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On *t*-Derivations of *BE*-algebras

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ABSTRACT
BE-algebra is a non-empty set X with a binary operation $*$ and a constant 1 which is denoted by $(X;*,1)$, and fulfill certain axioms. In this article, the concept of *t*-derivation in *BE*-algebra is defined and some of its properties are investigated. Then, the properties of the fixed set and kernel of a *t*-derivation in *BE*-algebra are identified based on their relation to the subalgebra and its elements.

KEYWORDS: *BE*-algebra; derivation; *t*-derivation; fixed set; kernel

I. INTRODUCTION

The study of algebraic structures is growing with the discovery of new algebraic structures. Imai and Iseki introduced the concepts of *BCK*-algebra [1] and *BCI*-algebra [2]. *BCK*-algebra is a proper subset of *BCI*-algebra. Hu and Li [3] introduced *BCH*-algebra as a generalization of *BCI*-algebra, so it could be written $BCK \subset BCI \subset BCH$. Later, H. S. Kim and Y. H. Kim [4] constructed *BE*-algebra as one form of generalization of *BCK*-algebra. *BE*-algebra $(X;*,1)$ is a non-empty set X with a binary operation $*$ and constant 1 that satisfies the following axioms: $(BE1)$ $x*x=1$, $(BE2)$ $x*1=1$, $(BE3)$ $1*x=x$, and $(BE4)$ $x*(y*z)=y*(x*z)$ for all $x,y,z\in X$. Other algebraic structures were introduced by Ahn and Han [5] which was named *BP*-algebra.

The concept of derivative is one of the important studies in calculus. Derivative express the rate of change of a bound variable due to a change in its free variable. The change in free variable is classified as a very small change. To determine derivatives of the results of two functions can be used product rule formula. Derivatives of *the functions u* and *v* can be written in Leibniz notation as follows:

$$\frac{d}{dx}(u\cdot v)=\frac{du}{dx}\cdot v+u\cdot\frac{dv}{dx}$$

Product rule formula can be developed or generalized under other conditions.

In abstract algebra, the product rule formula is used to define a derivation. Derivation is a function that maps a set to itself as the process of determining derivative from the product of two functions. The concept of derivation was first introduced in ring and near ring theories, later, Jun and Xin [6] applied the concept of derivation to *BCI*-algebra. A mapping d of *BCI*-algebra $(X;*,0)$ to itself is said to be *left-right* derivation ((*l*, *r*)-derivation) in X if for all $x,y\in X$ meets

$$d(x*y)=(d(x)*y)\wedge(x*d(y))$$

and d is said to be *right-left* derivation ((*r*, *l*)-derivation) in X if

$$d(x*y)=(x*d(y))\wedge(d(x)*y).$$

By define $x\wedge y=y*(y*x)$ for all $x,y\in X$. d is said to be derivation in X if it is (*l*, *r*)-derivation at once (*r*, *l*)-derivation in X.

The concept of derivation was also discussed in *BE*-algebra by Kim and Lee [7]. A self-map d of *BE*-algebra $(X;*,1)$ is called a derivation in X if for all $x,y\in X$ meets

$$d(x*y)=(x*d(y))\vee(d(x)*y).$$

by defining for all $x\vee y=(y*x)*x$ for all $x,y\in X$. Kim and Davvaz [8] discussed the concept of f-derivation in *BE*-algebra by involving an endomorphism f of *BE*-algebra. In addition, as a development of the concepts of derivation and f-derivation in *BE*-algebra, Kim in [9] and [10] also discussed the concepts of generalization of derivation and generalization of f-derivation in *BE*-algebra. The concept of generalization of derivation and f-derivation involves two self-maps in its definition.

Other types of derivations that have been discussed by researchers are the concept of *t*-derivation in *BP*-algebra by Siswanti et al. [11] and f_q-derivation in *BM*-algebra by Yattaqi et al. [12]. The constructing of *t*-derivation begins with define a mapping $d_t(x)=x*t$, for all $t,x\in$ *BP*-algebra $(X;*,0)$. Similarly, constructing f_q-derivation in *BM*-algebra, also involves a mapping similar to d_t, but added a mapping which is endomorphism f in *BM*-algebra.

Gemawati et al. [13] also discusses the concept of f_q-derivation in other algebraic structures, namely BN_1-algebra.

Based on the concept of derivation in *BE*-algebra by Kim and Lee [7] and the concept of *t*-derivation in *BP*-algebra by Siswanti et al. [11], in this paper, we discuss the concept of *t*-derivation in *BE*-algebra. Then, based on the concept determined the properties a fixed set and kernel of *t*-derivation in *BE*-algebra.

II. PRELIMINARIES

In this section, several definitions are given to construct the main results of research, namely the definition and theory of *BP*-algebra, *BE*-algebra, derivation in *BE*-algebra, and *t*-derivation in *BP*-algebra, which all of these concepts have been discussed in [4, 5, 7, 11].

Definition 2.1. [5] *BP*-algebra is a non-empty set X with a binary operations $*$ and a constants 0 that meet the following axioms:

($BP1$) $x * x = 0$,

($BP2$) $x * (x * y) = y$,

($BP3$) $(x * z) * (y * z) = x * y$,

for all $x, y, z \in X$.

Example 1. Let $X = \{0, 1, 2, 3\}$ be a set defined in Table 2.1.

Table 2.1: Cayley Table for (X; $*$, 0)

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

In Table 2.1, the main diagonal is 0, so it applies $x * x = 0$ (axiom (*BP*1) is fulfilled) and it can also be proven that the axiom (*BP*2) and (*BP*3) are fulfilled. Thus, $(X; *, 0)$ is *BP*-algebra.

Theorem 2.2. [5] If $(X; *, 0)$ is a *BP*-algebra, then

(i) $0 * (0 * x) = x$,

(ii) $0 * (y * x) = x * y$,

(iii) $x * 0 = x$,

(iv) If $x * y = 0$, then $y * x = 0$,

(v) If $0 * x = 0 * y$, then $x = y$,

(vi) If $0 * x = y$, then $0 * y = x$,

(vii) If $0 * x = x$, then $x * y = y * x$,

for all.x, $y \in X$

Proof. Theorem 2. 2 has been proven by Ahn and Han [5].

Definition 2.3. [11] Let $(X; *, 0)$ is a *BP*-algebra. A mapping d_t of X to itself is defined by $d_t\ (x) = x * t$ for all $t, x \in X$.

Definition 2.4. [11] Let $(X; *, 0)$ is a *BP*-algebra. A mapping d_t X to itself is called (l, r)-t-derivation in X if it satisfies

$$d_t(x * y) = (d_t(x) * y) \wedge (x * d_t(y))$$

for all x, $y \in X$ and it is called (r, l)-t-derivation in X if it satisfies

$$d_t(x * y) = (x * d_t(y)) \wedge (d_t(x) * y).$$

A Mapping d_t is called t-derivation in X if it is (l, r)-t-derivation at once (r, l)-t-derivation in X.

Definition 2.5. [4] An algebra $(X; *, 1)$ is said to be *BE*-algebra if it satisfies the following axioms:

($BE1$) $x * x = 1$,

($BE2$) $x * 1 = 1$,

($BE3$) $1 * x = x$,

($BE4$) $x * (y * z) = y * (x * z)$

for all $x, y, z \in X$.

Let $(X; *, 1)$ be a *BE*-algebra. Defined a relation $\leq$ on X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$.

Example 2. Let $X = \{1, a, b, c, d, 0\}$ be a set defined in Table 2.2.

Table 2.2: Cayley Table for (X; $*$, 1)

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Based on Table 2. 2 can be proven that $(X; *, 1)$ is a *BE*-algebra.

Definition 2.6. [4] Let $(X; *, 1)$ be a *BE*-algebra and F be a non-empty subset of X. F is said to be filter of X if

($F1$) $1 \in F$,

($F2$) $x \in F$ and $x * y \in F$ imply $y \in F$.

By Example 2, we obtain $F_1 = \{1, a, b\}$ is a filter of X, whereas $F_2 = \{1, a\}$ is not a filter of X, because $a \in F_2$ and $a * b \in F_2$, but $b \notin F_2$.

Definition 2.7. [4] A *BE*-algebra $(X; *, 1)$ is said to be self-distributive if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$

Example 3. Let $X = \{1, a, b, c, d, \}$ be a set defined in Table 2.3.

Table 2.3: Cayley table for (X; $*$, 1)

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Based on Table 2. 3, it can be proven that $(X; *, 1)$ is a *BE*-algebra satisfying self-distributive. Whereas, *BE*-algebra in Example 2 not self-distributive, due to $x = d, y = a$, and $z = 0$ obtained $d * (a * 0) = d * d = 1$, whereas $(d * a) * (d * 0) = 1 * a = a$.

Proposition 2. 8. [7] Let $(X; *, 1)$ be a *BE*-algebra, then the following identity applies for all x, y, $z \in X$.

$(P1)$ $x * (y * x) = 1$

$(P2)$ $x * ((x * y) * y) = 1,$

$(P1)$ Let $(X; *, 1)$ be a self-distributive *BE*-algebra. If $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$.

The concept of derivation in *BE*-algebra has been discussed in [7]. Let $(X; *, 1)$ be a *BE*-algebra. We define $x \vee y = (y * x) * x$ for all $x, y \in X$.

Definition 2.9. [7] A self-map d of *BE*-algebra $(X; *, 1)$ is called a derivation in X if $d(x * y) = (x * d(y)) \vee (d(x) * y)$ for all $x, y \in X$.

III. MAIN RESULT

In this section the concept of *t*-derivation in *BE*-algebra is defined based on the concept of derivation in *BE*-algebra [7] and *t*-derivation in *BP*-algebra [11]. Furthermore, the properties of *t*-derivation in *BE*-algebra are investigated. Finally, we define a fixed set and kernel of *t*-derivation in *BE*-algebra, as well as determined properties.

Definition 3.1. Let $(X; *, 1)$ be a *BE*-algebra. A self-map d_t of X is defined by $d_t\,(x) = x * t$ for all $t, x \in X$.

Lemma 3.2. Let $(X; *, 1)$ be a *BE*-algebra. If d_t be a self-map of X, then $d_1(x) = 1$ for all $x \in X$.

Proof. Let $(X; *, 1)$ be a *BE*-algebra. Since d_t is a self-map of X, then from *BE2* obtained $d_1(x) = x * 1 = 1$ for all $x \in X$.

Definition 3.3. Let $(X; *, 1)$ *BE*-algebra. A mapping d_t of X to itself is called a *t*-derivation in X if for all $x, y \in X$ satisfies

$$d_t(x * y) = (x * d_t(y)) \vee (d_t(x) * y).$$

The following is given the properties of the existence of *t*-derivation in *BE*-algebra.

Theorem 3.4. If $(X; *, 1)$ be a *BE*-algebra, then d_1 is a *t*-derivation in X.

Proof. Let $(X; *, 1)$ be a *BE*-algebra. By Lemma 3.2, *BE1*, *BE2*, and *BE3* obtained for all $x, y \in X$

$$\begin{aligned} d_1(x * y) &= 1 \\ &= 1 * 1 \\ &= (y * 1) * 1 \\ &= 1 \vee y \\ &= (x * 1) \vee (1 * y) \\ d_1(x * y) &= (x * d_1(y)) \vee (d_1(x) * y). \end{aligned}$$

Thus, it is proven that d_1 is a *t*-derivation in X.

Furthermore, we discuss some properties of *t*-derivation in *BE*-algebra.

Proposition 3.5. Let $(X; *, 1)$ be a *BE*-algebra. If d_t be a *t*-derivation in X, then

(i) $d_t(x) = d_t(x) \vee (t * x)$ for all $x \in X$.

(ii) $t * x \leq x * t$ for all $x \in X$.

Proof. Let $(X; *, 1)$ *BE*-algebra and d_t is a *t*-derivation in X.

(i) By *BE3* obtained $d_t(1) = 1 * t = t$, then

$$\begin{aligned} d_t(x) &= d_t(1 * x) \\ &= (1 * d_t(x)) \vee (d_t(1) * x) \\ d_t(x) &= d_t(x) \vee (t * x), \end{aligned}$$

for all $x \in X$.

(ii) By (i), *BE1*, and *BE4*, for all $x \in X$ obtained

$$\begin{aligned} (t * x) * (x * t) &= (t * x) * d_t(x) \\ &= (t * x) * (d_t(x) \vee (t * x)) \\ &= (t * x) * (((t * x) * d_t(x)) * d_t(x)) \\ &= ((t * x) * d_t(x)) * ((t * x) * d_t(x)) \\ (t * x) * (x * t) &= 1. \end{aligned}$$

Since $(t * x) * (x * t) = 1$, then it is proven that $t * x \leq x * t$ for all $x \in X$.

Proposition 3.6. Let $(X; *, 1)$ be a *BE*-algebra. If d_t is a *t*-derivation in X, then $d_t(x * d_t(x)) = d_t(d_t(x) * x)$ for all $x \in X$.

Proof. Let $(X; *, 1)$ be a *BE*-algebra. Since d_t is a *t*-derivation in X, then by *BE1* and *BE3* for all $x \in X$ obtained

$$\begin{aligned} d_t(x * d_t(x)) &= (x * d_t(d_t(x))) \vee (d_t(x) * d_t(x)) \\ &= (x * d_t(d_t(x))) \vee 1 \\ &= (1 * (x * d_t(d_t(x)))) * (x * d_t(d_t(x))) \\ &= (x * d_t(d_t(x))) * (x * d_t(d_t(x))) \\ d_t(x * d_t(x)) &= 1. \end{aligned}$$

On the other hand, by *BE1* and *BE2* obtained

$$\begin{aligned} d_t(d_t(x) * x) &= (d_t(x) * d_t(x)) \vee (d_t(d_t(x)) * x) \\ &= 1 \vee (d_t(d_t(x)) * x) \\ &= ((d_t(d_t(x)) * x) * 1) * 1 \\ d_t(d_t(x) * x) &= 1. \end{aligned}$$

Therefore, it is proven that $d_t(x * d_t(x)) = d_t(d_t(x) * x)$ for all $x \in X$.

Let $(X; *, 1)$ be a *BE*-algebra and d_t is a *t*-derivation in X. Define a fixed set of X by $Fix_{d_t}(X) = \{x \in X : d_t(x) = x\}$.

Proposition 3.7. Let $(X; *, 1)$ be a *BE*-algebra and d_t is a *t*-derivation in X.

(i) If $x, y \in Fix_{d_t}(X)$, then $x \vee y \in Fix_{d_t}(X)$ for all $y \in X$.

(ii) If $x \in Fix_{d_t}(X)$, then $(d_t \circ d_t)(x) = x$.

Proof. Let $(X; *, 1)$ be a *BE*-algebra and d_t is a *t*-derivation in X.

(i) Since $x \in Fix_{d_t}(X)$, then $d_t(x) = x$ for all $x \in X$. Then, since $z \vee z = (z * z) * z = 1 * z = z$ for all $z \in X$ and by *BE1* and *BE3* obtained

$$\begin{aligned} d_t(x \vee y) &= d_t((y * x) * x) \\ &= [(y * x) * d_t(x)] \vee [d_t(y * x) * x] \\ &= [(y * x) * x] \vee [((y * d_t(x)) \vee (d_t(y) * x)) * x] \\ &= [(y * x) * x] \vee [((y * x) \vee (y * x)) * x] \\ &= [(y * x) * x] \vee [(y * x) * x] \\ &= (y * x) * x \\ d_t(x \vee y) &= x \vee y. \end{aligned}$$

Hence, it is evident that $x \vee y \in Fix_{d_t}(X)$.

(ii) Let $x \in Fix_{d_t}(X)$, then $d_t(x) = x$. Such that

$$(d_t \circ d_t)(x) = d_t(d_t(x)) = d_t(x) = x.$$

Let $(X;*,1)$ be a *BE*-algebra and d_t is a *t*-derivation in X. Define a kernel of d_t by $Kerd_t = \{x \in X : d_t(x) = 1\}$.

Proposition 3.8. If $(X;*,1)$ be a *BE*-algebra, then $Kerd_1$ is a subalgebra.

Proof. Let $(X;*,1)$ be a *BE*-algebra. By Theorem 3.4 obtained d_1 is a *t*-derivation in X. Since $d_1(1) = 1 * 1 = 1$, then $1 \in Kerd_1$, such that $Kerd_1$ is a non-empty set. By *BE2* obtained $d_1(x) = x * 1 = 1$ and $d_1(y) = y * 1 = 1$ for all $x, y \in X$, then $x, y \in Kerd_1$. Then, by by *BE1*, *BE2*, and *BE3* we obtain

$$\begin{aligned} d_1(x*y) &= (x * d_1(y)) \vee (d_1(x) * y) \\ &= (x*1) \vee (1*y) \\ &= 1 \vee y \\ &= (y*1)*1 \\ d_1(x*y) &= 1. \end{aligned}$$

Thus, $x * y \in Kerd_1$. Therefore, it is proven that $Kerd_1$ is a subalgebra of X.

Proposition 3.9. Let $(X;*,1)$ be a *BE*-algebra and d_t is a *t*-derivation in X.

(i) If $x \in Kerd_t$, then $x \vee y \in Kerd_t$ for all $y \in X$.

(ii) If $y \in Kerd_t$, then $x * y \in Kerd_t$ for all $x \in X$.

Proof. Let $(X;*,1)$ be a *BE*-algebra and d_t is a *t*-derivation in X.

(i) Since $x \in Kerd_t$, then $d_t(x) = 1$. By *BE2* we get

$$\begin{aligned} d_t(x \vee y) &= d_t((y*x)*x) \\ &= [(y*x)*d_t(x)] \vee [d_t(y*x)*x] \\ &= [(y*x)*1] \vee [d_t(y*x)*x] \\ &= 1 \vee [d_t(y*x)*x] \\ &= [(d_t(y*x)*x)*1]*1 \\ d_t(x \vee y) &= 1. \end{aligned}$$

Hence, it is evident that $x \vee y \in Kerd_t$.

(ii) Let $y \in Kerd_t$, then $d_t(y) = 1$. By *BE2* obtained

$$\begin{aligned} d_t(x*y) &= (x * d_t(y)) \vee (d_t(x) * y) \\ &= (x*1) \vee (d_t(x)*y) \\ &= 1 \vee (d_t(x)*y) \\ &= ((d_t(x)*y)*1)*1 \\ d_t(x*y) &= 1. \end{aligned}$$

Thus, it is proven that $x * y \in Kerd_t$ for all $x \in X$.

IV. CONCLUSION

In this paper, the concept of *t*-derivation in *BE*-algebra is defined, as one form of development of the concept derivation in *BE*-algebra. The definition of *t*-derivation in *BE*-algebra begins with define a mapping d_t, that is a self-map in *BE*-algebra. Furthermore, it was investigated d_1 as the existence of *t*-derivation in *BE*-algebra, as well as obtained other properties of *t*-derivation. Finally, the properties of the fixed set and kernel of *t*-derivation in *BE*-algebra are obtained based on its elements, and it is obtained that the kernel d_1 is a subalgebra in *BE*-algebra.

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