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The Connection on Fiber Bundles

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ARTICLE INFO	ABSTRACT
Published Online: 13 July 2022 **Corresponding Author:** Abdel Radi Abdel Rahman Abdel Gadir Abdel Rahman	A connection is a device that defines the concept of parallel transport on a bundle, that is identifies fibers over nearby points. Fiber bundles form are the natural mathematical framework for the gauge filed theories. Also affine connection is the most elementary type of connection, a means of parallel transfer of tangent vectors on a manifold from one point to another. In any manifold with a positive dimension there is an infinite number of the affine connection; junctions are among the simplest methods to determine the differentiation of sections of vector bundles. Our goal in this paper is to identify the concept of connection in fiber bundles. We followed the analytical historical mathematical method and we found that the connection on the fiber bundle is a smooth distribution over the total bundle area, which is of central importance in modern geometry and leads to appropriate formulas for geometry constants.

KEYWORDS: Connection, Fiber Bundles, Vector Bundles, Principal Bundles.

1. INTRODUCTION

lifting path in S^2 to that bundle space. In section we found that such bundles actually arise in nature and are of considerable importance in areas not (apparently) related to mathematical physics. However, we also saw that there are, intact, many different ways to construct such circle bundles over the 2-sphere and it is not clear how one should make a selection from among these. But here's a coincidence for you Monopole field strengths g are "quantized" (Dirac Quantization Condition). In effect, there is one monopole for each integral (assuming there are any monopoles at all, of course). On the other hand, we have also pointed out that the principal U (1)-bundles over$S^2$are classified by the elements of the fundamental group π_1(U (1)) of the circle and, there is one principal U(1)-bundle over S^2 for each integer. This tantalizing one-to-one correspondence between monopoles and principal U (1)-bundles over S^2 suggests that the monopole strength may dictate the choice of the bundle with which to model it. Precisely how this choice is dictated is to be found in the details of the path lifting procedure to which we have repeatedly alluded. We will consider here only the simplest nontrivial case.

The Dirac Quantization Condition asserts that, for any charge q and monopole strength g, one must have qg = (½) n

In mathematics, and especially differential geometry and gauge theory, a connection is a device a notion of parallel transport on the bundle; that is, a way to "connect" or identify fibers over nearby points. A principal G-connection on a principal G-bundle P over a smooth manifold M is a particular type of connection which is compatible with the action of the group G. A principal connection can be viewed as a special case of an Ehresmann connection, and is sometimes called a principal Ehresmann connection. It gives rise to (Ehresmann) connections on any fiber bundle associated to P via the associated bundle construction. In particular, on any associated vector bundle the principal connection induces a covariant derivative, an operator that can differentiate sections of that bundle along tangent directions in the base manifold. Principal connections generalize to arbitrary principal bundles the concept of a linear connection on the frame bundle of a smooth manifold.

2. CONNECTIONS ON PRINCIPAL BUNDLES

Perhaps we should pause to recapitulate ended with some rather vague mutterings about an appropriate replacement for the classical vector potential of a monopole consisting of some sort of "bundle of circles aboveS^2" and a procedure for

Let π: E →M be a fiber bundle. The vertical bundle VE →Eis the sub bundle of TE →E defined by

$$VE = \{\xi \in TE \,|\, \pi_* \xi = 0\}. \qquad (6)$$

Its fibers V_pE :=$(VE)_p \subset T_p$E are called vertical subspaces.

Then V_pE =$T_p((E_{\pi(p)})$, so the vertical sub bundle is the set of all vectors in TE that are tangent to any fiber.

By the above definition, the covariant derivative defines for each section s: M → E and $x \in$ M a map

$$\nabla_{s(x)} : T_x M \to V_{s(x)} E.$$

We shall require the definition of parallel transport in fiber bundles to satisfy two (not quite independent) conditions:

1- The definition of $\nabla_X s$ in (3.1) is independent of γ except for the tangent vector ˙Υ (0) = X.

2- The map $\nabla_{s(x)} : T_x M \to V_{s(x)} E$ is linear.

Proposition 3.2 [3]

Suppose π: E→ M is a fiber bundle and for every path γ (t) ∈ M there is a smooth family of diffeomorphisms P_γ^t: $E_{\gamma(0)} \to E_{\gamma(t)}$ satisfying P_γ^o = Id and conditions (1) and (2). Then for each $x \in$ M and p ∈E_x, there is a unique linear injection

$$Hor_p : T_x M \to T_p E$$

Such that $Hor_p(\dot{\Upsilon}(0)) = \frac{d}{dt} P_\gamma^t(p) \big|_{t=0}$ for all paths with γ (0) = x. Moreover, the image of Hor_p is complementary to V_pE in T_pE.

Proof: Fix$x_0 \in M$ and $p_0 \in E_{x0}$. Then for any path γ (t) ∈M with γ (0) =x_0, the family of diffeomorphisms P_γ^t: $E_{x0} \to E_{\gamma(t)}$ is the flow of some vector field Y(t, p) on the total space of the pullback bundle $_\gamma^* E$. Choosing any section s: M →E with s (x_0) = p_0 and writing F (t, p) = $(P_\gamma^t)^{-1}$(p), we have

$$\nabla_{\dot{\Upsilon}(0)} s = \frac{d}{dt} F(t, s(\gamma(t))) \big|_{t=0} = \frac{\partial F}{\partial t}(0, p_0) + D_2 F(0, p_0) \circ T s(\dot{\Upsilon}(0))$$

$$= -Y(0, p_0) + T s(\dot{\Upsilon}(0)),$$

Thus

$$Hor_{p_0}(\dot{\Upsilon}(0)) = \frac{d}{dt} P_\gamma^t(p_0) \big|_{t=0} = Y(0, p_0) = T s(\dot{\Upsilon}(0)) - \nabla_{\dot{\Upsilon}(0)} s. \quad (8)$$

This expression is clearly a linear function of ˙Υ (0). It is also injective since $\nabla_{\dot{\Upsilon}(0)} s \in V_{p_0}$E if and only if ˙Υ (0) = 0, as we can see by applying π_*. The same argument shows (im Hor_{p_0}) ∩ V_{p_0}E = {0}, and since any non-vertical vector $\xi \in T_{p_0}E \backslash V_{p_0}$E can be written as T s (˙Υ (0)) for some path γ and section s, clearly

$$\text{im } Hor_{p_0} \oplus V_{p_0} E = T_{p_0} E. \qquad (9)$$

The moral is that parallel transport, if defined properly, determines for every p ∈ E a horizontal subspace T_pE := im Hor_p complementary to the vertical subspace V_pE. conversely, it's easy to see that choosing such complimentary subspace H_PE determines P_γ^t uniquely. This should be sufficient motivation for the following definition.

Definition 3.3 [3]

A connection on the fiber bundle π: E → M is a smooth distribution HE on the total space such that HE ⊕ VE = TE.

for some integer n. For a charge of unit strength (q=1) this becomes g = (½) n so that the smallest positive value for g (in the units we have tacitly adopted) is g = $\frac{1}{2}$

For this case, the potential 1-forms for the monopole are

$$A_N = \frac{1}{2}(1 - \cos \emptyset)\, d\theta \text{ on } U_N \subseteq S^2, \qquad (1)$$

and

$$A_s = \frac{1}{2}(1 + \cos \emptyset)\, d\theta \text{ on } U_s \subseteq S^2, \qquad (2)$$

A_N= A_+| U_N and A_s=A_-| U_s). Thus, on $U_s \cap U_N$, A_N- A_s= dθ so

$$A_N = A_s + d\theta \text{ on } U_s \cap U_N. \qquad (3)$$

At this point we must be the reader's indulgence. We are about to do something which is (quite properly) considered to be in poor taste. We are going to introduce what will appear to be a totally unnecessary complication. For reasons that we will attempt to explain once the deed is done, we replace the real-valued 1- forms A_N and A_s by the pure imaginary 1-forms A_N andA_s defined by

$$\mathcal{A}_N = -i\mathcal{A}_N \text{ on } U_N \text{ and } \mathcal{A}_s = -i\, \mathcal{A}_s \text{ on } U_s. \qquad (4)$$

Now (4) becomes $\mathcal{A}_N$=$\mathcal{A}_s$ - i dθ which, for no apparent reason at all, we prefer to write as

$$\mathcal{A}_N = e^{i\theta} \mathcal{A}_s e^{-i\theta} + e^{i\theta}\, de^{-i\theta} \qquad (5)$$

All of this algebraically quite trivial, of course, but the motivation is no doubt obscure (although one cannot help but notice that the transition functions for the Hop f bundle have put in an appearance). Keep in mind that our purpose in this preliminary chapter is to illustrate with the simplest case the general framework of gauge theory and that the process of generalization often requires that the instance being generalized undergo some cosmetic surgery first (witness the derivative of f: ℝ →ℝ at a ∈ R as a number $\acute{f}$(a), versus the derivative of f: $R^n \to R^m$at a ∈ R^n as a linear transformation Df_a:($R^n \to R^m$).

The process which led to the appropriate generalization in our case was particularly long and arduous and did not reach tuition until the 1950, with the work of Ehresmann [Ehr].

Ehresmann was attempting to generalize to the context of bundles such classical notions from differential geometry as "connection," "parallel translation," and "curvature," all of which had been elegantly formulated by Elie car fan in terms of the so-called "frame bundle".

3. CONNECTIONS ON FIBER BUNDLES

Before doing that, it helps to generalize slightly and consider an arbitrary fiber bundle π: E →M, with standard fiber F. now parallel transport along a path γ (t) ∈ M will be defined by a smooth family of diffeomorphisms P_γ^t: $E_{\gamma(0)} \to E_{\gamma(t)}$. Now however, we are differentiating paths through the fiber $E_x \cong$ F, which is generally not vector space, so $\nabla_{s(x)}$X is not in the fiber itself but rather in its tangent space $T_{s(x)}(E_x) \subset T_{s(x)}$E. Remember that the total space E is itself a smooth manifold, and has its own tangent bundle TE →E.

Definition 3.1[4]

(ii) $\nabla_{X+X's} = \nabla_X s + \nabla_{X'} s$.

(iii) $\nabla_X(f s) = f\nabla_X s$ + (X f) s, $\forall f \in C^\infty$(B).

(iv)∇_{fX}s = f$\nabla_X s$, $\forall$ f $\in C^\infty$(B).

The curvature 2-form F is defined by

F (X, Y) s = $\nabla_X \nabla_Y$s -$\nabla_Y \nabla_X$s - $\nabla_{[X,Y]}$s. (14)

In a natural basis

$$F_{\mu\nu}s = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) s. \quad (15)$$

Suppose that s_a, a = 1, 2… dim V is a local basis of section such that s = $s_a s^a$ let

$$s_a A^a_{\mu\, b} = \nabla_\mu s_b, \quad (16)$$

Then

$$(\nabla_\mu s)^a = \partial_\mu s^a + A^a_\mu b^{s^b}. \quad (17)$$

The connection 1-forms transform under change of basis in the obvious way

$$s^a \to \Lambda^a b^{s^b}, \qquad s_a \to s_b(\Lambda^{-1})^b a,$$

$$A_\mu{}^a{}_b \to \Lambda^a_e A^e_\mu (\Lambda^{-1})^c{}_b - (\Lambda^{-1})^a \partial_\mu \Lambda^c{}_b .$$

The curvature two-form is given by

$$F^{\ a}_{b\mu\nu} = \partial_\mu A_\nu{}^a{}_b - \partial_\nu A_\mu{}^a{}_b + [A_\mu, A_\nu]^a{}_b , \quad (18)$$

And transforms homogeneously

$$F^a_{b\mu\nu} \to \Lambda^a F^e (\Lambda^{-1})^c{}_b.$$

4. CONNECTIONS

To start with, we recall the notion of killing vector field, given a Lie group action (P, G, $\boldsymbol{\Psi}$), every element A of the Lie algebra g of G defines a vector filed A_*via the flow $\Psi_{\exp(tA)}$, that is,

$$(A_*)_p = \frac{d}{dt|0}\Psi_{\exp(tA)}(p) = \Psi'_p(A).$$

A_* is called the Killing vector field generated by A. Now, consider a principal fiber bundle (P, G, M, $\boldsymbol{\Psi}$, π). We denote the vertical distribution spanned by the killing vector fields of the G-action by V and call $V_p \subset T_p$P the vertical subspace of T_pP at $p\in$ P

Lemma 4.1[1]

The vertical distribution V has the following properties.

1- It is equivariant , that is, $V_{\psi a(p)} = \psi'_a(V_p)$.

2- The mapping

$$\boldsymbol{\psi}: P \times g \to V, (p, A) \mapsto = \psi'_a(A)$$

Is an isomorphism of vector bundles. In particular, the mappings= ψ'_a : g $\to V_p$ are isomorphism's of vector spaces.

3- For every p ∈ P, the vertical sub space V_p coincides with the tangent space of the fiber at p and, thus, with Ker (π'_p)

Definition 4.2: [1]

(Connection on principal fiber bundle) Let (P, G, M, $\boldsymbol{\psi}$, π) be a principal fiber bundle. A connection on P is a distribution Γ on P such that

1- $\Gamma_p \oplus V_p = T_p P$ for all p∈ P,

2- $\Gamma_{\psi a(p)} = \psi'_a(\Gamma_p)$ for all p∈ P and a∈ G.

Γ_p Is called the horizontal subspace at p.

A connection on principal bundle will be often referred to as a principal connection.

Remark 4.3:[1]

For any p ∈ E, the fiber H_PE $\subset T_P$E is called the horizontal subspace at p.

We can now recast all of the previous concepts in terms of horizontal subspace. Assume a connection (i.e. a horizontal sub bundle) has been chosen. Then for each $x\in$ M and p $\in E_x$, the linear map $\pi_*: T_P\text{E} \to T_x$M restricts to an isomorphism H_PE $\to T_x$M . Its inverse is called the horizontal lift

$$Hor_P: T_x\text{M} \to T_P\text{E} .$$

A path through the total space E is called horizontal if it is everywhere tangent to HE. Then given $x_0\in$ M and $p_0 \in E_{x0}$, any path γ (t) ∈ M with γ (0) = x_0 lifts uniquely to a horizontal path $\tilde{\gamma}$ (0) =p_0. This path is similarly called the horizontal lift of γ, and its tangent vectors satisfy

$$\frac{d}{dt}\tilde{\gamma}(t) = Hor_{\tilde{\gamma}(t)}(\dot{\gamma}(t)). \quad (10)$$

By considering horizontal lifts for all possible p $\in E_{x0}$, we obtain naturally the parallel transport diffeomorphism P^t_γ: $E_{x0} \to E_{\gamma(t)}$. Finally, (4.3.3) yields a convenient formula for the covariant derivative with respect to any vector X $\in T_x$M,

$$\nabla_{x^s} = T_s(X) - Hor_{s(x)}(X). \quad (11)$$

Note that since $\pi_* T_s$(X) = X, the second term on the right is simply the projection of T_s(X) to the horizontal subspace. We can express this more simply by defining the vertical projection

K: TE →VE,

which maps each T_PE to the vertical subspace V_PE by projecting along H_PE. Then

$$\nabla_{x^s} = K \circ T_s(X) \quad (12)$$

So the covariant derivative is literally the "vertical part" of the tangent map. For a section s (t) ∈ E along a path γ (t) ∈M, we have the analogous formula

$$\nabla_{t^s}(t) = K(\dot{s}(t)). \quad (13)$$

As one would expect, it is clear from this formula that s (t) is a horizontal lift of γ (t) if and only if $\nabla_{t^s} \equiv 0$.

The projection K: TE →VE is called a connection map, and it gives an equivalent definition for connections on fiber bundles.

Definition 3.4: [3]

A connection on the fiber bundle π: E →M is a smooth fiber wise linear map K: TE → VE such that K (ξ) = ξ for all ξ∈ VE.

The two definitions are related by setting HE = ker K.

Remark 3.5: [3]

The existence of connections for bundles on infinite dimensional manifolds is a far more intricate problem, because such manifolds do not generally admit smooth partitions of unity. However, more direct constructions of connections succeed in many interesting cases.

4. CONNECTIONS ON VECTOR BUNDLES

Let Γ (E, π, B, V, G) be the space of smooth sections of a vector bundle. A connection ∇ provides a map from T (M) × Γ (E, π, B, V, G) → Γ (E, π, B, V, G) written ∇ ×s, with X ∈ T (M) and s ∈Γ (E, π, B, V, G) such that

(i) ∇_X(s + s') = ∇_Xs + ∇_Xs'.

The subspace H_u of T_uP is called the horizontal subspace at u, and vectors in H_u are called horizontal tangent vectors at u.

Definition 4.8: [6]

Let π: P →M be a principal G-bundle. Then a g-valued 1-form

ω on P is called a connection form if it satisfies

(i) ω (u) $(\sigma(X)_u)$ = X for every u ∈ P and X ϵ g.

(ii) R_g^* ω = Ad (g^{-1}). ω for every g ∈G.

Proposition 4.9: [6]

If H is a connection in principal G-bundle π : P → M, then the map ω : P →Λ^1 (TP ;g) defined in 4.3.3 is a connection form on P with H_u = ker ω(u) for u ∈P.

Proof: formula (i) in the definition is just a reformulation of the formula

$\omega_u \circ \sigma_{u*}$= id g established in 4.3.3.

We next show that ω is a g-valued 1-form on P. let ℓ= $\{X_1,\ldots,X_m\}$ be a basis for g, and let u ∈P.

5. CONNECTION FORM

Let P be a principal bundle over V and Γ a connection on P. If dξ∈T_ξ, then Γ dξ∈$\mathfrak{N}_\xi$ and a→ ξ a is an isomorphism of $\mathcal{Y}$ onto$\mathfrak{N}_\xi$. We define a differential form γ on P with values in $\mathcal{Y}$ by setting γ (dξ) =a where (dξ) = ξa. In order to prove that γ is differentiable, it is enough to prove that γ takes differentiable vector fields into differentiable functions with value in $\mathcal{Y}$. We have γ (Z_a)= a for every a ∈$\mathcal{Y}$ and γ(X) =0 for every X∈$\mathfrak{J}$. Since the module (P) is generated by $\mathfrak{J}$ and the vector fieldsZ_a, γ(X) is differentiable for every differentiable vector filed X on P. Thus corresponding to every connection Γ on P, there exists one and only one form with values in $\mathcal{Y}$such that Γ (dξ) =ξγ (dξ) for every dξ∈T_ξ. It is easily seen that γ satisfies

1- γ (ξa) = a for every ξ∈ P and a ∈$\mathcal{Y}$

2- γ (dξs) = s^{-1}γ(dξ)s for dξ∈T_ξ and s ∈G.

A $\mathcal{Y}$- valued form on P satisfying 1- and 2-is called a connection form. Given a connection Γ for which γ is easy to see that there exists one and only one connection Γ for which γ is the associated form, i.e., Γ(dξ) = ξγ(dξ) for every dξ∈T_ξ.

6. AFFINE CONNECTIONS

Consider a smooth vector field Y on a manifold M and a tangent vector X_p∈T_pM at a point p in M. to define the directional derivative of Y in the directionX_p, it isnecessary to compare the values of Y in a neighborhood of p. If q is a point near p, in general it is not possible to compare the vectors Y_q and $Y_p$by taking the difference$Y_q - Y_p$, since they are in distinct tangent spaces. For this reason, the directional derivative of a vector field on an arbitrary manifold M cannot be defined in the same way, we extract form the directional derivative in $\mathbb{R}^n$certain key properties and call any map D: (M) × (M) ⟶ (M) with these properties an affine connection.

1-Bypoint1, every tangent vector X_p ∈ T_p Padmitsaunique decomposition into a horizontal component hor X_p ∈ Γ_p and a vertical component ver X_p ∈ V_p ,

$$X_p = \text{hor } X_p + \text{ver} X_p . \qquad (19)$$

Since both Γ and V are smooth, the mappings hor : TP ⟶ Γ and ver : TP ⟶ V are smooth .sthus , if X is a smooth vector filed on P, then hor X and ver X are smooth vector fields, too.

2- For a given connection Γ, the restriction of $\pi'$to the horizontal subspace Γ_p yields an isomorphism of Γ_p and $T_{\pi(p)}$M. thus, every vector field X on M admits a unique horizontal lift, that is, a vector field X^hon P with values in the horizontal distribution which is π-related to X. It is obtained by applying the inverse of the above isomorphism point wise to X. By construction, $X^h$is **ψ** invariant. Conversely, every **ψ**- invariant horizontal vector field on P is the horizontal lift of a vector field on M.

3- Every connection on a principal bundle P induces a connection on any bundle associated with P. indeed, let Γ be a connection on the principal bundle P (M, G) and let E= P × GF be an associated bundle. For f ∈ F.

Definition 4.4: [1]

(Connection form) Let (P, G, M, **ψ**, π) be a principal bundle and let Γ be a connection on P. The 1-form ω on P with values in g defined by

$$\omega_p \text{ (X) :=}(\Psi_p')^{-1} \text{ (verX)}, \qquad p \in P, \quad X \in T_p P$$

Is called the connection form of Γ.

As an immediate consequence of this definition, we obtain the following formula of the horizontal component of a tangent vector X ∈ T_p P**:**

$$\text{Hor X =X}- \Psi_p'(\omega(X)). \qquad (19)$$

Proposition 4.5: [6]

If π: P⟶M is a principal G-bundle, then

$$(R_g)*\boldsymbol{\sigma} \ (X) = \boldsymbol{\sigma} \ (Ad(g^{-1})X) \qquad (20)$$

For every x∈ X and g∈ G.

Proof: Since

$$R_g \circ \sigma_u(h) = uhg = ugg^{-1}hg = \sigma_{ug} \circ i_{g^{-1}}(h)$$

For u ∈ P and h ∈G, we have that

$$((R_g)*\boldsymbol{\sigma} \ (X))_{ug} = (R_g)*\boldsymbol{\sigma} \ (X)_u = (R_g)_* \circ \ \sigma_{u*}(X)$$

$$= \sigma_{ug*} \circ Ad \ (g^{-1})(X) = \boldsymbol{\sigma} \ (Ad \ (g^{-1})X)_{ug}. \qquad (21)$$

Corollary 4.6: [6]

If π: P⟶ M is a principal G-bundle, then

$$V_{ug} = (R_g)_* V_u \qquad (22)$$

For every u∈ P and g∈ G.

Proof: by proposition 4.1we have that

$$(R_g)_* \circ \ \sigma_{u*} = \sigma_{ug*} \circ Ad \ (g^{-1}) \qquad (23)$$

So that

$$(R_g)_* V_u = (R_g)_* \circ \ \sigma_{u*}(g) = \sigma_{ug*}(g) = V_{ug}.$$

Definition 4.7: [6]

A (principal) connection in a principal G-bundle π: P⟶ M is a distribution H on P such that

(i) T_uP= $H_u \oplus V_u$ for every u ∈ P.

(ii) H_{ug}= $(R_g)_* H_u$ for every u ∈ P and g ∈ G.

Even though spaces of forms with values in a bundle are easy to define there is no canonical analogue of the exterior derivative. There do however exist differential operators

$$D: \Omega^k(M;\zeta) \to \Omega^{k+1}(M;\zeta)$$

That satisfies familiar product formulas. These operators are called covariant derivatives (or connections) and are related to the notion of a connection on principal bundle, which we now define and study.

Let G be a compact Lie group. Recall that the tangent bundle τG has a canonical trivialization

$$\psi: G \times TG \to TG$$
$$(g, v) \to D(\ell g)(v)$$

Where for any g ∈ G, ℓg: G → G is the map given by left multiplication by g, and D (ℓg): $T_h G \to T_{gh} G$ is its derivative. r_g and D (r_g) will denote the analogous maps corresponding to right multiplication.

The differentials of right multiplication on G define aright action of G on the tangent bundle τ G. We claim that the trivialization ψ is equivariant with respect to this action, if we take as the right action of G on T_1G to be the ad joint action:

$$T_1 G \times G \to T_1 G$$
$$(v, g) \to D(\ell_{g-1})(v)\, D(r_g).$$

This bundle has following relevance. Let P^*(τ M): P^*(TM) → P be the Pull-back over the total space P of the tangent bundle of M. we have a subjective map of bundles

$$\tau P \to P^*(\tau M).$$

Define T_FP to be the kernel bundle of this map. So the fibreT_FP at a point y ∈ P is the kernel of the subjective linear transformation D p (y): T_yP →T_p(y) M. notice that the right action of G on the total space of the principal bundle P defines an action of G on the tangent bundle τ P, which restricts to an action of G on T_FP. Furthermore, by recognizing that the fibers are equivariantly homeomorphic to the Lie group G, the following is a direct consequence of the above considerations:

Proposition 7.2: [4]

T_FP is naturally isomorphic to the Pull-back of the ad joint bundle

$$T_F P \cong P^*(\text{ad}(p)).$$

Thus we have an exact sequence of G-equivariant vector bundles over P.

$$0 \to P^*(\text{ad}(p)) \to \tau P \xrightarrow{DP} P^*(TM) \to 0.$$

Recall that short exact sequences of bundles split as Whitney sums. A connection is a G-equivariant splitting of this sequence.

Definition 7.3: [4]

A connection on the principal bundle P is a G-equivariant splitting

$$\omega A : \tau P \to P^*(\text{ad}(P))$$

of the above sequence of vector bundles. That is, ωA defines a G-equivariant isomorphism

$$\omega A \oplus Dp: \tau P \to P^*(\text{ad}(P)) \oplus P^*(\tau M).$$

The following is an important description of the space of connections on P, (P).

Intuitively, an affine connection on a manifold is simply a way of differentiating vector fields on the manifold.

Mimicking the directional derivative in$\mathbb{R}^n$, we define the torsion and curvature of an affine connection on a manifold M. miraculously, both torsion and curvature are linear over C^∞functions in every argument.

There are infinitely many affine connections on any manifold. On a Riemannian manifold, however, there is a unique torsion- free affine connection compatible with the metric, called the Riemannian or Leiv-Civita connection. As an example, we describe the Riemannian connection on a surface in $\mathbb{R}^3$.

On an arbitrary manifold M, which is not necessarily embedded in a Euclidean space, we can define the directional derivative of a C^∞function f in the direction $X_p \in T_pM$ in the same way as before:

$$\nabla_{X_p} f = X_p f. \qquad (24)$$

However, there is no longer a canonical way to define the directional derivative of a vector filed Y.

Definition 6.1: [5]

An affine connection on a manifold M is an ℝ-bilinear map

$$\nabla: (M) \times (M) \to (M),$$

Written ∇_XY for ∇(X, Y), satisfying the two properties below: if $\mathcal{F}$ is the ring C^∞(M) of C^∞function on M, then for all X,Y ∈$\mathfrak{X}$(M),

i- $\nabla_X Y$ is $\mathcal{F}$-linear in X,

ii- (Leibniz rule) $\nabla_X Y$ satisfies the Leibniz rule in Y: for f ∈$\mathcal{F}$,

$$\nabla_X(fY) = (Xf)\, Y + f\nabla_X Y. \qquad (25)$$

Example 6.2: [5]

The directional derivative D_X of a vector filed Y on $\mathbb{R}^n$is an affine connection on$\mathbb{R}^n$, sometimes called the Euclidean connection on $\mathbb{R}^n$.

7. CONNECTIONS AND CURVATURE

In modern geometry, differential topology, and geometric analysis, one often needs to study not only smooth functions on a manifold, but more generally, spaces of smooth sections of a vector bundle Γ (ζ).(Notice that sections of bundles are indeed a generalization of smooth functions in that the space of sections of the n-dimensional trivial bundle over a manifold M, $\Gamma(\epsilon_n) = C^\infty(M; R^n) = \oplus_n C^\infty(M; R)$. similarly, one needs to study differential forms that take values in vector bundles. These are defined as follows.

Definition 7.1: [4]

Let ζ be a smooth bundle over a manifold M. a differential k-form with values in ζ is defined to be smooth section of the bundle of homomorphism's, Hom ($\Lambda^k(\tau(M)), \zeta) = \Lambda^k(\tau(M)^*) \otimes \zeta$.

We write the space of k-forms with values in ζ as

$$\Omega^k(M;\zeta) = \Gamma(\Lambda^k(\tau(M)^* \otimes \zeta). \qquad (26)$$

The zero forms are simply the space of sections, $\Omega^0(M;\zeta) = \Gamma(\zeta)$. Notice that if ζ is the trivial bundle $\zeta = \epsilon_n$, then one gets standard forms,

$$\Omega^k(M;\epsilon_n) = \Omega^k(M) \otimes R^n = \oplus_n \Omega^k(M). \qquad (27)$$

$F_A \in \Omega^1$(M; ad (p)).

These measures to what extent the splitting ωA commutes with the bracket operation on vector fields. More precisely, let x and y be vector fields on M. the connection ω_A defines an equivariant splitting of τP and hence defines a "horizontal lifting of these vector fields, which we denote by $\tilde{x}$and $\tilde{y}$ respectively.

Definition 7.6: [4]

The curvature $F_A \in \Omega^2$(M; ad (p)) is defined by

$$F_A (x, y) = \omega_A[\tilde{x}, \tilde{y}]. \qquad (30)$$

For those unfamiliar with bracket operation on vector fields, another important construction with connection is the associated covariant derivative which is defined as follows.

Definition 7.7: [4]

The covariant derivative induced by the connection ω_A

$$D_A: \Omega^0(\text{M; ad (p)}) \to \Omega^1(\text{M; ad (p)})$$

Is defined by

$$D_A (\sigma) (X) = [\tilde{x}, \sigma]. \qquad (31)$$

where X is a vector field on M.

The notion of covariant derivative, and hence connection, extends to vector bundles as well. Let ζ: $\mathcal{P}$: $E^{\zeta} \to$ M be a finite dimensional vector bundle over M.

Definition 7.8: [4]

A connection on ζ (or a covariant derivative) is a linear transformation

$$D_A: \Omega^0(\text{M}; \zeta) \to \Omega^1(\text{M}; \zeta)$$

That satisfies the Leibnitz rule

$$D_A(\text{f}\emptyset) = \text{df} \otimes \emptyset + \text{f}\, D_A(\emptyset) \qquad (32)$$

For any f $\in C^\infty$(M; R) and any $\emptyset \in \Omega^0$(M; ζ).

Now we can model the space of connections on a vector bundle, $\mathcal{A}$ (ζ) similarly to how we modeled the space of connection on a principal bundle $\mathcal{A}$ (P). Namely, given any two connections D_A and D_B on ζ and a function f $\in C^\infty$(M; R), one can take the convex combination f · D_A+ (1-f) ·D_B and obtain a new connection.

RESULTS

A connection on a fiber bundle is a device that defines the concept of parallel transport on the bundle, which is a smooth distribution over the total area of the bundle. The idea of connection enables us to transfer precise local geometric objects such as tangent vectors along a curve in a parallel and consistent manner. Connection is of central importance in modern geometric because it allows conducting comparison between the local geometry at one point and the geometry at another point. The connections lead to appropriate formulas for the geometric constants the affine connection is the most elementary type of connection and a means of parallel transfer of tangent vectors on a manifold from one point to another.

CONCLUSION

We dealt with the definitions of connection on principal, fibers and vector bundles, and we dealt with connection and

Proposition 7.4: [4]

The space of connections on the principal bundle P, $\mathcal{A}$ (P), is an affine space modeled on the infinite dimensional vector space of one forms on M with values in the bundle ad (P), Ω^1 (M; ad (P)).

Proof: consider two connection ωA and ωB

$$\omega A, \omega B: \tau P \to P^*(\text{ad (P)}).$$

Since these are splitting of the exact sequence, they are both the identity when restricted to P^*(ad (P)) ↪ τP.

Thus their difference, ωA-ωB is zero when restricted to P^*(ad (P)). By exact sequence it therefore as a composition

$$\omega A\text{-}\omega B: \tau P \to P^*(\tau M) \xrightarrow{\alpha} P^*(\text{ad (P)})$$

For some bundle homomorphism α:P^*(τM) → P^*(ad (P)).

That is, for every y ∈P, α defines a linear transformation

$$\alpha_y: P^*(TM)_y \to P^*(ad(P))_y.$$

Hence for every y ∈ P, defines a linear transformation

$$\alpha_y: T_{P(y)}\text{M} \to ad(P)_{P(y)}.$$

Furthermore, the fact that both ωA and ωB are equivariant splitting says that ωA-ωB is equivariant, which translates to the fact α_y only depends on the orbit of y under the G-action. That is,$\alpha_y = \alpha_{yg}$: $T_{P(y)}$M → $ad(P)_{P(y)}$ for every g ∈ G. thus α_y only depends on P(y) ∈ M.

Hence for every $x \in$ M, α defines, and is defined by, a linear transformation

$$\alpha_x: T_x\text{M} \to ad(P)_x.$$

Thus α may be viewed as a section of the bundle of homomorphism, Hom (TM, ad (P)), and hence is a one form,

$$\alpha \in \Omega^1(\text{M; ad (P)}.$$

Thus any two connections on P differ by an element in Ω^1(M; ad (P) in this sense.

Now reversing the procedure, an element β $\in \Omega^1$(M; ad (P) defines an equivariant homomorphism of bundles over P,

$$\beta: P^*(\tau M) \to P^*(\text{ad (P)}).$$

By adding the composition

$$\tau P \xrightarrow{DP} P^*(\tau M) \xrightarrow{\beta} P^*(\text{ad (P)})$$

to any connection (equivariant splitting)

$$\omega A: \tau P \to P^*(\text{ad (P)})$$

One produces a new equivariant splitting of τp, and hence a new connection. The proposition follows.

Remark 7.5: [4]

Even though the space of connections $\mathcal{A}$ (P) is affine, it is not, in general a vector space. There is no "zero" in $\mathcal{A}$ (P) since there is no pre-chosen, canonical connection. The one exception to this, of course, is when P is the trivial G-bundle,

$$\text{P} = \text{M} \times \text{G} \to \text{M}. \qquad (28)$$

In this case there is an obvious equvariant splitting of τP, which serves as the "zero" in $\mathcal{A}$ (P). Moreover in this case the ad joint bundle ad (P) is also trivial,

$$\text{Ad (P)} = \text{M} \times \text{g} \to \text{M}. \qquad (29)$$

Hence there is a canonical identification of the space of connection on the trivial bundle with Ω^1(M; g) = Ω^1(M) ⊗ g.

Let P: P →M be a principal G-bundle and let ωA ∈ (P) be a connection. The curvature F_A of ωA is two forms relationship to curvature, and we concluded that the idea of connection enables us to transfer tangent vectors along a curve or groups of curves in a parallel and consistent manner. Affine connection is the most elementary type of connection and a means of parallel transmission of these vectors.

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