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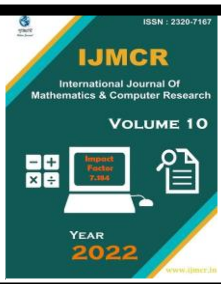

New Examples of Radical Join-Meet Ideals

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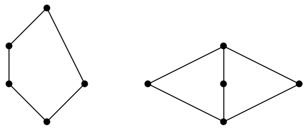
ABSTRACT
For non-distributive lattice L, it is not yet known classes of L with the property that the join-meet ideal of L is radical. In this paper, we give a partial answer to this problem.

KEYWORDS: Finite lattices, Non-distributive non-modular lattices, Join-meet ideal, Radical ideal, Compatible monomial order, Gr¨obner bases.
Mathematics Subject Classification. 13P10, 06A11, 06B05.

1. INTRODUCTION

Let L be a finite lattice and $K[L]$ be the polynomial ring over a field K whose variables are the elements of L. The ideal
$I_L = (\{f_{a,b} := x_a x_b - x_{a \vee b} x_{a \wedge b} \mid a,b \in L\}) \subset K[L]$
is called the join-meet ideal of L. It was introduced in 1987 by Hibi in [3]. As shown by [1] or [3], L is distributive if and only if I_L is a prime ideal. It follows from this result that I_L is radical when L is distributive. However, it is not yet completely known classes of non-distributive lattice L with the property that I_L is a radical ideal. On the other hand, for instance, it followed from [1] and [2] that there are some examples of non-distributive modular lattice such that I_L is a radical ideal.

Not all, here we briefly introduce three examples. First, the join-meet ideal of the pentagon lattice N_5 and diamond lattice M_5 is radical; see [1, Page 157] for detailed proof.



(a) The pentagon lattice N_5**(b)** The diamond lattice M_5
Figure 1. The Hasse diagram of the pentagon lattice N_5 and diamond lattice M_5

Second, for some integer $n \geq 1$, it exists a class of the distributive lattice of the divisors of $2 \cdot 3^n$ such that by including just one small diamond one get a radical join-meet ideal for the new lattice; see [2, Section 3] for detail.

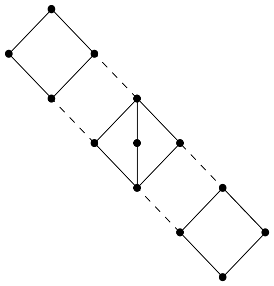


Figure 2. The Hasse diagram of the new non-distributive lattice made by including just one small diamond

In this paper, we introduce two new examples of non-distributive lattices L such that the join-meet ideal I_L is radical. Let k be non-negative integer with $k > 0$. For non-negative integer $n_1, \cdots, n_k \geq 1$, we denote by $L_k(n_1, \cdots, n_k)$ the finite lattice with the elements labeled as in Figure 3.

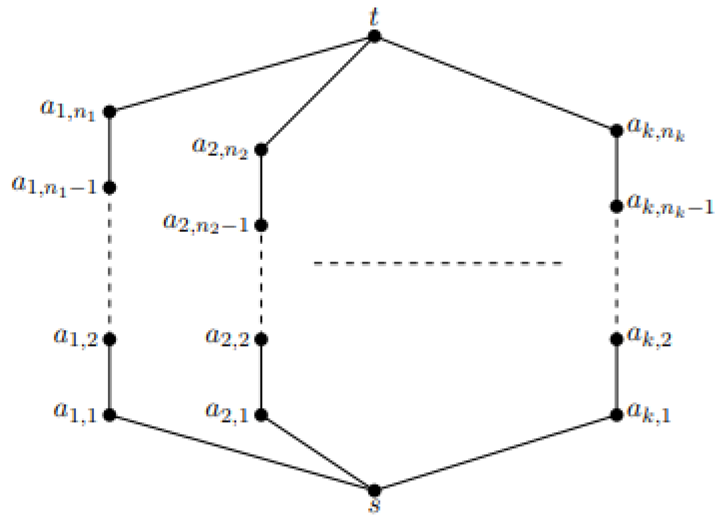


Figure 3. The Hasse diagram of new finite lattices

A finite lattice $L_k(n_1,\cdots,n_k)$ looks larger version of finite lattices introduced by [1, Problems 6.13] in the terms of the appearance of the Hasse diagram.

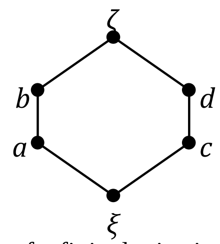


Figure 4. The Hasse diagram of a finite lattice introduced in [1, Problems 6.13]

Then, the following question arises. Is the join-meet ideal $I_{L_k(n_1,\cdots,n_k)}$ radical ? Unfortunately, we couldn't answer this question. On the other hand, for $k = 2,3$, the following results were obtained.

Theorem 1.1. *The join-meet ideal $I_{L_2(n_1,n_2)}$ is radical.*

Theorem 1.2. *For*

$$(n_1,n_2,n_3) = \begin{cases} (k_1,1,1), & 2 \le k_1 \le 10, \\ (k_2,k_3,1), & 2 \le k_2 \le 4, 2 \le k_3 \le 4, \\ (k_4,k_5,k_6), & 2 \le k_4,k_5,k_6 \le 3, \end{cases}$$

the join-meet idea $I_{L_3(n_1,n_2,n_3)}$ is radical.

By using new examples $L_2(n_1,n_2)$, we obtained new non-distributive non-modular lattices $L_2(n_1,n_2)[k',i_1,i_2]$ for non-negative integer k',n_1,n_2 satisfying certain conditions. We also obtained new distribtuive lattices O_{n_1}. Then, the following results were obtained.

Theorem 1.3. *The join-meet ideal $I_{L_2(n_1,n_2)[k',i_1,i_2]}$ is radical.*

Theorem 1.4. *The join-meet ideal I_{O_n} is radical.*

We checked that Theorem 1.3 and 1.4 are similar to [2, Theorem 3.3] in terms of the opposite approach. Unfortunately, since O_{n_1} is a distributive lattice, note that it not new exmaples. The detailed definition of $L_2(n_1,n_2)[k',i_1,i_2]$ and O_{n_1} is given in section 4.

This paper is organized as follows. In section 2, we introduce the proof of Theorem 1.1 and 1.2. In section 3, we introduce the conjectures that occur naturally by Theorem 1.1 and 1.2. Then, we give some thoughts. In section 4, we introduce the proof of Theorem 1.3 and 1.4. In section 5, we introduce topics related to O_{n_1}. The keywords of it are number theory and gorenstein ring. Note that it has little to do with the gist of this paper.

Below, unless otherwise noted, in order to avoid the complexity of notation, we denote

$a_{1,i} = a_i$ for $1 \le i \le n_1$, $a_{2,i} = b_i$ for $1 \le i \le n_2$, $a_{3,i} = c_i$ for $1 \le i \le n_3$.

Furthermore, in order to match the logical calculation, let a_i, b_i and c_i satisfy

$a_i,b_i,c_i = s$ for $i \le 0$, $a_i = t$ for $n_1 + 1 \le i$, $b_i = t$ for $n_2 + 1 \le i$, $c_i = t$ for $n_3 + 1 \le i$.

2. THE PROOF OF THEOREM 1.1 AND THEOREM 1.2

In this section, we introduce the proof of theorem 1.1 and theorem 1.2. Hereafter, in order to avoid the complexity of notation, let $n_1 = n$, $n_2 = m$, $n_3 = r$.

2.1. **The proof of Theorem 1.1.** Let

$$\begin{aligned} G(n,m) &= \{f_{a,b} = ab - st \mid a,b \in L_2(n,m)\}, \\ A_n &= \{a_i st - a_1 st \mid 1 < i \le n\}, \\ B_m &= \begin{cases} \emptyset, & m = 1, \\ \{b_i st - b_1 st \mid 1 < i \le m\}, & m > 1. \end{cases} \end{aligned}$$

The outline of the proof of Theorem 1.1 is to show that $I_{L2(n,m)}$ is squarefree with respect to the inverse lexicographic order induced by

$$s \prec a1 \prec \cdots \prec an \prec b1 \prec \cdots \prec bm \prec c1 \prec \cdots \prec cr \prec t. \quad (2.1.1)$$

To prove this claim, for $n > 1$, we show that the set $G_{n,m} \cup A_n \cup B_m$ is a Gr¨obner basis of $I_{L2(n,m)}$ with repsect to $\prec$. Below, by using **Buchberger's criterion**, we show each case when $m = 1$ and when it is not. Note that it is clearly that $I_{L2(1,1)}$ is radical; see Figure 5 , [1, Theorem 6.10(Dedekind)] and [1, Theorem 6.21].

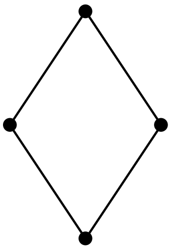


Figure 5. The Hasse diagram of a finite lattice $L_2(1,1)$

2.1.1. The case $m = 1$. First, for u and v belonging to $G_{n,1}$, we show that the S- polynomial $S(u,v)$ reduces to 0. Let i and j be non-negative integer with $1 \le i,j \le n$. Let $u_{i,j}$ denote the S-polynomial $S(a_ib_1 - st, a_jb_1 - st)$. If $i = j$, then we have $u_{i,j} = u_{i,i} = 0$. On the other hand, if $i \neq j$, we have

$u_{i,j} = a_j(a_ib_1 - st) - a_i(a_jb_1 - st) = -a_jst + a_ist.$ (2.1.2)

Thus, computational result of $u_{i,j}(i \neq j)$ is as Table 1. Hence, we showed that $S(u,v)$ reduces to 0.

Table 1. Computational result of $u_{i,j}(i \neq j)$

Value of i	Value of j	A standard expression of $u_{i,j}(i \neq j)$
$i = 1$	$j = 1$	0
	$j > 1$	$-(a_jst - a_1st)$
$i > 1$	$j = 1$	$a_ist - a_1st$
	$j > 1$	$(a_ist - a_1st) - (a_jst - a_1st)$

Second, for u and v belonging to $\in A_n$, we show that $S(u,v)$ reduces to 0. Let i and j be nonnegative integer with $2 \le i,j \le n$. Let $u_{i,j}$ denote the S-polynomial $S(a_ist - a_1st, a_jst - a_1st)$. If $i = j$, then we have $u_{i,j} = u_{i,i} = 0$. On the other hand, if $i \neq j$, then we have

$$u_{i,j} = a_j(a_ist - a_1st) - a_i(a_jst - a_1st) = -a_ja_1st + a_ia_1st = a_1(a_ist - a_1st) - a_1(a_jst - a_1st).$$

Hence, we showed that the $S(u,v)$ reduces to 0.

Finally, for (u,v) belonging to $G_{n,1} \times A_n$, we show that $S(u,v)$ reduces to 0. Let i and j be non-negative integer with $1 \le i \le n$, $1 < j \le n$. Let $u_{i,j}$ denote the S- polynomial

$S(a_ib_1 - st, a_jst - a_1st)$. If $i = j$, then we have $u_{i,i} = st(a_ib_1 - st) - b_1(a_ist - a_1st) = a_1b_1st - s^2t^2 = st(a_1b_1 - st)$. On the ohter hand, if $i \neq j$, then we have $u_{i,j} = a_jst(a_ib_1 - st) - a_ib_1(a_jst - a_1st) = -a_js^2t^2 + a_ia_1b_1st = a_1st(a_ib_1 - st) - st(a_jst - a_1st)$.

Hence, we showed that $S(u,v)$ reduces to 0.

Therefore, We showed that the set $G_{n,m} \cup A_n \cup B_1$ is a Gr¨obner basis of $I_{L2(n,1)}$ with repsect to $\prec$.

2.1.2. The case $m > 1$. First, for u and v belonging $G_{n,m}$, we show that the S- polynomial $S(u,v)$ reduces to 0. Let i, j, k and r be non-negative integer with $1 \le i \le n$, $1 \le j \le m$, $1 \le k,r \le m$. Let $u_{i,j,k,r}$ denote the S-polynomial $S(a_ib_j - st, a_kb_r - st)$. If $i = k$, then we have $u_{i,j,i,r} = -b_rst + b_jst$. Thus, computational result of $u_{i,j,i,r}$ from table 2. Hence, $u_{i,j,i,r}$ reduces to 0.

Table 2. Computational result of $u_{i,j,i,r}$

Value of j	Value of r	A standard expression of $u_{i,j,i,r}$
$j = 1$	$r = 1$	0
	$r > 1$	$-(b_r st - b_1 st)$
$j > 1$	$r = 1$	$b_j st - b_1 st$
	$r > 1$	$(b_j st - b_1 st) - (b_r st - b_1 st)$

On the other hand, for $j = r$, it follows that $u_{i,j,i,r}$ reduces to 0 by rewriting b_1 to b_j in (2.1.2) and using table 1. Therefore, we showed that $S(u,v)$ reduces to 0.

Second, for u and v belonging to $A_n \cup B_m$, we show that $S(u,v)$ reduces to 0. Since $S(u,v)(u,v \in A_n)$ and $S(u,v)(u,v \in B_m)$ reduce *to* 0 from the discussion in case $m = 1$, it suffices to prove that $S(u,v)$ reduces to 0, where $(u,v) \in A_n \times B_m$.

Now, let i and j be non-negative integer with $1 < i \leq n, 1 < j \leq m$. Let $u_{i,j}$ denote the S-polynomial $S(a_i st - a_1 st, b_j st - b_1 st)$. Then, the polynomial $u_{i,j}$ is computed as follows: $u_{i,j} = b_j(a_i st - a_1 st) - a_i(b_j st - b_1 st) = -a_1 b_j st + a_i b_1 st = st(a_i b_1 - st) - st(a_1 b_j - st)$.
Hence, we showed that $S(u,v)$ reduces to 0.

Finally, for (u,v) belonging to $G_{n,m} \times A_n \cup B_m$, we show that $S(u,v)$ reduces to 0. Let i, j, k and r be non-negative integer with $1 \leq i \leq n$, $1 \leq j, \leq m$, $1 < k \leq n$, $1 < r \leq m$. Let $u_{i,j,k}$ be the S-polynomial $S(a_i b_j - st, a_k st - a_1 st)$ and $u_{i,j,r}$ the S-polynomial $S(a_i b_j - st, b_k st - b_1 st)$. At first, about computational result of $u_{i,j,k}$, if $k = i > 1$, then we have $u_{i,j,k} = st(a_i b_j - st) - b_j(a_i st - a_1 st) = a_1 b_j st - s^2 t^2 = st(a_1 b_j - st)$. Hence, $u_{i,j,i}$ reduces to 0. On the other hand, if $i \neq k$, then it follows that $\mathrm{in}_{\prec}(a_i b_j - st) = a_i b_j$ and $\mathrm{in}_{\prec}(a_k st - a_1 st) = a_k st$ are relatively prime. Hence, for $i \neq k$, $u_{i,j,k}$ reduces to 0 with respect to $a_i b_j - st$, $a_k st - a_1 st$.
Next, about the computational result of $u_{i,j,r}$, if $r = j > 1$, then we have $u_{i,j,k} = st(a_i b_j - st) - a_i(b_j st - b_1 st) = a_i b_1 st - s^2 t^2 = st(a_i b_1 - st)$. Hence, $u_{i,j,j}$ reduces to 0. On the other hand, if $j \neq r$, it follows that $\mathrm{in}_{\prec}(a_i b_j - st) = a_i b_j$ and $\mathrm{in}_{\prec}(b_r st - b_1 st) = b_r st$ are relatively prime. Hence, for $j \neq r$, $u_{i,j,r}$ reduces to 0 with respect to $a_i b_j - st$, $b_r st - b_1 st$.

From the discussion of computational result of $u_{i,j,k}$ and $u_{i,j,r}$, we showed that $S(u,v)$ reduces to 0.

Therefore, We showed that the set $G_{n,m} \cup A_n \cup B_m$ is a Gr¨obner basis of $I_{L2(n,m)}$ with repsect to $\prec$.

2.1.3. Conclusion. The set $G_{n,m} \cup A_n \cup B_m$ is a Gr¨obner basis of $I_{L2(n,m)}$ with repsect to $\prec$.
Thus, we have $\mathrm{in}_{\prec}(I_{L2(n,m)}) = (\{a_i b_j \mid 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{a_i st \mid 1 \leq i \leq n\} \cup \{b_i st \mid 1 \leq i \leq m\})$.
Hence, $\mathrm{in}_{<}(I_{L2(n,m)})$ is squarefree with respect to $\prec$. Therefore, $I_{L2(n,m)}$ is radical.

2.2. The proof of Theorem 1.2. Let denote the following ideals:

$$
\begin{aligned}
En,m,r &\overset{=}{=} (a_1 - a_n, \cdots, a_1 - a_2, a_1 - b_m, \cdots, a_1 - b_1, a_1 - c_r, \cdots, a_1 - c_1, st - a_1^2),\\
Xn,m &= (s, a_1, \cdots, a_n, b_1, \cdots, b_m) \cap (a_1, \cdots, a_m, b_1, \cdots, b_m, t),\\
Ym,r &= (s, b_1, \cdots, b_m, c_1, \cdots, c_r) \cap (b_1, \cdots, b_m, c_1, \cdots, c_r, t),\\
Zn,r &= (s, a_1, \cdots, a_n, c_1, \cdots, c_r) \cap (a_1, \cdots, a_n, c_1, \cdots, c_r, t).
\end{aligned}
$$

The outline of proof is to show that all primary ideals appearing in the primary decomposition of $I_{L3}(n,m,r)$ is prime ideal.
First, by using Risa/Asir [4], we have

$$
\begin{aligned}
{}^{I}L_3(n,1,1) &= En,1 \cap Xn,1 \cap Y1,1 \cap Zn,1 && \text{for } n = 2,3,\cdots,10, && (2.2.1)\\
{}^{I}L_3(n,1,1) &= En,m \cap Xn,m \cap Ym,1 \cap Zn,1 && \text{for } 2 \leq n \leq 4,\ 2 \leq m \leq 4, && (2.2.2)\\
IL_3(n,m,r) &= En,m,r \cap Xn,m \cap Ym,r \cap Zn,r && \text{for } 2 \leq n,m,r \leq 3. && (2.2.3)
\end{aligned}
$$

We comment a little here. The above results were obtained by doing something like the following computation:
Listing 1. The computation of the primary decomposition of $I_{L3(3,1,1)}$ with Risa/Asir
[4]

```
load("primdec")$
primadec([a_1 * b_1 - s * t , a_1 * c_1 - s * t , a_2 * b_1 - s * t , a_2
    * c_1 - s * t , a_3 * b_1 - s * t , a_3 * c_1 - s * t , b_1 * c_1 - s
    * t],[t,c_1,b_1,a_3,a_2,a_1,s]);
[[[a_1-a_2,a_1-a_3,a_1-b_1,a_1-c_1,t*s-a_1^2],[a_1-a_2,a_1-a_3,a_1-b_1,a_1-
    c_1,t*s-a_1^2]],[[a_1,a_2,a_3,b_1,t],[a_1,a_2,a_3,b_1,t]],[[s,a_1,a_2,
    a_3,b_1],[s,a_1,a_2,a_3,b_1]],[[a_1,a_2,a_3,c_1,t],[a_1,a_2,a_3,c_1,t
    ]],[[s,a_1,a_2,a_3,c_1],[s,a_1,a_2,a_3,c_1]],[[b_1,c_1,t],[b_1,c_1,t
    ]],[[s,b_1,c_1],[s,b_1,c_1]]]
```

By (2.2.1), (2.2.2) and (2.2.3), we have

$$\begin{aligned}\sqrt{I_{L_3(n,1,1)}} &= \sqrt{E_{n,1,1}} \cap \sqrt{X_{n,1} \cap Y_{1,1} \cap Z_{n,1}} \quad \text{for } n = 2,3,\cdots,10,\\ \sqrt{I_{L_3(n,1,1)}} &= \sqrt{E_{n,m,1}} \cap \sqrt{X_{n,m} \cap Y_{m,1} \cap Z_{n,1}} \quad \text{for } 2 \le n \le 4,\ 2 \le m \le 4,\\ \sqrt{I_{L_3(n,m,r)}} &= \sqrt{E_{n,m,r}} \cap \sqrt{X_{n,m} \cap Y_{m,r} \cap Z_{n,r}} \quad \text{for } 2 \le n,m,r \le 3.\end{aligned}$$

Then, we have

$$\sqrt{I_{L_3(n,1,1)}} = \sqrt{E_{n,1,1}} \cap X_{n,1} \cap Y_{1,1} \cap Z_{n,1} \quad \text{for } n = 2,3,\cdots,10, \tag{2.2.4}$$

$$\sqrt{I_{L_3(n,1,1)}} = \sqrt{E_{n,m,1}} \cap X_{n,m} \cap Y_{m,1} \cap Z_{n,1} \quad \text{for } 2 \le n \le 4, 2 \le m \le 4, \tag{2.2.5}$$

$$\sqrt{I_{L_3(n,m,r)}} = \sqrt{E_{n,m,r}} \cap X_{n,m} \cap Y_{m,r} \cap Z_{n,r} \quad \text{for } 2 \le n,m,r \le 3. \tag{2.2.6}$$

In fact, It is clear from the following lemma.

Lemma 2.1. *Let* $\{i_1,\cdots,i_s\}$ *be a subset of* L*, where* $i_1 < i_2 < \cdots < i_s$*. Let* I *be the ideal* $(i_1,\cdots,i_s)$*. Then,* I *is prime ideal.*

Proof. Let a and b be the elemetns of $K[L]$ such that ab belongs to I. Suppose neither a nor b belongs to I. Then, a and b belong to the polynomial ring over K whose variables are the elements of $L\backslash\{i_1,\cdots,i_s\}$. Therefore, since a and b do not contain the variables $i_1,i_2,\cdots,i_s$, it contradicts that ab belongs to I. Hence, a is in I or b is in I. Therefore, I is prime ideal.

Now, by using Risa/Asir [4] , we computed the prime decomposition of $\sqrt{E_{n,m,r}}$ appearing in the right-hand side of (2.2.4) , (2.2.5) and (2.2.6). It was as follows:

$$\sqrt{E_{n,m,r}} = \begin{cases} E_{n,1,1}, & n = 2,3,\cdots,10,\\ E_{n,m,1}, & 2 \le n \le 4, 2 \le m \le 4,\\ E_{n,m,r}, & 2 \le n,m,r \le 3.\end{cases}$$

Hence, it follows from (2.2.4) , (2.2.5) and (2.2.6) that we have

$$\begin{aligned}\sqrt{I_{L_3(n,1,1)}} &= E_{n,1,1} \cap X_{n,1} \cap Y_{1,1} \cap Z_{n,1} \quad \text{for } n = 2,3,\cdots,10,\\ \sqrt{I_{L_3(n,1,1)}} &= E_{n,m,1} \cap X_{n,m} \cap Y_{m,1} \cap Z_{n,1} \quad \text{for } 2 \le n \le 4, 2 \le m \le 4,\\ \sqrt{I_{L_3(n,m,r)}} &= E_{n,m,r} \cap X_{n,m} \cap Y_{m,r} \cap Z_{n,r} \quad \text{for } 2 \le n,m,r \le 3.\end{aligned}$$

Therefore, we proved Theorem 1.2.

3. CRYSTAL CONJECTURE

In this section, we introduce the conjectures that occur naturally by Theorem 1.1 and 1.2. It is as follows:

Conjecture 3.1 (Crystal conjecture)**.** *The join-meet ideal* $I_{L_k(n_1,n_2,\cdots,n_k)}$ *is radical.*

We consider that Conjecture 3.1 is positive. The reason is as follows. By the proof of theorem 1.1, it was confirmed the existence of monomial order $\prec$ which is satisfying in $(L_2(n_1,n_2)) = \sqrt{\text{in}_{\prec}(L_2(n_1,n_2))}$. Furthermore, the method of constructing $\prec$ was simple. $\prec$ Hence, for $k = 3$, we can conjecture that there may be such a monomial order. Therefore, the following conjecture naturally occurs:

Conjecture 3.2. *For* $k \ge 3$ *and* $(n_1,\cdots,n_k) \neq (1,\cdots,1)$*, it exists a monomial order* $\prec'$ *such that* $I_{L_k(n_1,\cdots,n_k)}$ *is squarefree with respect to* $\prec'$*.*

Remark 3.3 (Reason for imposing $(n_1,\cdots,n_k) \neq (1,1,1)$)**.** By [1, Theorem 6.10 (Dedekind)], since we have $L_3(1,1,1) = M_5$, a finite lattice $L_k(1,1,1,\cdots,1)$ is non-distributive modular lattice. Hence, by [2, Theorem 1.3], it do not exist a monomial order such that $I_{L_k(1,\cdots,1)}$ is squarefree. From such a fact, it imposes $(n_1,\cdots,n_k) \neq (1,1,1)$ in Conjecture 3.2.

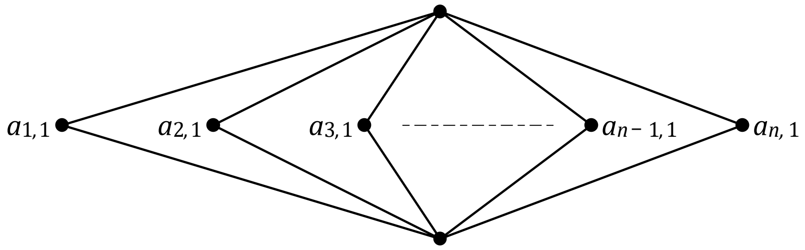


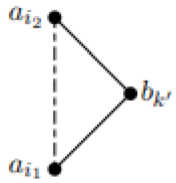
Figure 6. The Hasse diagram of non-distributive modular lattice $L_k(1,1,1,\cdots,1)$

In Conjecture 3.2, if the method of constructing $\prec'$ can be formulated as an algorithm that does not depend on k, Conjecture 3.1 be resolved. Hence, it is worth working on Conjecture 3.2. However, in the case $k = 3$, although we computed a lot with Risa/Asir [4], we are not yet checked a monomial order $\prec'$ which is satisfying $\mathrm{in}_{\prec'}(L_3(n_1,n_2,n_3)) = \sqrt{\mathrm{in}_{\prec'}(L_3(n_1,n_2,n_3))}$ for $(n_1,n_2,n_3) = (1,1,1)$. From this calculation experiment, unfortunately, Conjecture 3.2 may be negative. On the other hand, we can consider positively that it is very important result in terms of squarefree of join-meet ideal.

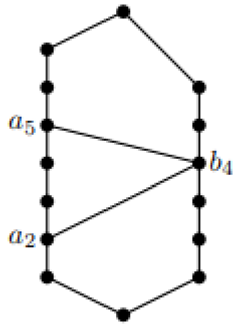
4. INVARIANCE OF RADICALITY BY ADDING NEW RELATIONSHIP

In this section, at first, we introduce a new finite lattice $L_2(n_1,n_2)[k',i_1,i_2]$ and O_{n_1} which is created by adding a new relationship to $L_2(n_1,n_2)$. Next, we prove Theorem 1.3 and 1.4. This result is similar to [2, Theorem 3.3] in terms of the opposite approach and it claims strongly invariance of radicality by adding a new relationship. Hereafter, we explain each $L_2(n_1,n_2)[^{k'},i_1,i_2]$ and O_{n_1} separately.

4.1. **A finite lattice** $L_2(n_1,n_2)[k',i_1,i_2]$**.** Let $n_1 \geq 5$ and $n_2 \geq 5$. Let i_1,i_2 be non-negative integer and let $i_1 > 1$, $4 < i_2 < n_2$ and $i_2-i_1 \geq 2$. Let k' be non-negative integer which satisfies $3 \leq k' \leq n_2 - 2$ and $k' \neq n_1,n_2$. We denote $L_2(n_1,n_2)[k',i_1,i_2]$ by $L_2(n_1,n_2)$ which satisfies $a_{i1} < b_{k'}$, $b_{k'} < a_{i2}$. Figure 7 (A) displays The Hasse diagram of a poset $\{a_{i1},\cdots,a_{i2},b_{k'}\}$. By [1, Theorem 6.10 (Dedekind)], note that $L_2(n_1,n_2)[k',i_1,i_2]$ be a non-modular lattice. In fact, since $b_1 < b_2 < b_3 < \cdots < b_{k'}$ and since $a_1,a_2,\cdots,a_{i1}-1$ are incomparable to $b_1,b_2,b_3,\cdots,b_{k'}$ respectively, it exists a sublattice $\{s,a_1,b_1,b_2,b_{k'}\}$ of $L_2(n_1,n_2)[k',i_1,i_2]$ is isomorphic to the pentagon lattice N_5.



(a) New additionship



(b) A finite lattice $L_2(7,6)[4,2,5]$

Figure 7. The Hasse diagram of a poset $\{a_{i1},\cdots,a_{i2},b_{k}'\}$ and a finite lattice $L_2(7,6)[4,2,5]$

Before introducing the lemma to prove Theorem 1.3, we need to introduce some notation. Let

$$
\begin{aligned}
G_1(i_1,k') &= \{a_ib_j - sb_{k'} \mid 1 \leq i \leq i_1, 1 \leq j \leq k'-1\},\\
G_2(i_1,i_2) &= \{a_ib_{k'} - a_{i_1}a_{i_2} \mid i_1+1 \leq i \leq i_2-1\},\\
G_3(i_2,k') &= \{a_ib_j - b_{k'}t \mid i_2 \leq i \leq n_1, k'+1 \leq j \leq n_2\},\\
A_1(i_1) &= \{a_isb_{k'} - a_1sb_{k'} \mid 2 \leq i \leq i_1\},\\
A_2(i_1,i_2) &= \{a_ia_{i1}a_{i2} - a_{i_1+1}a_{i1}a_{i2} \mid i1+2 \leq i \leq i2-1\},\\
A_3(i_2) &= \{a_ib_{k'}t - a_{i2}b_{k'}t \mid i_2+1 \leq i \leq n_1\},\\
B_1(k') &= \{b_isb_{k'} - b_1sb_{k'} \mid 2 \leq i \leq k'-1\},\\
B_2(k') &= \{b_ib_{k'}t - b_{k'}+1b_{k'}t \mid k'+2 \leq i \leq n_2\}
\end{aligned}
$$

and

$$
\begin{aligned}
L_{i_1,k'} &= \{s,a_1,\cdots,a_{i_1},b_1,\cdots,b_{k'-1},b_{k'}\},\\
L_{i_1,i_2,k'} &= \{a_{i_1},a_{i_1+1},\cdots,a_{i_2-1},a_{i_2},b_{k'}\},\\
L_{i_2,k'} &= \{a_{i_2},\cdots,a_{n_1},b_{k'},b_{k'+1},\cdots,b_{n_2},t\}
\end{aligned}
$$

Note that a system of generators of $IL2(n1,n2)[k',i1,i2]$ is $G1(i1,k') \cup G2(i1,i2) \cup G3(i2,k')$.

Lemma 4.1. *For* $i_2 - i_1 > 2$, *the set*
$G_1(i_1,k') \cup G_2(i_1,i_2) \cup G_3(i_2,k') \cup A_1(i_1) \cup A_2(i_1,i_2) \cup A_3(i_2) \cup B_1(k') \cup B_2(k')$
is a Gröbner basis of $I_{L2(n1,n2)[k',i1,i2]}$ *with respect to the inverse lexicographic order induced by* (2.1.1).

Proof. By Theorem 1.1, $G_1(i_1,k') \cup A_1(i_1) \cup B_1(k')$ is a Gröbner basis with respect to $\prec$ of $L_{i1,k'}$ and $G_2(i_1,i_2) \cup A_1(i_1,i_2)$ is a Gröbner basis with respect to $\prec$ of $L_{i1,i2,k'}$. Also, it follows from Theorem 1.1 that $G_3(i_2,k') \cup A_3(i_2) \cup B_2(k')$ is a Gröbner basis with respect to $\prec$ of $L_{i2,k'}$. Hence, it follows from [1, Lemma 1.27] that the S-polynomials which we only have to check are

$$S(a_isb_{k'}-a_1sb_{k'},a_jb_{k'}t-a_{i_2}b_{k'}t) \quad \text{for } 2\le i\le i_1, i_2+1\le j\le n_1, \tag{4.1.1}$$
$$S(a_isb_{k'}-a_1sb_{k'},b_jb_{k}t-b_{k'+1}b_{k'}t) \quad \text{for } 2\le i\le i_1, k'+2\le j\le n_2, \tag{4.1.2}$$
$$S(a_ib_{k'}-a_{i_1}a_{i_2},b_jsb_{k'}-b_1sb_{k'}) \quad \text{for } i_1+1\le i\le i_2-1, 2\le j\le k'-1 \tag{4.1.3}$$
$$S(a_ib_{k'}-a_{i_1}a_{i_2},b_jb_{k'}t-b_{k'+1}b_{k'}t) \quad \text{for } i_1+1\le i\le i_2-1, k'+2\le j\le n_2 \tag{4.1.4}$$
$$S(a_ib_{k'}t-a_{i_2}b_{k'}t,b_jsb_{k'}-b_1sb_{k'}) \quad \text{for } i_2+1\le i\le n_1, 2\le j\le k'-1, \tag{4.1.5}$$
$$S(b_isb_{k'}-b_1sb_{k'},b_jb_{k'}t-b_{k'+1}b_{k'}t) \quad \text{for } 2\le i\le k'-1, k'+2\le j\le n_2. \tag{4.1.6}$$

Now, the result of computation of (4.1.1),⋯ ,(4.1.6) is as follows:

$$\begin{aligned}
S(a_isb_{k'}-a_1sb_{k'},a_jb_{k'}t-a_{i_2}b_{k'}t) &= -sa_1a_jb_{k'}t+sa_ia_{i_2}b_{k'}t\\
&= a_{i_2}t(a_isb_{k'}-a_1sb_{k'})-sa_1(a_jb_{k'}t-a_{i_2}b_{k'}t)\\
&\quad \text{for } 2\le i\le i_1, i_2+1\le j\le n_1,\\
S(a_isb_{k'}-a_1sb_{k'},b_jb_{k'}t-b_{k'+1}b_{k'}t) &= -sa_1b_jb_{k'}t+sa_ib_{k'+1}b_{k'}t\\
&= b_{k'+1}t(a_isb_{k'}-a_1sb_{k'})-sa_1(b_jb_{k'}t-b_{k'+1}b_{k'}t)\\
&\quad \text{for } 2\le i\le i_1, k'+2\le j\le n_2,\\
S(a_ib_{k'}-a_{i_1}a_{i_2},b_jsb_{k'}-b_1sb_{k'}) &= -sa_{i_1}a_{i_2}b_j+sa_ib_1b_{k'}\\
&\quad sb_1(a_ib_{k'}-a_{i_1}a_{i_2})+sa_{i_2}(a_{i_1}b_1-sb_{k'})-sa_{i_2}(a_{i_1}b_j-sb_{k'})\\
&\quad \text{for } i_1+1\le i\le i_2-1, 2\le j\le k'-1\\
S(a_ib_{k'}-a_{i_1}a_{i_2},b_jb_{k'}t-b_{k'+1}b_{k'}t) &= -a_{i_1}a_{i_2}b_jt+a_ib_{k'+1}b_{k'}t\\
&= b_{k'+1}t(a_ib_{k'}-a_{i_1}a_{i_2})+a_{i_1}t(a_{i_2}b_{k'+1}-bk't)-a_{i_1}t(a_{i_2}b_j-bk't)\\
&\quad \text{for } i_1+1\le i\le i_2-1, k'+2\le j\le n_2\\
S(a_ib_{k'}t-a_{i_2}b_{k'}t,b_jsb_{k'}-b_1sb_{k'}) &= -sa_{i_2}b_jb_{k'}t+sa_ib_1b_{k'}t\\
&= sb_1(a_ib_{k'}t-a_{i_2}b_{k'}t)-a_{i_2}t(b_jsb_{k'}-b_1sb_{k'})\\
&\quad \text{for } i_2+1\le i\le n_1, 2\le j\le k'-1,\\
S(b_isb_{k'}-b_1sb_{k'},b_jb_{k'}t-b_{k'+1}b_{k'}t) &= -sb_1b_jb_{k'}t+sb_ib_{k'}b_{k'+1}t\\
&= b_{k+1}t(b_isb_{k'}-b_1sb_{k'})-sb_1(b_jb_{k'}t-b_{k'+1}b_{k'}t)\\
&\quad \text{for } 2\le i\le k'-1, k'+2\le j\le n_2.
\end{aligned}$$

Thus, S-polynomials (4.1.1), ⋯ , (4.1.6) reduce to 0. Hence, for $i_2-i_1>2$, the set $G_1(i_1,k')\cup G_2(i_1,i_2)\cup G_3(i_2,k')\cup A_1(i_1)\cup A_2(i_1,i_2)\cup A_3(i_2)\cup B_1(k')\cup B_2(k')$ is a Gröbner basis of $I_{L_2(n_1,n_2)[k',i_1,i_2]}$ with respect to $\prec'$. □

Lemma 4.2. *For* $i_2-i_1=2$, *the set*

$$G_1(i_1,k')\cup G_2(i_1,i_1+2)\cup G_3(i_1+2,k')\cup A_1(i_1)\cup A_3(i_1+2)\cup B_1(k')\cup B_2(k')$$

is a Gr¨obner basis of $I_{L_2(n_1,n_2)[k',i_1,i_1+2]}$ *with respect to the inverse lexicographic order induced by* (2.1.1).

Proof. Since $i_2=i_1+2$, we have $G_2(i_1,i_1+2)=\{a_{i_1+1}b_{k'}-a_{i_1}a_{i_1+2}\}$. Hence, we have Theorem 4.2 by [1, Lemma 1.27] and the computational result of (4.1.1),(4.1.2),(4.1.5) and (4.1.6).

Now, we prove Theorem 1.3.

The proof of Theorem 1.3. By Theorem 4.1 and 4.2, the join-meet ideal $I_{L_2(n_1,n_2)[k',i_1,i_2]}$ is squarefree with respect to the inverse lexicographic order induced by (2.1.1). Hence, it is radical.

4.2. **A finite lattice** O_{n_1}**.** Let $n=n_1$. Let i be odd number. We denote O_n by $L_2(n,n)$ which satisfies $a_i<b_{i+1}<a_{i+2}$. Figure 8 (A) displays the Hasse diagram of $\{a_i,a_{i+1},a_{i+2},b_i,b_{i+1},b_{i+2}\}$, where i is odd number.

(a) New additionship **(b)** A finite lattice O_5

Figure 8. The Hasse diagram of a poset $\{a_i,a_{i+1},a_{i+2},b_i,b_{i+1},b_{i+2}\}$ and a finite lattice O_5, where i is odd number

Let prove Therorem 1.4. The outline of proof of it is to show that a system of generators of the join-meet ideal I_{O_n} is a Gr¨obner basis of I_{O_n} with respect to the inverse lexicographic order induced by (2.1.1). In short, we show that O_n is a distributive lattice. Note that $\prec$ is a rank reverse lexicographic order on $K[O_n]$; see [1, Example 6.16] for definitions.

The proof of Therorem 1.4. At first, we clarify a system of generators of the join-meet ideal I_{O_n}. Let R be a system of generators of I_{O_n}. Since $a_1,\cdots,a_n$ are incomparable with $b_1,\cdots,b_n$ respectively, then we have

$$\begin{aligned} R &= \{f_{a,b} \mid a,b \in L \text{ such that } a \text{ and } b \text{ are incomparable}\} \\ &= \{f_{a_1,b_j} \mid 1 \le j \le n\} \cup \cdots \cup \{f_{a_n,b_j} \mid 1 \le j \le n\} \\ &= \bigcup_{i=1}^{n} \{f_{a_i,b_j} \mid 1 \le j \le n\}. \end{aligned}$$

Hence, we must consider the following cases:

Case 1: The calculation of f_{ai},b_j for $i \equiv 0(\mathrm{mod}2)$,

Case 2:The calculation of f_{ai},b_j for $i \equiv 1(\mathrm{mod}2)$.

(Case 1) Let i be even nummber. Now, a finite lattice O_n satisfies the following inequality: $s \le b_1 \le b_2 \le \cdots \le b_i-2 \le a_i-1 \le a_i \le a_{i+1} \le b_{i+2} \le \cdots \le b_n \le t.$ (4.2.1)

By (4.2.1), we have

$b_1 \le \cdots \le b_i-2 \le a_i \le b_{i+2} \le \cdots \le b_n.$ (4.2.2)

Hence, it follows from (4.2.2) that we have $f_{ai},b_j = 0$ for $j \le i-2$, $i+2 \le j$. On the other hand, since a_i is incomparable to b_i-1, b_i, b_{i+1} respectively, we must consider the calculation of $a_i \vee b_\ell$, $a_i \wedge b_\ell$, where $\ell = i-1,i,i+1$.

First, in the case $j = i-1$, since

$$bi-2 \le ai-1 \le ai \le ai+1, \quad bi-2 \le bi-1 \le bi \le ai+1, \text{we have}$$
$$a_i \vee b_i-1 = a_{i+1},\ a_i \wedge b_i-1 = b_i-2. \text{ Hence, for } j = i-1, \text{ we have}$$
$$fa_i,b_j = fa_i,b_{i-1} = aibi-1 - ai+1bi-2.$$

Second, in the case $j = i$, since

$$ai-1 \le ai \le ai+1,\ ai-1 \le bi \le ai+1,$$
$$\text{we have } a_i \vee b_i = a_{i+1},\ a_i \wedge b_i = a_i-1. \text{ Hence, for } j = i, \text{ we have}$$
$$fa_i,b_j = fa_i,b_i = aibi - ai-1ai+1.$$

Finally, in the case $j = i+1$, since

$$ai-1 \le ai \le ai+1 \le bi+2, \quad ai-1 \le bi \le bi+1 \le bi+2,$$
$$\text{we have } a_i \vee b_{i+1} = b_{i+2},\ a_i \wedge b_{i+1} = a_i-1. \text{ Hence, for } j = i+1, \text{ we have}$$
$$fa_i,b_j = fa_i,b_{i+1} = aibi+1 - ai-1bi+2.$$

Therefore, the polynomial f_{ai},b_j is as follows:

$$f_{a_i,b_j} = \begin{cases} 0, & j \le i-2, \\ a_i b_{i-1} - a_{i+1} b_{i-2}, & j = i-1, \\ a_i b_i - a_{i-1} a_{i+1}, & j = i, \\ a_i b_{i+1} - a_{i-1} b_{i+2}, & j = i+1, \\ 0, & j \ge i+2. \end{cases} \qquad (4.2.3)$$

Hence, for $i \ne j$, we have $f_{ai},b_j = 0$ by (4.2.5). On the other hand, for $i = j$, since a_i and b_i are incomparable, we have $f_{ai},b_i = a_ib_i - b_i-1b_{i+1}$ by (4.2.4). Hence, the polynomial f_{ai},b_j is as follows:

(Case 2) Let i be odd nummber.Now, O_n satisfies the following inequality:

$$b_{i-1} \leq a_i \leq b_{i+1}. \tag{4.2.4}$$

By (4.2.4), we have

$$s \leq b_1 \leq \cdots \leq b_{i-1} \leq a_i \leq b_{i+1} \leq \cdots b_n \leq t. \tag{4.2.5}$$

Hence, for $i \neq j$, we have $f_{a_i,b_j} = 0$ by (4.2.5). On the other hand, for $i = j$, since a_i and b_i are incomparable, we have $f_{a_i,b_i} = a_ib_i - b_{i-1}b_{i+1}$ by (4.2.4). Hence, the polynomial f_{a_i,b_j} is as follows:

$$f_{a_i,b_j} = \begin{cases} a_ib_i - b_{i-1}b_{i+1}, & i = j, \\ 0, & i \neq j. \end{cases} \tag{4.2.6}$$

Therefore, R consists of (4.2.3) and (4.2.6).

Next, we show that R is Gr¨obner basis of I_{On} with respect to compatible monomial order $\prec$. Let j and r be non-negative integer. Let check that S- polynomials

$S(fa_i,b_j,fa_k,b_\ell)$ for $i,k \equiv 0(\text{mod}2)$,

$S(fa_i,b_j,fa_k,b_\ell)$ for $i,k \equiv 1(\text{mod}2)$,

$S(fa_i,b_j,fa_k,b_\ell)$ for $i \equiv 1(\text{mod}2), k \equiv 0(\text{mod}2)$.

Reduce to 0 with respect to generators of R.

First, we check that $S(f_{ai},b_j,f_{ak},b_\ell)$ reduces to 0 with respect to generators of R, where $i,k \equiv 0$ (mod 2). It follows from (4.2.3) that we have

$$\text{in}_\prec(f_{a_i,b_j}) = \begin{cases} 0, & j \leq i-2, \\ a_ib_{i-1}, & j = i-1, \\ a_ib_i, & j = i, \\ a_ib_{i+1}, & j = i+1, \\ 0, & j \geq i+2. \end{cases}$$

From above equation, for each $i = k$ and $i \neq k$, it is necessary to consider the calculation of $S(fa_i,b_j,fa_k,b_\ell)$.

(The case $i = k$) Since a initial monomial of f_{ai},b_j and f_{ai},b_ℓ are as follows:

$$\text{in}_\prec(f_{a_i,b_j}) = \begin{cases} 0, & j \leq i-2, \\ a_ib_{i-1}, & j = i-1, \\ a_ib_i, & j = i, \\ a_ib_{i+1}, & j = i+1, \\ 0, & j \geq i+2, \end{cases} \quad \text{in}_\prec(f_{a_i,b_\ell}) = \begin{cases} 0, & \ell \leq i-2, \\ a_ib_{i-1}, & \ell = i-1, \\ a_ib_i, & \ell = i, \\ a_ib_{i+1}, & \ell = i+1, \\ 0, & \ell \geq i+2. \end{cases}$$

Thus, for $j = \ell = i-1, i, i+1$, we have $S(f_{a_i,b_j}, f_{a_i,b_\ell}) = 0$. Hence, we only have to check out that S-polynomial

$$S(f_{a_i,b_j}, f_{a_i,b_\ell}) = \begin{cases} S(f_{a_i,b_{i-1}}, f_{a_i,b_i}), & (j,\ell) = (i-1,i), \\ S(f_{a_i,b_{i-1}}, f_{a_i,b_{i+1}}), & (j,\ell) = (i-1,i+1), \\ S(f_{a_i,b_i}, f_{a_i,b_{i+1}}), & (j,\ell) = (i,i+1). \end{cases} \tag{4.2.7}$$

reduces to 0. Now, it follows from (4.2.7) that we have

$$S(f_{a_i,b_j}, f_{a_i,b_\ell}) = \begin{cases} a_{i+1}f_{a_{i-1},b_{i-1}}, & (j,\ell) = (i-1,i), \\ b_{i+2}f_{a_{i-1},b_{i-1}} - b_{i-2}f_{a_{i+1},b_{i+1}}, & (j,\ell) = (i-1,i+1), \\ -a_{i-1}f_{a_{i+1},b_{i+1}}, & (j,\ell) = (i,i+1). \end{cases}$$

Hence, $S(f_{ai},b_j,f_{ai},b_\ell)$ reduces to 0.

(The case $i \neq k$) Suppose $i < k$ and let $\varepsilon = k - i$. Then, a initial monomial of f_{ai},b_j and f_{ak},b_ℓ are as follows:

$$\mathrm{in}_\prec(f_{a_i,b_j}) = \begin{cases} 0, & j \le i-2, \\ a_ib_{i-1}, & j = i-1, \\ a_ib_i, & j = i, \\ a_ib_{i+1}, & j = i+1, \\ 0, & j \ge i+2, \end{cases} \quad \mathrm{in}_\prec(f_{a_k,b_\ell}) = \begin{cases} 0, & \ell \le i+\varepsilon-2, \\ a_{i+\varepsilon}b_{i+\varepsilon-1}, & \ell = i+\varepsilon-1, \\ a_{i+\varepsilon}b_{i+\varepsilon}, & \ell = i+\varepsilon, \\ a_{i+\varepsilon}b_{i+\varepsilon+1}, & \ell = i+\varepsilon+1, \\ 0, & \ell \ge i+\varepsilon+2. \end{cases}$$

Then, we must consider that we look for ε such thatand ℓ satisfy $b_j = b_\ell$ in the above calculation result. Below, we consider the following cases:

Case 2.1 : $bi-1 = bi+\varepsilon- 1,bi+\varepsilon,bi+\varepsilon+1$,

Case 2.2 : $bi = bi+\varepsilon- 1,bi+\varepsilon,bi+\varepsilon+1$,

Case 2.3 : $bi+1 = bi+\varepsilon- 1,bi+\varepsilon,bi+\varepsilon+1$.

(Case 2.1) In this case, sincei $- 1 = i + \varepsilon - 1, i + \varepsilon, i + \varepsilon + 1$, we have $\varepsilon = 0,-1,-2$. Since $\varepsilon > 0$, then $\varepsilon = 0,-1,-2$ can't satisfy Case 2

(Case 2.2) In this case, since $i = i+\varepsilon-1,i+\varepsilon,i+\varepsilon+1$, we have $\varepsilon = -1,0,1$. Since $\varepsilon > 0$ and since it is even, then $\varepsilon = -1,0,1$ can't satisfy Case 2.2.

(Case 2.3) In this case, since $i + 1 = i + \varepsilon - 1,i + \varepsilon,i + \varepsilon + 1$, we have $\varepsilon = 0,1,2$. Since $\varepsilon > 0$ and since it is even, then $\varepsilon = 2$ only satisfies Case 2.3.

From three cases, at first, for $\varepsilon > 2$, since in$\prec$ (f_{ai},b_j) and in$\prec$$(f_{ak},b_\ell)$ are relatively prime, $S(f_{ai},b_j,f_{ak},b_\ell)$ reducess to respect to f_{ai},b_j,f_{ak},b_ℓ. Next, for $\varepsilon = 2$, we have

$$\prec(f_{a_i,b_j}) = \begin{cases} 0, & j \le i-2, \\ a_ib_{i-1}, & j = i-1, \\ a_ib_i, & j = i, \\ a_ib_{i+1}, & j = i+1, \\ 0, & j \ge i+2, \end{cases} \quad \mathrm{in}_\prec(f_{a_k,b_\ell}) = \begin{cases} 0, & \ell \le i, \\ a_{i+2}b_{i+1}, & \ell = i+1, \\ a_{i+2}b_{i+2}, & \ell = i+2 \\ a_{i+2}b_{i+3}, & \ell = i+3 \\ 0, & \ell \ge i+4. \end{cases}$$

Hence, for

$$(j,\ell) = (i-1,i+1),(i-1,i+2),(i-1,i+3),(i,i+1),(i,i+2),(i,i+3),(i+1,i+2),(i+1,i+3),$$

since in$\prec$ (f_{ai},b_j) and in$_\prec$$(f_{ak},b_\ell)$ are relatively prime, $S(f_{ai},b_j,f_{ak},b_\ell)$ reduces to 0 with respect to f_{ai},b_j,f_{ak},b_ℓ. On the other hand, for $(j,\ell) = (i + 1,i + 1)$, since we have

$$S(fai,bi+1,fai+2,bi+1) = ai+3fai,bi - ai-1fai+2,bi+2,$$

$S(f_{ai},b_{i+1},f_{ai+2},b_{i+1})$ reduces to 0 with respect to f_{ai},b_i, f_{ai+2},b_{i+2}. Therefore, we checked that $S(f_{ai},b_j,f_{ak},b_\ell)$ reduces to 0 with respect to generators of R.

Second, we check that $S(f_{ai},b_j,f_{ak},b_\ell)$ reduces to 0 with respect to generators of R, where $i,k \equiv 1 \pmod 2$. For $i \neq j$ or $k \neq \ell$, it follows from (4.2.6) that $S(f_{ai},b_j,f_{ak},b_\ell)$ reduces to 0 with respect to f_{ai},b_j,f_{ak},b_ℓ. On the other hand, for $i = j$ and $k = \ell$, since in$\prec$$(f_{ai},b_i)$ $= a_ib_i$ and in$\prec$ $(f_{ak},b_k) = a_kb_k$, we have to consider the calculation of $S(f_{ai},b_i,f_{ak},b_k)$ for each $i = k$ and $i \neq k$.

(The case $i = k$) It follows from $i = k$ that we have $S(f_{ai},b_i,f_{ak},b_k) = 0$. (The case $i \neq k$) It follow from $i \neq k$ that in (f_{ai},b_i) and in$\prec$$(f_{ak},b_k)$ are relatively prime. Hence, $S(f_{ai},b_i,f_{ak},b_k)$ reduces to 0 with respect to f_{ai},b_i, f_{ak},b_k.

Therefore, we checked that $S(f_{ai},b_j,f_{ak},b_\ell)$ reduces to 0 with respect to generators of R.

Finally, we check that $S(f_{ai},b_j,f_{ak},b_\ell)$ reduces to 0 with respect to generators of R, where $i \equiv 1 \pmod 2$ and $k \equiv 0 \pmod 2$. By (4.2.3) and (4.2.6), we have

$$\mathrm{in}_\prec(f_{a_i,b_j}) = \begin{cases} a_ib_i, & i = j, \\ 0, & i \neq j, \end{cases} \quad \mathrm{in}_\prec(f_{a_k,b_\ell}) = \begin{cases} 0, & \ell \le k-2, \\ a_kb_{k-1}, & \ell = k-1, \\ a_kb_k, & \ell = k, \\ a_kb_{k+1}, & \ell = k+1, \\ 0, & \ell \ge k+2. \end{cases}$$

From above result, if $i \neq j$, then we have $S(f_{a,b},f_{a,b}) = -f_{a,b}$. On the other hand, for $i = j$, we have

$$S(f_{a_i,b_j}, f_{a_k,b_\ell}) = \begin{cases} -b_{k-2}f(a_k,b_k), & i=j=\ell=k-1, \\ -b_{k+2}f(a_k,b_k), & i=j=\ell=k+1. \end{cases}$$

Hence, we checked that $S(f_{a_i,b_j},f_{a_k,b_\ell})$ reduces to 0 with respect to generators of R.

Therefore, we showed that R is a Gröbner basis of I_{O_n} with respect to $\prec$. By [1, Theorem 6.17], O_n is a distributive lattice. Hence, it follows from [1, Theorem 6.21] that I_{O_n} is radical.

5. TOPICS RELATED TO SPECIAL FINITE LATTICE

In this section, we introduce topics related to a distributive lattice O_{n1}. Note that it has little to do with the gist of this paper.

5.1. **Number-theoretic characterization.** In this subsection, we introduce the relationship between O_{n1} and number theory. By Theorem 1.4, a finite lattice O_{n1} is distributive lattice. On the other hand, it looks abstract as the structure of the set and it has a difficult shape. However, it is not. By the following theorem, we can see O_{n1} as number-theoretic finite lattice whose shape is very easy.

Theorem 5.1. *Let p and q be prime number with $p \neq q$. For non-negative integer k, let*

$$L_{p,q,k} = \bigcup_{r=1}^{k} C_{p,q,r}$$

ordered by divisibility, where

$$C_{p,q,1} = \{1, p, q, p^2, pq, p^2q\}, \quad C_{p,q,r} = \{p^{r-1}q^r, p^rq^r, p^{r+1}q^r, p^{r+1}q^{r-1}\}, r > 1.$$

Then, O_{2k} is isomorphic to $L_{p,q,k}$.

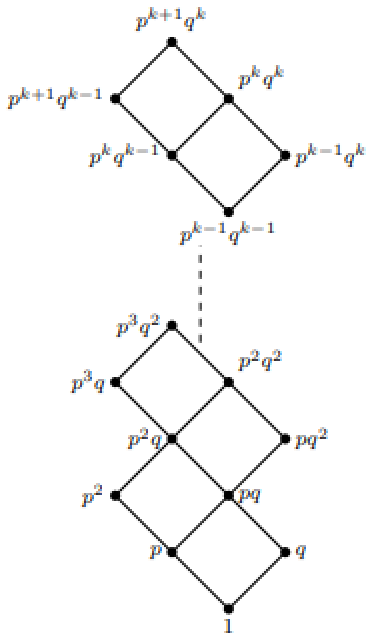


Figure 9. The Hasse diagram of $L_{p,q,k}$

Proof. At first, we prepare some things necessary to prove Theorem 5.1.

Let $L_{p,q,k} = L_k$. Let define the map $h_{1,k} : O_{2k} \longrightarrow L_k$ by setting

$$h_{1,k}(s) = 1, \quad h_{1,k}(a_1) = p,$$
$$h_{1,k}(a_{2r}) = p^{r+1}q^{r-1}, \quad h_{1,k}(b_{2r-1}) = p^{r-1}q^r \quad \text{for } r = 1, \cdots, k,$$
$$h_{1,k}(a_{2r+1}) = h_{1,k}(a_{2r}) \vee h_{1,k}(b_{2r}), \quad h_{1,k}(b_{2r}) = h_{1,k}(a_{2r-1}) \vee h_{1,k}(b_{2r-1}) \quad \text{for } r = 1, \cdots, k.$$

We define the map $h_{2,k} : L_k \longrightarrow O_{2k}$ by setting

$$h_{2,k}(1) = s, \quad h_{2,k}(p) = a_1,$$
$$h_{2,k}(p^{r+1}q^{r-1}) = a_{2r}, \quad h_{2,k}(p^{r-1}q^r) = b_{2r-1} \quad \text{for } r = 1, \cdots, k,$$
$$h_{2,k}(p^{r+1}q^r) = a_{2r+1}, \quad h_{2,k}(p^rq^r) = b_{2r} \quad \text{for } r = 1, \cdots, k.$$

Now, we prove Theorem 5.1. First, we show that $h_{1,k}$ is bijective. Since

$$\begin{aligned} L_k &= \{1,p,q,p^2,pq,p^2q\} \cup \left(\bigcup_{r=2}^{k} C_{p,q,r} \right) \\ &= \{h_{1,k}(s),h_{1,k}(a_1),f(b_1),f(a_2),f(b_2)=f(a_1)\vee f(b_1),f(a_3)=f(a_2)\vee f(b_2)\} \cup \left(\bigcup_{r=2}^{k} C_{p,q,r} \right), \end{aligned}$$

the mapping $h_{1,k}$ is bijective for $k = 1$. On the other hand, in the case $k > 1$, since

$$\begin{aligned} & C_{p,q,r} \\ = \; & \{h_{1,k}(a_{2r}),h_{1,k}(b_{2r-1}),h_{1,k}(b_{2r}) = h_{1,k}(a_{2r-1})\vee h_{1,k}(b_{2r-1}),h_{1,k}(a_{2r+1}) = h_{1,k}(a_{2r})\vee h_{1,k}(b_{2r})\} \end{aligned}$$

for $r = 2,\cdots,k$, the set $C_{p,q,r}$ can be described inductively by $\{h_{1,k}(s),h_{1,k}(a_1),h_{1,k}(b_1),h_{1,k}(a_2),h_{1,k}(b_2),h_{1,k}(a_3$ and $C_{p,q,r'}$, where $1 \le r' \le r-1$. Hence, for $k > 1$, $h_{1,k}$ is bijective. Therefore, $h_{1,k}$ is bijective.

Second, we show that $h_{1,k}$ is order-preserving. By definition of $h_{1,k}$, it follows from a subset $\{a_i\}_{i=1}^{2k}$ of O_{2k} that we have

$$\begin{aligned} h_{1,k}(a_1) = p < p^2 = h_{1,k}(a_2), \quad h_{1,k}(a_2) = p^2 < h_{1,k}(a_3) = p^2q, \\ \cdots, h_{1,k}(a_{2k-1}) = p^kq^{k-1} < p^{k+1}q^{k-1} = h_{1,k}(a_{2k}). \end{aligned}$$

Hence, $h_{1,k}(a_i) < h_{1,k}(a_{i+1})$ for $i = 1,2,\cdots,2k-1$). Also, it follows from a subset $\{b_i\}_{i=1}^{2k}$ that we get

$$\begin{aligned} h_{1,k}(b_1) = q < pq = h_{1,k}(b_2), \quad h_{1,k}(b_2) = pq < h_{1,k}(b_3) = pq^2, \\ \cdots, h_{1,k}(b_{2k-1}) = p^{k-1}q^k < p^kq^k = h_{1,k}(b_{2k}). \end{aligned}$$

Hence, for $i = 1,\cdots,2k-1$, we have $h_{1,k}(b_i) < h_{1,k}(b_{i+1})$. On the other hand, L_k satisfies the following inequality:

$$\begin{aligned} & h_{1,k}(a_1) = p < pq = h_{1,k}(b_2), \quad h_{1,k}(b_2) = pq < p^2q = h_{1,k}(a_3), \\ & \cdots, h_{1,k}(a_r) = p^rq^{r-1} < p^rq^r = h_{1,k}(b_{r+1}), \quad h_{1,k}(b_{r+1}) = p^rq^r < p^{r+1}q^r = h_{1,k}(a_{r+2}), \\ & \cdots, h_{1,k}(a_{2k-1}) = p^kq^{k-1} < p^kq^k = h_{1,k}(b_{2k}), \quad h_{1,k}(b_{2k}) = p^kq^k < p^{k+1}q^k = h_{1,k}(t) \\ & \text{for } r = 1,3,\cdots,2k-1 \end{aligned}$$

Hence, we have $h_{1,k}(a_r) < h_{1,k}(b_{r+1}) < h_{1,k}(a_{r+2})$ for $i = 1,3,\cdots,2k-1$. Thus, for a and b belonging to O_{2k} with $a < b$, we have $h_{1,k}(a) < h_{1,k}(b)$. Therefore, we showed that $h_{1,k}$ is order-preserving.

Third, we show that $h_{2,k}$ is the inverse mapping of $h_{1,k}$. Let $h_{21,k}$ be the composite mapping $h_{2,k} \circ h_{1,k}$ and let $h_{12,k}$ the composite mapping $h_{1,k} \circ h_{2,k}$. For an arbitrary element belonging to O_{2k}, we have the following result:

$h_{21},k(s)$	=	$h2,k(h1,k(s)) = h2,k(1) = s$,
$h_{21},k(a1)$	=	$h2,k(h1,k(a1)) = h2,k(p) = a1$,
$h21,k(a2r)$	=	$h2,k(h1,k(a2r)) = h2,k(pr+1qr-1) = a2r$ for $r = 1,2,\cdots,k$,
$h21,k(a2r+1)$	=	$r+1$ r $h2,k(h1,k(a2r+1)) = h2,k(p\ q\ ) = a2r+1$ for $r = 1,2,\cdots$
$h21,k(b2r-1)$	=	$,k$, $h_{2,k}(h_{1,k}(b_{2r}-1)) = h_{2,k}(p\ ^{r}-^{1}q^r) = b_{2r}-1$ for $r = 1,2,\cdots,k$,
$h21,k(b2r)$	=	r r $h_{2,k}(h_{1,k}(b_{2r})) = h_{2,k}(p\ q\ ) = b_{2r}$ for $r = 1,2,\cdots,k$.

Hence, we have $h_{21,k} = \mathrm{id}_{O2k}$. On the other hand, for an arbitrary element belonging to L_k, we have the following result:

$h_{12},k(1)$	=	$h1,k(h2,k(1)) = h1,k(s) = 1$,
$h_{12},k(p)$	=	$h1,k(h2,k(p)) = h1,k(a1) = p$,
$h_{12,k}(pr+1qr-1)$	=	$h1,k(h2,k(pr+1qr-1)) = h1,k(a2r) = pr+1qr-1$, for $r = 1,2,\cdots,k$,
$h12,k(pr-1qr)$	=	$^{r-1}\ ^{r}$)) $= h_{1,k}(b_{2r}-1) = p^{r+1}q^{r-1}$ for $r = 1,2,\cdots,k$, $h_{1},k(h_{2},k(p$ q
$h12,k(pr+1qr)$	=	$h1,k(h2,k(pr+1qr)) = h1,k(a2r+1) = pr+1qr$ for $r = 1,2,\cdots,k$, $h_{1,k}(h_{2,k}(p^rq^r)) = h_{1,k}(b_{2r}) = p^rq^r$
$h12,k(prqr)$	=	for $r = 1,2,\cdots,k$.

Hence, we have $h_{12,k} = \mathrm{id}_{Lk}$. Therefore, $h_{2,k}$ is the inverse mapping of $h_{1,k}$.

Finally, we show that $h_{2,k}$ is order-preserving. Since $s < a_1 < \cdots < a_{2k} < t$ and $s < b_1 < \cdots < b_{2k} < t$, we have

$$\begin{aligned} & h_{2,k}(1) < h_{2,k}(p), \\ & h_{2,k}(p^rq^{r-1}) < h_{2,k}(p^{r+1}q^{r-1}) < h_{2,k}(p^{r+1}q^r) \quad \text{for } r = 1,2,\cdots,k, \\ & h_{2,k}(p^{r-1}q^{r-1}) < h_{2,k}(p^{r-1}q^r) < h_{2,k}(p^rq^r) \quad \text{for } r = 1,2,\cdots,k. \end{aligned}$$

On the other hand, for $r = 1,3,\cdots,2k-1$, since $a_r < b_{r+1} < a_{r+2}$, we have $h_{2,k}(p^r q^{r-1}) < h_{2,k}(p^r q^r) < h_{2,k}(p^{r+1} q^r)$. Hence, $h_{2,k}$ is order-preserving. Therefore, O_{2k} is isomorphic to L_k.

In terms of appearance of shape, Theorem 5.1 means that pulling each two vertex 1 and $p^{k+1}q^k$ of hasse diagram of $L_{p,q,k}$ deformes the appearance of shape from $L_{p,q,k}$ to O_{2k}. We can consider that this deformation in appearance of shape is very natural in everyday life. Figure 10 displays the deformation in appearance of shape from $L_{p,q,2}$ to O_4.

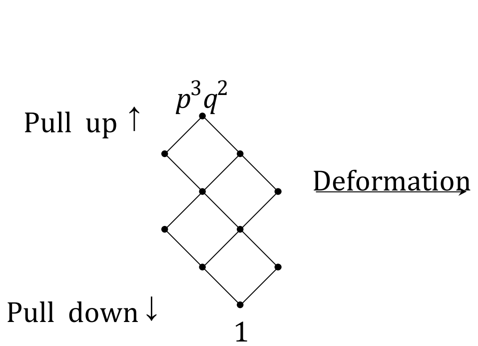


Figure 10. Deformation by pulling each two vertex 1 and p^3q^2

5.2. Gorenstein ring. In this subsection, we give a non-trivial answer to the question of whether the Hibi ring $R_K[O_{n1}] = K[O_{n1}]/I_{On}1$ is Gorenstein. Let $n_1 = n$. Let P_n denote the subposet of O_n consisting of all join-irreducible elements of O_n. By [1, Theorem 6.4 (Birkhoff)], we have $O_n = J(P_n)$. Then, we obtain the following result.

Theorem 5.2. *For $n \geq 4$, the Hibi ring $R_K[O_n]$ is not Gorenstein.*

Proof. Suppose that

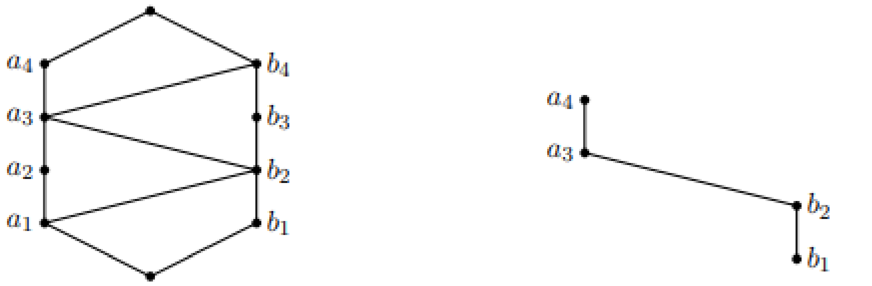
$$P_4 = \{a_1,a_2,a_4,b_1,b_3\}$$

is pure. Then, it follows from [1, Lemma 6.12] that P_4 possesses a rank function ρ. Since a_4 covers a_2 and a_2 covers a_1 in P_4, we have

$$\rho(a_4) = \rho(a_2) + 1 = \rho(a_1) + 2 = 2. \qquad (5.2.1)$$

On the other hand, since $b_1 < b_2 < a_3 < a_4$ in O_4, a_4 covers b_1 in P_4. Thus, we have

$$\rho(a_4) = \rho(b_1) + 1 = 1. \qquad (5.2.2)$$



(a) A finite lattice O_4 **(b)** A chain $\{b_1, b_2, a_3, a_4\}$ of O_4

Figure 11. The Hasse diagram of O_4 and a chain $\{b_1,b_2,a_3,a_4\}$

Hence, (5.2.1) and (5.2.2) contradict the uniqueness of ρ. Therefore, P_4 is not pure.

For $n \geq 4$, P_4 is a subposet of P_n. Since P_4 is not pure,≥ P_n is not pure for $n \geq 4$. Hence,□ it follows from [3] that $R_K[O_n]$ is not Gorenstein for n 4.

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