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Pythagorean-Like Formulas for any Triangle

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ARTICLE INFO	ABSTRACT
Published Online: 15 December 2022 Corresponding Author: **Francesco Laudano**	We propose two generalizations of Pythagoras' theorem from which the cosine law immediately follows.

KEYWORDS: Pythagoras' theorem, Cosine Law, Cevian, Converse of Pythagorean Theorem

I. INTRODUCTION

In [1], L. Hoehn, by using the tangent-secant theorem, (sometimes called Steiner's Theorem), obtains an elegant proof of Pythagoras' theorem and also tries a Pythagorean-like formula for isosceles triangles. In [2], the author generalizes the previous result to any triangle.

In the following we extend this results through the concept of cevian, providing some Pythagorean-like formulas valid for each triangle (Theorems 1 and 2), from which immediately follows the law of cosines.

Moreover, based on the same idea, we try a simple proof of the converse of Pythagoras' theorem (Theorem 3).

II. PYTAGOREAN-LIKE FORMULAS

Theorem 1. Let a, b, c, with $a \geq b$, be the lengths of the sides of a triangle. Let d be the length of the projection of a onto c, obtained by the cevian congruent to b. Then, $a^2=b^2+cd$.

In particular, if the angle opposite to the side a is right, then $a^2=b^2+c^2$.

Proof. Given a triangle ABC with $BC=a \geq AC=b$ and $AB=c$, construct the circle with centre C and radius b as shown in Figures 1 and 2. By using the two secant theorem, we have $(a+b)(a-b)=cd$, then, $a^2=b^2+cd$.

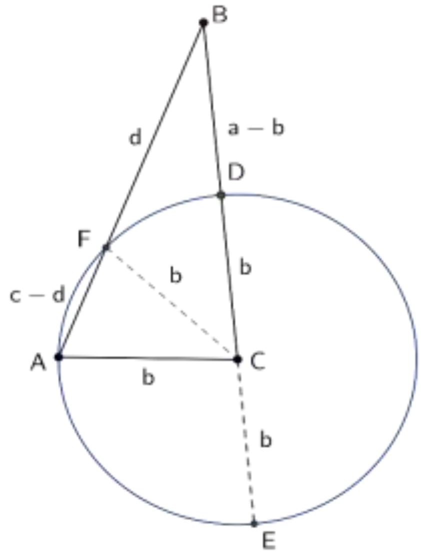

Fig. 1: Case $a \leq c$, $(a+b)(a-b)=cd$, i.e. $a^2=b^2+cd$

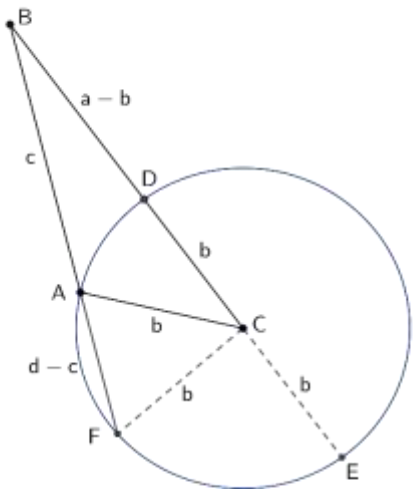

Fig. 2: Case $a > c$, $(a+b)(a-b)=cd$, i.e. $a^2=b^2+cd$

In particular, if $B\hat{A}C$ is a right angle, then $c = d$, so we have Pythagoras' theorem. □

Similarly, in the case $BC = a \le AC = b$, by using the intersecting chords theorem as shown in Figures 3 and 4, we can see that $DB : AB = BF : BE$, i.e. $(b\text{-}a) : c = d : (b+a)$. Therefore we can prove the following statement.

Theorem 2. Let a, b, c, with $a \le b$, be the lengths of the sides of a triangle. Let d be the length of the projection of a onto c, obtained by the cevian congruent to b. Then, $a^2 = b^2 - cd$.
In particular, if the angle opposite to the side b is right, then $a^2 = b^2 - c^2$.

Proof.

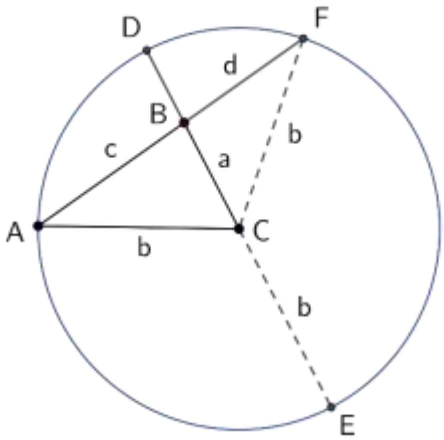

Fig. 3: Case $b \ge c$, $(b\text{-}a)(b+a) = cd$, i.e. $a^2 = b^2 - cd$

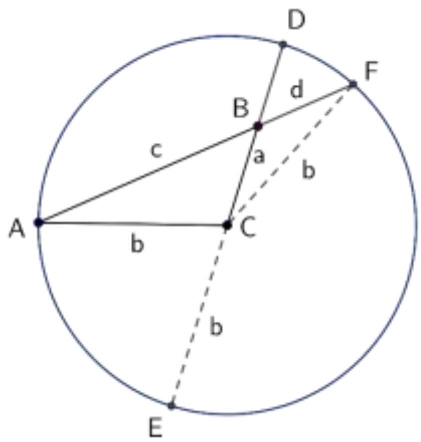

Fig. 4: Case $b < c$, $(b\text{-}a)(b+a) = cd$, i.e. $a^2 = b^2 - cd$.

Analyzing the diagrams in Figure 1, we can also derive that $c\text{-}d=2b \cos B\hat{A}C$. Similarly, analyzing the diagrams in Figure 2, we can see that $c+d=2b \cos B\hat{A}C$. In each of the previous cases we have $a^2=b^2+c(c\text{-}2b \cos B\hat{A}C)$, from which follows the cosine law, that for completeness we enunciate below.

Corollary. Let a, b, c be the lengths of the sides of a triangle and let α the width of the angle opposite to the a side, then $a^2=b^2+c^2\text{-}2bc \cos \alpha$.

Finally, thinking by exclusion about the diagrams in Figure 1, we can also try a simple proof of the converse of Pythagoras' Theorem.

Theorem 3. Let a, b, c be the lengths of the sides of a triangle and let $a^2=b^2+c^2$. Then the angle opposite to the side a is right.

Proof. Given a triangle ABC, with $BC=a$, $AC=b$, $AB=c$ and $a^2=b^2+c^2$, we have $a > b$. Then the circle with center C and radius b intersects the BC side in a point D, as in the previous Figures 1 and 2.
Let F be the other point where this circle intersects the AB side (or its extended side), and let $FB=d$. By $FC=b$, follows that FB is the projection of a onto c obtained by the cevian congruent to b. Then, for Theorem 1, we have $a^2=b^2+cd$ and a quick comparison with the hypothesis $a^2=b^2+c^2$ yields the conclusion that $c=d$.
Then A and F are coincident, so the segment AB is tangent to the circle, which implies that the angle $B\hat{A}C$ is right.

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