_i = F_i(x,u) \leq F_i\left(x, \overline{u}_s^{(0)}\right) & \text{in } \Omega \\ u_i(x) = g_i(x) \ , \ 1 \leq i \leq N & \text{on } \partial\Omega. \end{cases}$$

By (4.1) (with $m = 1$ and $u_i^{(1)} = \overline{u}_i^{(1)}$ for $1 \leq i \leq N$), we have

$$F_i\left(x, \overline{u}_s^{(0)}\right) = -\Phi_i\left[\overline{u}_i^{(1)}\right] + \gamma_i \overline{u}_i^{(1)},$$

then

$$-\Phi_i[u_i] + \gamma_i u_i \leq -\Phi_i\left[\overline{u}_i^{(1)}\right] + \gamma_i \overline{u}_i^{(1)}.$$

By lemma 2, we obtain $u_i \leq \overline{u}_i^{(1)}$ for $1 \leq i \leq N$, i.e., $u_s \leq \overline{u}_s^{(1)}$. This gives us, by Lemma 1, $F_i(x,u) \leq F_i\left(x, \overline{u}_s^{(1)}\right)$. By an induction, we get

$$F_i(x,u) \leq F_i\left(x, \overline{u}_s^{(1)}\right) \leq \cdots \leq F_i\left(x, \overline{u}_s^{(m)}\right),$$

which implies $u_s \leq \overline{u}_s^{(m)}$.
Letting $m \to \infty$ and using relation (5.1) lead to $\underline{u}_s \leq u \leq \overline{u}_s$, which proves the minimal and maximal property. Finally, if $\underline{u}_s = \overline{u}_s \ (\equiv u_s^*)$ then this maximal-minimal property ensures that u_s^* is the unique positive solution in S^*.

REFERENCES

1. N. Alaa, S. Mesbahi, W. Bouarifi. Global existence of weak solutions for parabolic triangular reaction diffusion systems applied to a climate model. An. Univ. Craiova Ser. Mat. Inform., 42 (1) (2015) 80-97.
2. N. Alaa, S. Mesbahi, A. Mouida, W. Bouarifi. Existence of solutions for quasilinear elliptic degenerate systems with L^1 data and nonlinearity in the gradient. Electron. J. Diff. Equ., 142 (2013) 1-13.
3. P. Alvarez-Caudevilla, Y. Du, R. Peng. Qualitative Analysis of a Cooperative Reaction-Diffusion System in a Spatiotemporally Degenerate Environment. SIAM Journal on Mathematical Analysis, 46 (2014) 499-531.
4. J. R. Anderson. Local existence and uniqueness of solutions of degenerate parabolic equations. Comm. Partial Differential Equations, 16 (1991) 105-143.
5. J. Deuel, P. Hess. Nonlinear parabolic boundary value problems with upper and lower solutions. Israel J. Math., 29 (1978) 92-104.
6. A. Einav, J. Morgan, B.Q. Tang. Indirect diffusion effect in degenerate reaction-diffusion systems. SIAM Journal on Mathematical Analysis, 2020, arXiv:2001.00852 [math.AP].
7. W. Fitzgibbon, M. Langlais, J. Morgan, A degenerate reaction-diffusion system modeling atmospheric dispersion of pollutants. Journal of Mathematical Analysis and Applications, 307 (2005) 415-432.
8. O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural'ceva. Linear and Quasi-Linear Equations of Parabolic Type. Amer. Math. Soc., Providence, RI, 1968 (English transl.).
9. P. Lei, S.N. Zheng. Global and nonglobal weak solutions to a degenerate parabolic system. J. Math. Anal. Appl., 324 (2006) 177-198.
10. S. Mesbahi. Analyse mathématique de systèmes de réaction diffusion quasi-linéaires. Editions Universitaires Européennes, 2019. ISBN-13 : 978-6138492917.
11. S. Mesbahi, N. Alaa. Mathematical analysis of a reaction diffusion model for image restoration. An. Univ. Craiova Ser. Mat. Inform., 42 (1) (2015) 70-79.
12. S. Mesbahi, N. Alaa. Existence result for triangular reaction diffusion systems with L^1 data and critical growth with respect to the gradient. Mediterr. J. Math., 10 (2013) 255-275.
13. J.D. Murray. Mathematical Biology I : An Introduction. Springer-Verlag, New York Inc.; 3rd edition, 2002.
14. J.D. Murray. Mathematical Biology II : Spatial Models and Biochemical Applications. Springer-Verlag, New York Inc.; 3rd edition, 2002.
15. C.V. Pao, W. H. Ruan. Positive solutions of quasilinear parabolic systems with Dirichlet boundary condition. J. Differential Equations, 248 (2010) 1175-1211.
16. C.V. Pao. Quasilinear parabolic and elliptic equations with nonlinear boundary conditions. Nonlinear Anal., 66 (2007) 639-662.
17. C.V. Pao. Nonlinear Parabolic and Elliptic Equations. North Carolina State University, Springer US, 1992.
18. C.V. Pao, L. Zhou, X.J. Jin. Multiple solutions of a boundary value problem in enzyme kinetics. Advances in Appl. Math., 6 (1985) 209-229.
19. K.I. Saffidine, S. Mesbahi. Existence Result for Positive Solution of a Degenerate Reaction-Diffusion System via a Method of Upper and Lower Solutions. Nonlinear Dyn. Syst. Theory, 21 (4) (2021) 434--445.
20. X.S. Wang, X.Q. Zhao. Pulsating waves of a partially degenerate reaction-diffusion system in a periodic habitat. J. Differential Equations, 259 (2015) 7238-7259.