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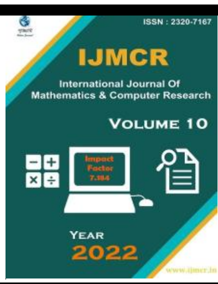

Fixed Point Theorems for (ψ,α) Type Expansive Mappings in b-Metric Spaces

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ABSTRACT
In this paper, we confine over selves to obtain some results on fixed point theorems for a new category of expansive mappings called (ψ,α) expansive mapping in b- metric spaces. Our results are with much shorter proof and generalize many existing results in the literature. We also have given some examples to support our results.

KEYWORDS: Fixed Point**,** Expansive Mappings, b- Metric Space etc.
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I. INTRODUCTION

During the last 95- years a lot of fixed point theorems have been established and we find that Banach contraction principle is at the base of the most of these results established so far. The concept of metric spaces has been generalized in many directions. The notion of a b-metric space was introduced by Czerwik in [11, 12] and during the last few years by many authors a lot of fixed point theorems have been proved in b-metric spaces. Recently, Samet et. al**.**[30] studied a new class of (α, ψ) type contraction andα- admissible mapping

The following definitions are required in sequel.

2. PRELIMINARIES

Definition 2.1 ([6]) Let X be a nonempty set. A mapping d : $X \times X \to [0, \infty)$ is called b-metric if there exists a real number $b \geq 1$ such that for every x, y, z ∈ X, we have

(i) d(x, y) = 0 if and only if x = y
(ii) d(x, y) = d(y, x)
(iii) $d(x, z) \leq b[d(x, y) + d(y, z)]$

In this case the pair (X, d) is called a b-metric space.

There exists many examples in the literature (see[6-8])(BS) showing that every metric function is a b-metric function with b =1, while the converse is not true, i.e. the class b- metric is effectively larger than that of ordinary metric spaces.

Definition 2.2 [15] Let $\{x_n\}$ be a sequence in a b-metric space (X, d).

(i) $\{x_n\}$ is called b-convergent if and only if there is x ∈ X such that $d(x_n, x) \to 0$ as $n \to \infty$.
(ii) $\{x_n\}$ is called b-Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

A b-metric space is said to be complete if and only if each b-Cauchy sequence in this space is b-convergent.

Recently, Samet et al.[30] considered the following family of functions and presented the new notion of (α - ψ)-contractive and α - admissible mappings.

Definition 2.3 ([31]) Let φ denote the family of all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following :

(i) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each t > 0, where ψ^n is the nth iterate of ψ;
(ii) ψ is non-decreasing.

Definition 2.4 ([31]) Let (X, d) be a metric space and $T : X \to X$ be a self-mapping. T is said to be an (α, ψ) – contractive mapping if there exists two functions $\alpha : X \times X \to [0, \infty)$ and ψ ∈ φ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \text{ for all } x, y \in X.$$

Definition 2.5 ([31]) Let (X, d) be a metric space and $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. T is said to be α-admissible if

$$x, y \in X,\ \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Now we present an example of α-admissible mappings.

Example2.6 Let X be the set of all non-negative real numbers. Let us define the mapping

$$\alpha : X \times X \to [X, \infty) \text{ by}$$

$$\alpha(x, y) = \begin{cases} 2^{x-y}, & if\ x \geq y \\ 0, & if\ x < y \end{cases}$$

and define the mapping $T : X \to X$ by $Tx = 2^x$ for all $x \in X$. Then T is α-admissible.

Let K denote all functions $\psi : [X, \infty) \to [X, \infty)$ which satisfy the following properties:

(i) ψ is non-decreasing;

(ii) $\sum_{n=1}^{\infty} \psi^{n}(a) < \infty$ for each $a > 0$, where ψ^{n} is the nth iterate of ξ;

(iii) $\psi(a + b) = \psi(a) + \psi(b)$

Lemma 2.7 [31] *If* ψ: [X, ∞) → [X, ∞) *is a non-decreasing function, then for each a* > 0,

$$\lim_{n\to\infty} \psi^{n}(a) \text{ implies } \psi(a) < a .$$

Definition 2.8 [26] Let (*X*, *d*) be a metric space and *T* : *X* → *X* be a given mapping. We say that *T* is (ψ, α)-expansive mappings if there exist two functions ψ : [X, ∞) → [X, ∞) and $\alpha : X \times X \to$ [X, ∞) such that

$\psi(d(Tx,Ty)) \geq \alpha(x, y)d(x, y)$ for all x, y ∈ X.

In what follows, we present the main results of Samet *et al.* [11].

Theorem 2.9 [31] Let (X, d) be a complete metric space and T : X → X be an α-ψ contractive mapping satisfying the following conditions:

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is continuous.

Then T has a fixed point, that is, there exists x ∈ X such that Tx = x

Theorem 2.10 [31] Let (X, d) be a complete metric space and T : X → X be an α-ψ- contractive mapping satisfying the following conditions:

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.

Then T has a fixed point.

Samet et al. [31] added the following condition (H) to the hypotheses of Theorem 2.10 and Theorem 2.10 to assure the uniqueness of the fixed point:

(H) For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

We introduce here a new notion of (ψ, α)-expansive mappings as follows:

Let K denote the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ which satisfy the properties:

(i) ψ is non-decreasing ;

(ii) $\sum_{i,j=1}^{\infty} b^{i} \psi^{j}(a) < +\infty$ for each a > 0, where ψ^{j} is the j[th] iterate of ψ ;

(iii) $\psi(a + b) = \psi(a) + \psi(b)$, ∀ a, b ∈ [0, +∞)

Lemma 2.11 [31] If $\psi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function, then for each a > 0,

$$\lim_{n\to+\infty} \psi^{n}(a) = 0 \text{ implies } \psi(a) < a.$$

Definition 2.12 [26] Let (X, d) be a metric space and T : X → X be a given mapping. We say that T is an (ψ,α) -expansive mapping if there exist two functions $\psi \in K$ and $\alpha : X \times X \to [0, +\infty)$ such that

(A) $\psi(d(Tx,Ty)) \geq \alpha(x,y)d(x,y)$

for all x, y ∈ X.

Remark 2.13 If T : X → X is an expansion mapping, then T is an (ψ, α)-expansive mapping, where $\alpha(x, y) = 1$ for all x, y ∈ X and $\psi(a) = sa$ for all $a \geq 0$ and some s ∈ [0,1).

Throughout this paper we shall making use of the standard notations and terminologies of nonlinear analysis.

3. MAIN RESULTS

Theorem 3.1 Let (X, d) be a complete b- metric space with constant $b \geq 1$ and T : X → X be a bijective, (ψ, α) - expansive mapping satisfying the following conditions:

(i) T^{-1} is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$;

(iii) T is continuous.

Then T has a fixed point, that is, there exists z ∈ X such that Tz = z.

Proof : Let us define the sequence $\{x_n\}$ in X by $x_n = Tx_{n+1}$, for all n ∈ N,
where $x_0 \in X$ is such that $\alpha(x_0, T^{-1}x_0) \geq 1$. Now, if $x_n = x_{n+1}$ for any n ∈ N, then from the definition $\{x_n\}$ it can be easily seen that x_n is a fixed point of T. Hence, without loss of generality, we may assume $x_n \neq x_{n+1}$ for each n ∈ N.

It is given that $\alpha(x_0, x_1) = \alpha(x_0, T^{-1}x_0)$. Recalling that T^{-1} is α-admissible, therefore, we have

$\alpha(T^{-1}x_0, T^{-1}x_1) = \alpha(x_0, x_1) \geq$
1

Using mathematical induction, we obtain

(3.1.1) $\alpha(x_n, x_{n+1}) \geq 1$ for all n ∈ N.

Using (3.1.1) and applying the inequality (A) with $x = x_n$, and $y = x_{n+1}$, we obtain

$d(x_n, x_{n+1}) \leq \alpha(x_n, x_{n+1})d(x_n, x_{n+1}) \leq \psi(d(Tx,Ty)) = \psi(d(x_{n-1},x_n))$

Therefore, by repetition of the above inequality, we have that

$$d(x_n, x_{n+1}) \leq \psi^{n}(d(x_0,x_1)), \text{ for all } n \in N.$$

For any $n > m \geq 0$, we have

$d(x_m, x_n) \leq bd(x_m, x_{m+1}) + b^2d(x_{m+1}, x_{m+2}) + . . . + b^{n-m}d(x_{n-1}, x_n)$

$\leq (b\psi^{m} + b^2\psi^{m+1} + . . . + b^{n-m}\psi^{n-1})d(x_0, x_1)$

From $\sum_{i,j=1}^{\infty} b^{i} \psi^{j}(a) < +\infty$ for each a > 0, it follows that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). So, there exists z ∈ X such that $x_n \to z$ as $n \to +\infty$. From the continuity of *T*, it follows that $x_n = Tx_{n+1} \to Tz$ as $n \to +\infty$. By the uniqueness of the limit, we get *z* = *Tz*, that is, *z* is a fixed point of *T*. This completes the proof.

In what follows, we prove that Theorem 3.1 still holds for T not necessarily continuous, assuming the following condition:

(B): If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\} \to u \in X$ as $n \to +\infty$, then

(3.1.2) $\alpha(x_0, T^{-1}x_0) \geq 1$, for all n.

Theorem 3.2 If in Theorem 3.4 we replace the continuity of T by the condition (B), then the result holds true.

Proof: Following the proof of Theorem 3.1, we know that $\{x_n\}$ is a sequence in X such that

$d(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\} \to z \in X$ as $n \to +\infty$. Now, from the hypothesis (3.1.2), we have

(3.2.1) $\alpha(T^{-1}x_n, T^{-1}z) \geq 1$, for all $n \in N$.

From (3.2.1), (3.1.2) and b-triangular inequality, we have

$$d(T^{-1}z, z) \leq b(d(T^{-1}z, x_{n+1}) + d(x_{n+1}, z)) = b\alpha(T^{-1}z, T^{-1}x_n)d(T^{-1}z, T^{-1}x_n) + bd(x_{n+1}, z)$$
$$\leq b\psi(d(x_n, z) + bd(x_{n+1}, z))$$

Continuity of ψ at t = 0 implies that $(T^{-1}z, z) = 0$ as $n \to +\infty$. That is, $T^{-1}z = z$.

Consider, $Tz = T(T^{-1}z) = (TTT^{-1})z = z$. This gives an end to the proof. Q

We now present some examples in support of our results.

Example 3.3 Let X = [0, +∞) endowed with standard metric d(x, y) = |x – y| for all x, y ∈ X. Define the mappings T : X → X and α : X × X → [0, +∞) by

$$T(x) = \begin{cases} 2x - \frac{7}{4}, & x \geq 1, \\ \frac{x}{4}, & x \in [0,1) \end{cases} \quad \text{and} \quad \alpha(x, y) = \begin{cases} 0, & x, y \in [0,1), \\ \frac{x}{4}, & otherwise \end{cases}$$

Clearly, T is an (ψ,α)-expansive mapping with $\psi(t) = t/4$ for all $t \geq 0$.

In fact, for all x, y ∈ X, we have $\frac{1}{4}d(Tx, Ty) \geq \alpha(x, y)d(x, y)$.

Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$. In fact, for $x_0 = 1$, we have $\alpha(1, T^{-1}1) = 1$.

Obviously, T is continuous, and so it remains to show that T^{-1} is α-admissible. For this, let x, y ∈ X such that α(x, y) ≥ 1. This implies that x ≥ 1 and y ≥ 1, and by the definitions of T^{-1} and α, we have

$T^{-1}x = \frac{x}{2} + \frac{7}{8} \geq 1$, $T^{-1}y = \frac{y}{2} + \frac{7}{8} \geq 1$ and $\alpha(T^{-1}x, T^{-1}y) = 1$,

Then T^{-1} is α-admissible.

Now, all the hypotheses of Theorem 3.4 are satisfied. Consequently, T has a fixed point. In this example, 0 and $\frac{7}{4}$ are two fixed points of T.

Example 3.4 Let X = [0, ∞) endowed with the standard metric d(x, y) = |x – y| for all x, y ∈ X. Define the mappings T : X → X and α : X × X → [0, ∞) by

$$T(x) = \begin{cases} \sqrt{x}, & x \geq 1, \\ \frac{x}{4}, & x \in [0,1) \end{cases} \quad \text{and} \quad \alpha(x, y) = \begin{cases} 0, & x, y \in [0,1), \\ \frac{x}{4}, & otherwise \end{cases}$$

Due to the discontinuity of T at 1, Theorem 3.1 is not applicable in this case. Clearly, T is an (ψ, α)-expansive mapping with $\psi(t) = t/4$ for all $t \geq 0$. In fact, for all x, y ∈ X, we have

$\frac{1}{4}d(Tx, Ty) \geq \alpha(x, y)d(x, y)$.

Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$. In fact, for $x_0 = 1$, we have $\alpha(1, T^{-1}1) = 1$

Now, let x, y ∈ X such that α(x, y) ≥ 1.

This implies that x ≥ 1, y ≥ 1 and by the definition of T^{-1} and α, we have

$T^{-1}x = x^{1/4} \geq 1$, $T^{-1}y = y^{1/4} \geq 1$ and $\alpha(T^{-1}x, T^{-1}y) = 1$. Then T^{-1} is α-admissible.

Finally, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for a n and $\{x_n\} \to x \in X$ as $n \to \infty$.

Since $\alpha(x_n, x_{n+1}) \geq 1$ for all *n*, by the definition of *α*, we have $x_n \geq 1$ for all *n* and $x \geq 1$. Then $\alpha(T^{-1}x_n, T^{-1}x) = 1$.

Therefore, all the required hypotheses of Theorem 3.2 are satisfied, and so *T* has a fixed point. Here, 0 and 1 are two fixed points of *T*.

Remark 3.5 As in the previous example, the expansion mapping theorem is not applicable in this case either. To ensure the uniqueness of the fixed point in Theorem 3.1 and 3.2 we consider the condition:

(C): For all $z_1, z_2 \in X$, there exists z ∈ X such that $\alpha(z_1, z) \geq 1$ and $\alpha(z_2, z) \geq 1$.

Theorem 3.6 Adding the condition (B) respectively to the hypotheses of Theorem 3.1 and Theorem 3.2, we get the uniqueness of the fixed point .

Proof Theorem 3.1 and 3.2, the set of fixed points is non-empty. We shall show that if z_1 and z_2 are two fixed points of T , that is, $T(z_1) = z_1$ and $T(z_2) = z_2$, then $z_1 = z_2$.

From the condition (C), there exists z ∈ X such that

(3.6.1) $\alpha(z_1, z) \geq 1$and $\alpha(z_2, z) \geq 1$

Recalling the α-admissible property of T^{-1}, we obtain from (3.6.1)

(3.6.2) $\alpha(z_1, T^{-1}z) \geq 1$ and $\alpha(z_2, T^{-1}z) \geq 1$, for all n ∈N

Therefore, by repeatedly applying the α-admissible property of T^{-1}, we get

(3.6.3) $\alpha(z_1, T^{-n}z) \geq 1$ and $\alpha(z_2, T^{-n}z) \geq 1$, for all n ∈ N..

Using the inequalities (3.2.1) and (3.6.3), we obtain

$\alpha(z_1, T^{-n}z) \leq \alpha(z_1, T^{-n}z)d(z_1, T^{-n}z) \leq \psi(d(z_1, T^{-n+1}z))$

On repeating the above inequality implies we obtain

$\alpha(z_1, T^{-n}z) \leq \psi^n(d(z_1, z))$, for all n ∈ N.

Thus, we have $T^{-n}z \to z$ as $n \to \infty$.

Using the similar steps as above, we obtain $T^{-n}z \to z_2$ as $n \to \infty$.

Now, the uniqueness of the limit of $T^{-n}z$

gives us $z_1 = z_2$.
This completes the proof.

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