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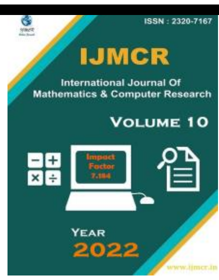

A New Approach on the Two-Dimensional Differential Transform

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ABSTRACT

In recent years, new formulas of the two-dimensional differential transform have been proven by using the definition of the transform. In this work, we use a new approach based on the definition of the transform and the summation properties to prove the two-dimensional differential transform of the product of two functions, then we used this result to establish other useful formulas. This study shows that this procedure can be used to find formulas for many complicated terms. This enables us to apply the differential transform method on many types of partial differential equations. To demonstrate this approach, we applied the dimensional differential transform method on selected equations and compared our results with analytical solutions obtained by other methods

KEYWORDS: Two-dimensional differential transform; Partial differential equations; Differential transform method

I. INTRODUCTION

The concept of differential transform was first introduced by Zhou [1]. Then Chen and Ho in [2] developed the deferential transform on partial deferential equations to obtained closed form series solutions for linear and nonlinear initial value problems. Ref. [3] introduced new useful formulas for one-dimensional differential transform and applied the differential transform method on selected ordinary differential equations. In [4], we reviewed the two-dimensional differential transform (2-DT) and applied the differential transform method on selected nonlinear partial differential equations. The main aim of this work is to use the summation properties to prove the two-dimensional differential transform of the product of two functions and then use the product formula to find other formulas. This procedure as we will see in the numerical examples will help us to apply the differential transform method on many types of partial differential equations. To

do that, we introduce in the next section, the two-dimensional differential transform and the review the proofs of some basic formulas which based on the definition of differential transform [5, 6].

II. Basic Definitions and Formulas

To introduce the 2-DT, we assume that $w(x, y)$ be a $C^{\infty}(\Omega)$ function and (x_0, y_0) be any point of Ω, where Ω is an open domain of R^2, and then we defined the Taylor series of the function $w(x, y)$ about (x_0, y_0) as

$$w(x,y)=\sum_{k,h=0}^{\infty}\left[\frac{w^{(k,h)}(x)}{k!h!}\right]_{x=(x_0,y_0)}(x-x_0)^k(y-y_0)^h \quad (1)$$

Then the two-dimensional differential transform, which denoted by D_T, is defined as following:

Definition: (2.1): Let $w(x,y)$ be an analytic function about (x_0, y_0), then the 2-DT, of $w(x,y)$ is defined as**:**

$$D_T\{w(x,y)\}:=W(k,h)=\left[\frac{w^{(k,h)}(x,y)}{k!h!}\right]_{(x_0,y_0)}, \quad (2)$$

Hence, if we chose $(x_0, y_0)=(0,0)$, then the definition (2) reduces to

$$D_T\{w(x,y)\} := W(k,h) = \left[\frac{w^{(k,h)}(x,y)}{k!h!}\right]_{(0,0)}. \qquad (3)$$

Definition: (2.2): The inverse differential transform of $W(k,h)$ is defined as

$$D_T^{-1}\{W(k,h)\} = w(x,y) := \sum_{k,h=0}^{\infty} W(k,h)x^k y^h \qquad (4)$$

Next we assume that $U(h,k)$, $V(h,k)$ and $W(h,k)$ are the differential transform of the analytic functions $u(x,y)$, $v(x,y)$ and $w(x,y)$ respectively. Then by using the definition (3) and (4), we can establish the following theorems:

Theorem (2.1)

$$D_T\{\alpha u(x,y)+\beta v(x,y)\} = \alpha D_T\{u(x,y)\} + \beta D_T\{v(x,y)\} \qquad (5)$$

The proof of the linearity property (5) follows immediately from the definition (3). Similarly we can prove the linearity property of the inverse differential transform.

Theorem (2.2)

$$D_T\{w^{(r,s)}(x,y)\} = \frac{(k+r)!}{k!}\frac{(h+s)!}{h!}W(k+r,h+s) \qquad (6)$$

To prove the differential transform of the (r-s)-derivatives (6), we use the formula

$$w^{(r,s)}(x,y) = \sum_{k,h=0}^{\infty}\left(\frac{w^{(k+r,h+s)}(x,y)}{k!h!}\right)_{\substack{x=0\\y=0}} x^k y^h \qquad (7)$$

This establishes the formula (6) as following

$$D_T\{w^{(r,s)}(x,y)\} = D_T\left\{\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\left(\frac{w^{(k+r,h+s)}(x,y)}{k!h!}\right)_{\substack{x=0\\y=0}} x^k y^h\right\}$$

$$= D_T\left\{\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{(h-r)!}{k!}\frac{(h-s)!}{h!}\left(\frac{w^{(k+r,h+s)}(x,y)}{(h-r)!(h-s)!}\right)_{\substack{x=0\\y=0}} x^k y^h\right\}$$

$$= \frac{(h-r)!}{k!}\frac{(h-s)!}{h!}W(k+r,h+s)$$

Theorem (2.3):

$$D_T\{x^i y^j\} = \delta(k,i)\delta(h,j) = \begin{cases}1 & k=i \;\&\; h=j\\ 0 & otherwise\end{cases}, \qquad (8)$$

The proof of this theorem follows immediately from identity

$$x^i y^j \equiv \sum_{k,h=0}^{\infty}\delta(k,i)\delta(h,j)x^k y^h. \qquad (9)$$

III. The 2-DT Of the product of two functions

In this section we introduce the formula of the 2-DT of the product of two functions proved in [5, 6]. The proof of this formula was based on the definition of the 2-DT. As a new approach, we use the summation properties to prove the product formula, and then used this procedure to establish many other formulas for complicated functions. This approach enables us to apply the differential transform method on many types of equations. To do that let us introduce the following theorem [6].

Theorem (3.1):

Let $u(x,y)$, $v(x,y)$ and $w(x,y)$ be an analytic functions, with $w(x,y) = u(x,y)\cdot v(x,y)$ then

$$D_T\{w(x,y)\} := W(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)V(k-r,s), \qquad (10)$$

Where $U(k,h)$ and $V(k,h)$ are the 2-DT of the functions $u(x,y)$ and $v(x,y)$ respectively. To prove this theorem, we used the procedure of [5, 6], which based on the definition (3). By definition, we can find

$$W(0,0) = \frac{1}{0!0!}\left[u(x,y)v(x,y)\right]_{\substack{x=0\\y=0}} = U(0,0)V(0,0)$$

$$W(1,0) = \frac{1}{1!0!}\frac{\partial}{\partial x}\left[u(x,y)v(x,y)\right]_{\substack{x=0\\y=0}}$$

$$= \left[u^{(1,0)}(x,y)v(x,y) + u(x,y)v^{(1,0)}(x,y)\right]_{\substack{x=0\\y=0}}$$

$$= U(1,0)V(0,0) + U(0,0)V(1,0)$$

$$W(2,0) = \frac{1}{2!0!}\frac{\partial^2}{\partial x^2}\left[u(x,y)v(x,y)\right]_{\substack{x=0\\y=0}}$$

$$= U(2,0)V(0,0) + U(1,0)V(1,0) + U(0,0)V(2,0)$$

$$W(0,1)=\frac{1}{0!1!}\frac{\partial}{\partial y}\left[u(x,y)v(x,y)\right]_{\substack{x=0\\y=0}}$$

$$=U(0,1)V(0,0)+U(0,0)V(0,1)$$

$$W(1,1)=\frac{1}{1!1!}\frac{\partial^2}{\partial x\partial y}\left[u(x,y)v(x,y)\right]_{\substack{x=0\\y=0}}$$

$$=U(1,1)V(0,0)+U(1,0)V(0,1)+$$
$$U(0,1)V(1,0)+U(0,0)V(1,1)$$

$$W(1,2)=\frac{1}{1!2!}\frac{\partial^3}{\partial x\partial y^2}\left[u(x,y)v(x,y)\right]_{\substack{x=0\\y=0}}$$

$$=U(1,2)V(0,0)+U(1,1)V(0,1)+U(1,0)V(0,2)+$$

$$U(0,2)V(1,0)+U(0,1)V(1,1)+U(0,0)V(1,2)$$

$$W(0,2)=\frac{1}{0!2!}\frac{\partial^2}{\partial y^2}\left[u(x,y)v(x,y)\right]_{\substack{x=0\\y=0}}$$

$$=U(0,2)V(0,0)+U(0,1)V(0,1)+U(0,0)V(0,2)$$

$$W(2,1)=\frac{1}{2!1!}\frac{\partial^3}{\partial x^2\partial y}\left[u(x,y)v(x,y)\right]_{\substack{x=0\\y=0}}$$

$$=U(2,1)V(0,0)+U(2,0)V(0,1)+U(1,1)V(1,0)+$$

$$U(1,0)V(1,1)+U(0,1)V(2,0)+U(0,0)V(2,1)$$

$$W(2,2)=\frac{1}{2!2!}\frac{\partial^4}{\partial x^2\partial y^2}\left[u(x,y)v(x,y)\right]_{\substack{x=0\\y=0}}$$

$$=U(2,2)V(0,0)+U(2,1)V(0,1)+U(2,0)V(0,2)+$$

$$U(1,2)V(1,0)+U(1,1)V(1,1)+U(1,0)V(1,2)+$$

$$U(0,2)V(2,0)+U(0,1)V(2,1)+U(0,0)V(2,2)$$

Next, if we continuous this procedure, they conclude the formula (10), [5, 6].

Our approach of establishing the formula (10) is based on the definition (3) and the summation properties. To explain that, we assume

$$u(x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{u^{(r,s)}(0,0)}{r!s!}x^r y^s,$$

$$v(x,y)=\sum_{\tilde r=0}^{\infty}\sum_{\tilde s=0}^{\infty}\frac{u^{(\tilde r,\tilde s)}(0,0)}{\tilde r!\tilde s!}x^{\tilde r} y^{\tilde s} \tag{11}$$

Then, we can write

$$w(x,y)=\sum_{r,s=0}^{\infty}\sum_{\tilde r,\tilde s=0}^{\infty}\frac{u^{(r,s)}(0,0)}{m!n!}\frac{v^{(\tilde r,\tilde s)}(0,0)}{\tilde r!\tilde s!}x^{r+\tilde r}y^{s+\tilde s}. \tag{12}$$

Next, we set

$$r+\tilde r=k, \quad s+\tilde s=h \tag{13}$$

This enables us to write

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{r=0}^{k}\sum_{\tilde s=0}^{h}\frac{u^{(r,h-\tilde s)}(0,0)}{r!(h-\tilde s)!}\frac{v^{(k-r,\tilde s)}(0,0)}{(k-r)!\tilde s!}x^k y^h$$

$$=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{r=0}^{k}\sum_{\tilde s=0}^{h}\frac{u^{(r,h-\tilde s)}(0,0)}{r!(h-\tilde s)!}\frac{v^{(k-r,\tilde s)}(0,0)}{(k-r)!\tilde s!}x^r y^{h-\tilde s}x^{k-r}y^{\tilde s}.$$

Or simply

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{h}\frac{u^{(r,h-s)}(0,0)}{r!(h-s)!}\frac{v^{(k-r,s)}(0,0)}{(k-r)!s!}x^r y^{h-s}x^{k-r}y^{s}.$$

This established the formula (10). Note that, making use of (13), we can write (12) in the form

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{h}\frac{u^{(r,s)}(0,0)}{r!s!}\frac{v^{(k-r,h-s)}(0,0)}{(k-r)!(h-s)!}x^k y^h$$

$$=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{h}\frac{u^{(r,s)}(0,0)}{r!s!}\frac{v^{(k-r,h-s)}(0,0)}{(k-r)!(h-s)!}x^r y^s x^{k-r}y^{h-s}.$$

This leads to the formula

$$D_T\{w(x,y)\}=\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,s)V(k-r,h-s), \tag{14}$$

Furthermore, by setting $h-s=\tilde s$, we can prove that the formulas (10) and (14) are equivalent.

Note that, by using the product formula (10), we can establish many other useful formulas, which enables us to apply the differential transform method on many partial deferential equations. To explain that, in the following section, we apply this procedure on selected partial deferential equations.

IV. NUMERICAL EXAMPLES

Example (4.1)

Consider the initial value problem

$$u_t = uu_x \qquad (15)$$

$$u(x,0) = 1 + x^2. \qquad (16)$$

The given initial value problem (15-16) satisfies the assumptions of the Cauchy-Kovalevsky theorem, and the Taylor series of the exact solution can be found in [7]. To solve (15-16) by using the 2-DT method, we first introduce the formula

$$D_T\{u(x,t)u_x(x,t)\} = \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)\,U(r,h-s)U(k-r+1,s), \qquad (17)$$

Next, we write

$$D_T\{u(x,t)\} = U(k,h),$$
$$D_T\{u_x(x,t)\} = (k+1)U(k+1,h), \qquad (18)$$

Then by using the formula (10), we can easily establish the formula (17).

Now, we are able to apply the two-dimensional transform on the both sides of (15). Hence using the formulas (6) and (17) yield the iteration formula

$$U(k,h+1) = \frac{1}{h+1}\left[\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)(k-r+1)U(k-r+1,s)\right] \qquad (19)$$

For the initial condition (16), we have

$$U(k,0) = \delta(k,0) + \delta(k,2) \qquad k = 0,1,2,\ldots$$
$$U(0,0) = 1, \qquad U(2,0) = 1, \qquad (20)$$
$$U(k,0) = 0, \qquad k = 1,3,4,\ldots$$

And the iteration formula (19), leads to the series solution

$$u(x,t) = 1 + x^2 + 2xt + t^2 + \cdots \qquad (21)$$

Hence, the Taylor series (21) of exact solution can be found in ref. [28], page 106.

Example (4.2)

Consider the initial value problem

$$u_{tt} - u_{xx} - x^2 u = x \qquad (22)$$

$$u(x,0) = 0, \qquad (23)$$

$$u_t(x,0) = 0. \qquad (24)$$

To solve the initial value problem (22-23-24), we first introduce the formula

$$D_T\{x^i u_x(x,t)\} = \sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r,i,)\delta(h-s,0)(k-r+1)U(k-r+1,s) \qquad (25)$$

To prove (25), we write

$$D_T\{x^i\} = \delta(k,i)\delta(h,0),$$
$$D_T\{u_x(x,t)\} = (k+1)U(k+1,h). \qquad (26)$$

Then by using the formula (10), we can easily establish the formula (26). Now, we are able to apply the two-dimensional transform on the both sides of (22). Hence using the formula (6), and (17) yield the iteration formula

$$(h+2)(h+1)U(k,h+2) - (k+2)(k+1)U(k+2,h) - \left(\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r,2)\delta(h,s)U(k-r,s)\right) = \delta(k,1) \qquad (27)$$

Using the initial conditions (23) and (24), we have

$$U(k,0) = 0 \qquad k = 0,1,2,\ldots \qquad (28)$$

$$U(k,1) = 0, \qquad k = 0,1,2,\ldots \qquad (29)$$

Next, we substitute (28-29) into (27) and simplify. This leads to the iteration formula

$$U(k,h+2) = \frac{1}{(h+2)(h+1)}\left[(k+2)(k+1)U(k+2,h) + \left(\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r,2)\delta(h,s)U(k-r,s)\right) + \delta(k,1)\right] \qquad (30)$$

Hence, the iteration formula (3.22), gives

$$U(1,2)=\tfrac{1}{2},\quad U(3,4)=\tfrac{1}{24},\quad U(1,6)=\tfrac{1}{120},$$

And the others are zeros. Therefore, the close form solution can be easily written in the form

$$u(x,t)=\frac{xt^2}{2}+\frac{x^3t^4}{24}+\frac{xt^6}{120}. \qquad (31)$$

Which is the exact solution found by the method of successive approximation in [8].

V. CONCLUSION

In this paper, we reviewed the concept of two-dimensional differential transform, properties and some formulas. Then we used an approach based on the summation properties to prove the dimensional differential transform of the product of two functions of two variables, and then used this product formula to establish other useful formulas. This study showed that this procedure can be used to find the differential transform of many complicated terms appears on the differential equation. This approach, as we showed in the numerical examples, enables us to apply the differential transform method on many types of partial differential equations. This also showed that this approach reduces the computational difficulties of finding formulas for many complicated functions. At the end, we reveal that the differential transform method is very efficient, simple and can be applied to many complicated linear and non-linear partial differential equations.

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