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Fractional Differential and Integrating Equations by Numerical Solution

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ABSTRACT

The main objective of this paper is to investigate a new fractional mathematical model that includes a nonsingular derivative factor. The basic properties of the new model including non-negative, finite solution, numerical simulations are shown, and some discussions from mathematical perspectives are given. Then, the optimal control problem for the new model is determined by introducing several variables. Solving fractional order differential equations in an accurate, reliable, and efficient manner is more difficult than in the case of standard integer order; In addition, most computational tools do not provide built-in functionality for this type of problem. In this paper, we review two of the most effective numerical methods for solving fractional-order problems, Static and solving nonlinear systems included in the implicit. Methods. We, therefore, present a set of MATLAB procedures specifically designed to solve three families of partial order problems: partial differential equations (FDEs), Some examples are provided to illustrate the use of the procedures.

KEYWORDS: Fractional Derivative, Fractional Differentiation, Fractional Calculus, Numerical Schemes, Conversion to Single-Order Systems.

1. INTRODUCTION

This paper is about problems arising in the field of fractional calculus - branch Mathematics that is, in a sense, as old as classical calculus as we know it. The origins [1] can be traced back to the end of the seventeenth century, the time when Newton and Leibniz developed the foundations of differentiation and Integrated account. Leibniz introduced the symbol

$$\frac{d^n f(x)}{dx^n}$$

to denote the nth derivative of a function f. When he reported this in a letter to de $l'Hospital$ (apparently with the implicit assumption that $n \in N$), de $l'Hospital$ replied: What does $\frac{d^n}{dx^n} f(x)$ mean if $n = \frac{1}{2}$. This letter from, written in 1695, is accepted as the first occurrence of what we today call a fractional derivative, and the fact that specifically for $n = \frac{1}{2}$. A fraction (rational number) actually gave rise to the name of this part of mathematics. This name has remained in use ever since, even though it is well known by now that there is no reason to restrict n to the set of rational numbers. Indeed, any real number – rational or irrational – will do just as well, at least for the analytical considerations that we shall concentrate on. Using ideas of ordinary calculus, we can differentiate a function, say, to the $1th$ or $2th$ order. We can also establish a meaning or some potential applications of the results. However, can we differentiate the same function to, say, the order better still, can we establish a meaning or some potential applications of the results, we may not achieve that through ordinary calculus, but we may through Numerical fractional calculus a more generalized form of calculus [2] . As a matter of fact, even complex numbers may be allowed, but this is well beyond the scope of this paper. Numbers of very interesting and applications of fractional differential equations in physics, chemistry, engineering, finance, and other sciences that have been developed in the last few decades. Some early examples are given. [3] and the classical papers of [4], [5], and [6, 7]. The concept of integration and differentiation is familiar to all who have studied elementary calculus. We know, for instance, that if $f(x) = x^2$ then integrating $f(x)$ to the 1^{st} order results in $\int f(x)\, dx = \frac{1}{3} x^3 + c_1$ and integrating the same function to the 2^{nd} order results in $\int[\int f(x)dx]dx = \frac{1}{12}x^4 + c_1 + c_2$. Similarly, $\frac{df(x)}{dx} = 2x$ and $\frac{d^2f(x)}{dx^2} = 2$ However, what if we wanted to integrate our function $f(x)$ to the $\frac{1}{2^{th}}$ order, or find its $\frac{1}{2^{th}}$ order derivative, How could we define our operations? Better still, would our results have a meaning or an application comparable to that of the familiar integer order operations?

2. FRACTIONAL CALCULUS AND APPLICATIONS

The Numerical Solution of Fractional Differential Equations: A Survey and a Software Tutorial [8] aims to provide a tutorial for the numerical solution of fractional differential equations (FDEs). Solving differential equations of fractional As the starting point for introducing fractional-order operators, we consider the Riemann–Liouville (RL) integral; for a function $y(t) \in L^1[t_0, T]$ (as usual, L^1 is the set of Lebesgue integrable functions), the fractional integral of order $\alpha > 0$ and origin at t_0 is defined as:

$$J_{t_0}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1}\, y(\tau) d\tau. \quad (1)$$

It provides a generalization of the standard integral, which, indeed, can be considered a particular case of the integral (1) when $\alpha = 1$. The left inverse of $J_{t_0}^{\alpha}$ is the fractional derivative:

$$\hat{D}_{t_0}^{\alpha} y(t) = D^m J_{t_0}^{m-\alpha} y(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t_0}^{t} (t-\tau)^{m-\alpha-1}\, y(\tau) d\tau \quad (2)$$

where $m = d\alpha^e$ is the smallest integer greater or equal to α and D^m, $y(m)$. An alternative definition of the fractional derivative, obtained after interchanging differentiation and integration in Equation (2), is the so-called Caputo derivative, which, for a sufficiently differentiable function, namely for: $y \in A^m[t_0, T]$ ($y(m-1)$ continuous, is given by:

$$D_{t_0}^{\alpha} y(t) = J_{t_0}^{m-1} D^m y(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-\tau)^{m-\alpha-1}\, y^{(m)}(\tau) d\tau \quad (3)$$

We observe that also $D_{t_0}^{\alpha} y(t)$ is a left inverse of the integral, namely

$$D_{t_0}^{\alpha} J_{\alpha_0}^{t} y = y, J_{t_0}^{\alpha} D_{t_0}^{\alpha} = y(t) - T_{m-1}(y, t_0)(t), \quad (4)$$

Where $T_{m-1}(y, t_0)(t)$ is the Taylor polynomial of degree $m-1$ for the function $y(t)$ centered at t_0 that is:

$$\sum_{k=0}^{m-1} \frac{(t-t_0)^k}{k!} y^{(k)}(t_0). \quad (5)$$

In general, from the content of the combination of [9] we can also note that for any $\beta > \alpha$, it holds:

$$J_{t_0}^{\beta} D_{t_0}^{\alpha} y(t) = J_{t_0}^{\beta} \hat{D}_{t_0}^{\alpha} [y(t) - T_{m-1}(y, t_0)(t)] = J_{t_0}^{\beta-\alpha} [y(t) - T_{m-1}(y, t_0)(t)] \quad (6)$$

Numerical methods for solving systems of FDEs, as well as of multi-order type and multi-term FDEs, are presented. Some aspects related to the efficient implementation of the methods are discussed and the corresponding MATLAB routines are made available. Numerical Solution of Multiterm Fractional Differential Equations Using the Matrix Mittag–Leffler Functions, by [10] focuses on a numerical approach to solve Multiterm Fractional Differential Equations (MTFDEs), that is, equations involving derivatives of different orders. They are very common to model many important processes, particularly for multi rate systems.

2.1 An Algorithm for Single-Term Equations

The method can be called indirect because, rather than discretizing the differential

Equation:

$$D_{t_0}^{\alpha} y(t) = f(t, y(t)) \quad (7)$$

with appropriate initial conditions

$$D^{\alpha} y(0) = y_0^{\alpha}, \qquad \alpha = 0,1, \dots, (n) - 1 \quad (8)$$

it requires some preliminary analytical manipulation, namely an application to convert the initial value problem for the differential equation into an equivalent Volterra integral equation,

$$y(x) = \sum_{k=0}^{m-1} \frac{x^k}{k!} D^k y(0) + \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} f(t, y(t)) dt \quad (9)$$

where $m = n$. We shall therefore now look at a method for the numerical solution of (9). The algorithm that we shall consider can be interpreted as a fractional variant of the classical second-order Adams–Bashforth–Moulton method. It has been introduced and briefly, more information is given in [11]. Some additional results for a specific initial value problem are contained in [12], a detailed mathematical analysis is provided in [13], and additional practical remarks can be found in [14]. Numerical experiments and comparisons with other methods are reported in [15, 16] Here we shall give an even more detailed analysis under quite general assumptions. We use the nodes $t_j, (j = 0,1, \dots, k+1)$ and interpret the function $(t_{k+1} - z)^{n-1}$ as a weight function for the integral. In other words, we apply the approximation

$$\int_0^{t_{k+1}} (t_{k+1} - z)^{n-1}\, g(z) dz \approx \int_0^{t_{k+1}} (t_{k+1} - z)^{n-1}\, g_{k+1}(z) dz \quad (10)$$

where g_{k+1} is the piecewise linear interpolant for g with nodes and knots chosen at the $t_j, j = 0,1,2, \dots, k+1$. The function values of the integrand g, taken at the points t_j [17, 18] . Specifically, we find that we can write the integral on the right-hand side of (10) as

$$\int_0^{t_{k+1}} (t_{k+1}-z)^{n-1}\, g_{k+1}(z)dz = \sum_{j=0}^{k+1} a_{j,k+1}\, g(t_j) \quad (11)$$

$$a_{j,k+1} = \int_0^{t_{k+1}} (t_{k+1}-z)^{n-1}\, \emptyset_{j,k+1}(z)dz \quad (12)$$

$$\emptyset_{j,k+1}(z) = \begin{cases} (z-t_{j-1}) \big/ (t_j - t_{j-1}) & if\, t_{j-1} < z \le t_j \\ (t_{j+1}-z) \big/ (t_{j+1}-t_j) & if\ t_j < z \le t_{j+1} \\ 0 & \end{cases} \quad (13)$$

This is clear because the functions $\emptyset_{j,k+1}$ satisfy

$$\emptyset_{j,k+1}(t_\mu) = \begin{cases} 0 & if\ j \ne \mu \\ 1 & if\ j = \mu \end{cases}$$

and that they are continuous and piecewise linear with breakpoints at the nodes t, so that they must be integrated exactly by our formula. for an arbitrary choice of the t_j, (12) and (13) produce

$$a_{0,k+1} = \frac{(t_{k+1}-t_1)^{n+1} + t_{k+1}^n (nt_1 + t_1 - t_{k+1})}{t_1 n(n+1)} \quad (15)$$

$$a_{j,k+1} = \frac{(t_{k+1}-t_{j-1})^{n+1} + (t_{k+1}-t_j)^n \left(n(t_{j-1}-t_j) + t_{j-1} - t_{k+1}\right)}{(t_{k+1}-t_j) n(n+1)}$$

$$+ \frac{(t_{k+1}-t_{j+1})^{n+1} - (t_{k+1}-t_j)^n \left(n(t_j - t_{j+1}) - t_{j+1} + t_{k+1}\right)}{(t_{k+1}-t_j) n(n+1)} \quad (16)$$

If $1 \le j \le k$ and

$$a_{k+1,k+1} = \frac{(t_{k+1}-t_k)^n}{n(n+1)} \quad (17)$$

In the case of equi -spaced nodes ($t_j = h_j$ with some fixed h), these relations reduce to

$$a_{j,k+1} = \begin{cases} \dfrac{h^n}{n(n+1)} (k^{n+1} - (k-n)(k+1)^n) & if\ j = 0 \\ \dfrac{h^n}{n(n+1)} \begin{pmatrix} (k-j+2)^{n+1} + (k-j)^{n+1} \\ -2(k-j+1)^{n+1} \end{pmatrix} & if\ 1 \le j \le k \\ \dfrac{h^n}{n(n+1)} & if\ j = k+1 \end{cases} \quad (18)$$

This then gives us our corrector formula (the fractional variant of the one-step Adams–Moulton method), which is

$$y_{k+1} = \sum_{j=0}^{m-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(n)} \left(\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^P) \right) \quad (19)$$

The idea we use to generalize the one-step Adams–Bash forth method is the same as the one described above for the Adams–Moulton technique: We replace the integral on the right-hand side of (9) by the product rectangle rule

$$\int_0^{t_{k+1}} (t_{k+1}-z)^{n-1} g(z)dz \approx \sum_{j=0}^{k} b_{j,k+1} g(t_j) \quad (20)$$

Where

$$b_{j,k+1} = \int_{t_j}^{t_{k+1}} (t_{k+1}-z)^{n-1} dz = \frac{(t_{k+1}-t_j)^n - (t_{k+1}-t_{j+1})^n}{n} \quad (21)$$

We can be derived in a way like the method used in the derivation of (16). However, here we are dealing with a piecewise constant approximation and not a piecewise linear one, and hence we must replace the (hat-shaped) functions $b_{j,k+1}$ by functions being of constant value 1 on $[t_j, t_{j+1}]$ and 0. On the remaining parts of the interval $[0, t_{k+1}]$ [17, 18] in the equispaced case, we have the simpler expression

$$b_{j,k+1} = \frac{h^n}{n} ((k+1-j)^n - (k-j)^n) \quad (22)$$

Thus, the predictor y_{k+1}^P is determined by the fractional Adams–Bash forth method:

$$y_{k+1}^P = \sum_{j=0}^{m-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(n)} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j). \quad (23)$$

Our basic algorithm, the fractional Adams–Bash forth–Moulton method, is therefore completely described now by (23) and (18) with the weights $a_{j,k+1}$ and $b_{j,k+1}$ being defined according to (16) and (20), respectively [19,20].

Lemma 2.1. Assume that the solution y of the initial value problem is such that:

$$\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} D_0^n\, y(t)dt - \sum_{j=0}^{k} b_{j,k+1} D_0^n y(t_j)\right| \le C_1 t_{k+1}^{\gamma_1} h^{\delta_1}$$

$$and$$

$$\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} D_0^n\, y(t)dt - \sum_{j=0}^{k} a_{j,k+1} D_0^n y(t_j)\right| \le C_2 t_{k+1}^{\gamma_2} h^{\delta_2}$$

With some $\gamma_1, \gamma_2 \ge 0$ and $\delta_1, \delta_2 > 0$. Then, for some suitably, chosen $T > 0$, we have:

$$\max_{0\le j\le N} |y(t_j) - y_j| = O(h^q)$$

Where $q = min\{\delta_1 + n, \delta_2\}\ and\ N = \left(\frac{T}{h}\right)$

Proof:

We will show that, for sufficiently small h,

$$|y(t_j) - y_j| \le Ch^q \qquad (24)$$

for all $j \in \{0,1,\dots,N\}$, where C is a suitable constant. The proof will be based on mathematical induction. In view of the given initial condition, the induction basis ($j = 0$) is presupposed [21,22] . Now assume that (24) is true for $j = 0,1,\dots,k$ for some $k \le N-1$. We must then prove that the inequality also holds for $j = k+1$. To do this, we first look at the error of the predictor y_{k+1}^P. By construction of the predictor we find that:

$$|y(t_{k+1}) - y_{k+1}^p| = \frac{1}{\Gamma(n)}\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} f(t, y(t))\,dt - \sum_{j=0}^{k} b_{j,k+1} f(t, y(t))\right|$$

$$\le \frac{1}{\Gamma(n)}\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} D_0^n\, y(t)dt - \sum_{j=0}^{k} b_{j,k+1} D_0^n y(t_j)\right| + \frac{1}{\Gamma(n)}\sum_{j=0}^{k} b_{j,k+1}\ \left|f\left(t_j, y(t_j) - f(t_j, y_j)\right)\right|$$

$$\le \frac{C_1 t_{k+1}^{\gamma_1} h^{\delta_1}}{\Gamma(n)} + \frac{1}{\Gamma(n)}\sum_{j=0}^{k} b_{j,k+1} LCh^q$$

$$\le \frac{C_1 t_{k+1}^{\gamma_1} h^{\delta_1}}{\Gamma(n)} + \frac{CLT^n}{\Gamma(n+1)} h^q \qquad (25)$$

Here we have used the Lipschitz property of f , the assumption on the error of the rectangle formula, and the facts that, by construction of the quadrature formula underlying the predictor, $b_{j,k+1} > 0$ for all j and k and:

$$\sum_{j=0}^{k} b_{j,k+1} = \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} dt = \frac{1}{n} t_{k+1}^n \le \frac{1}{n} T^n.$$

Based on the bound (25) for the predictor error we begin the analysis of the corrector error [23,24]. We recall the relation (17) which we shall use for$j = k+1$ and find, arguing in a similar way to above, that:

$$|y(t_{k+1}) - y_{k+1}| = \frac{1}{\Gamma(n)}\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} f(t, y)dt - \sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) - a_{k+1,k+1} f(t_{k+1}, y_{k+1}^P)\right|$$

$$\le \frac{1}{\Gamma(n)}\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} D_0^n\, y(t)dt - \sum_{j=0}^{k+1} a_{j,k+1} D_0^n y(t_j)\right|$$

$$+ \frac{1}{\Gamma(n)}\sum_{j=0}^{k+1} a_{j,k+1} \left|f\left(t_j, y(t_j)\right) - f(t_j, y_j)\right|$$

$$+ \frac{1}{\Gamma(n)} a_{k+1,k+1} |f(t_{k+1}, y(t_{k+1})) - f(t_{k+1}, y_{k+1}^p)|$$

$$\le \frac{C_2 t_{k+1}^{\gamma_2} h^{\delta_2}}{\Gamma(n)} + \frac{CL}{\Gamma(n)} h^q \sum_{j=0}^{k} a_{k+1,k+1} \frac{L}{\Gamma(n)}\left(\frac{C_1 T^{\gamma_1}}{\Gamma(n)} h^{\delta_1} + \frac{CLT^n}{\Gamma(n+1)} h^q\right)$$

$$\left(\frac{C_2 T^{\gamma_2}}{\Gamma(n)} + \frac{CLT^n}{\Gamma(n+1)} + \frac{C_1 LT^{\gamma_1}}{\Gamma(n)\Gamma(n+1)} + \frac{CL^2 T^n}{\Gamma(n)\Gamma(n+1)} h^n\right) h^q$$

in view of the nonnegativity of γ_1 and $\gamma_2 2$ and the relations $\delta_2 \le q$ and

$\delta_1 + n \leq q$. By choosing T sufficiently small, we can make sure that the second summand in the parentheses is bounded by $C/2$. Having fixed this value for T, we can then makethe sum of the remaining expressions in the parentheses smaller than $C/2$ too (forsufficiently small h) simply by choosing C sufficiently large[25]. It that the entire upper bound does not exceed Ch^q [26,27]. As a first application of this Lemma, we assume that the given data is such that the solution y itself is sufficiently differentiable. As mentioned above, the result depends on whether

$n > 1$ or $n < 1$.

Lemma2.2 An interesting observation here is that by choosing a larger number of corrector iterations, we essentially leave the computational complexity unchanged: A corrector iteration is of the form:

$$y_{j+1}^{(\ell)} = \sum_{r=0}^{n-1} \frac{r_{k+1}^r}{r!} y_0^{(r)} + \frac{h^n}{\Gamma(n+2)} f\left(t_{j+1}, y_{j+1}^{(\ell+1)}\right) + \frac{h^n}{\Gamma(n+2)} \sum_{r=0}^{j} a_{r,j+1} f(t_r, y_r)$$

Here $y_{j+1}^{(\ell)}$ denotes the approximation after corrector steps, $y_{j+1}^{(\ell)} = y_{j+1}^P$ is the predictor, and $y_{j+1} = y_0^{(r)}$ is the final approximation after µ corrector steps that we use. We can rewrite this as:

$$y_{j+1}^{(\ell)} = \beta_{j+1} + \frac{h^n}{\Gamma(n+2)} f\left(t_{j+1}, y_{j+1}^{(\ell-1)}\right)$$

where

$$\beta_{j+1} = \sum_{r=0}^{n-1} \frac{r_{k+1}^r}{r!} y_0^{(r)} + \frac{h^n}{\Gamma(n+2)} \sum_{r=0}^{j} a_{r,j+1} f(t_r, y_r)$$

is independent of. Thus, the total arithmetic complexity of the corrector part of the $(j+1)^{st}$ step (taking us from t_j to t_{j+1}) is $O(j)$ for the calculation of β_{j+1} plus $O(\ell)$ for the µ corrector steps, which (since µ is constant) is asymptotically the same as the complexity in the case $\ell = 1$.

Lemma 2.3. Let $n > 0$ and assume that $d_j h \in C^k[0,T]$ for some $k \geq 3$ and some suitable T. Then:

$$y(T) - y_{T/h} = \sum_{j=1}^{k_1} d_j h^{2j} + \sum_{j=1}^{k_2} d_j h^j + O(h^{k_3})$$

where k_1, k_2 and k_3 are certain constants depending only on k and satisfying

$k_3 > max(2k_1, k_2 + n)$.

2.2 Numerical Schemes for Multi-Term Equations

We extension the numerical methods to multi-term equations. The most important theoretical properties of these multi-term equations we restrict our attention to equations of the form

$$D_0^{n_k} y(x) = f\big(x, y(x)\big), D_0^{n_1} y(x), D_0^{n_2} y(x), \ldots, D_0^{n_{k-1}} y(x) \qquad (26)$$

(Where $0 < n_1 < n_2 < \ldots < n_k$) with a suitable function $f\big(x, y(x)\big)$ and initial conditions:

$$y^{(j)}(0) = y_0^{(j)}, \quad j = 0,1, \ldots, [n_k] - 1 \qquad (27)$$

3. Conversion to Single-Order Systems

In this way we transform the given initial value problem into a system of equations of the form:

$$D_0^\gamma y_0(x) = y_1(x),$$
$$D_0^\gamma y_1(x) = y_2(x),$$
$$\vdots$$
$$D_0^\gamma y_{N-2}(x) = y_{N-1}(x),$$
$$D_0^\gamma y_{N-1}(x) = f\left(x, y_0(x), {}^{y_{n_1}}/_{\gamma(x)}, \ldots, {}^{y_{n_{k-1}}}/_{\gamma(x)}\right), \qquad (28)$$

Together with the initial conditions:

$$y_j(0) = \begin{cases} y_0^{(j\gamma)} & if\ j\gamma \in \mathbb{N}_0 \\ 0 & else \end{cases} \qquad (29))$$

with the precise choice of the new parameters γ and N as appropriate. We have thus formally obtained an equation of the type:

$$D_0^\gamma Y(x) = F\big(x, Y(x)\big), \qquad if \quad Y(0) = Y_0 \qquad (30)$$

with certain vector-valued functions F (known) and Y (unknown) and an initial condition vector Y_0, a single-term equation of order γ with vector-valued data. Thus we calculate an approximate solution for this system, for the sake of simplicity, we shall restrict ourselves to the Adams–Bash forth–Moulton scheme developed above [25]. The first component of the solution vector is then the required approximate solution for the original equation. We illustrate the procedure taken from [12].

Example 3.1. Solve the nonlinear three-term equation

$$D_0^{1.355}y(x) = -x^{0.1}\frac{Q_{1.454}(-x)}{Q_{1.554}(-x)}e^x y(x) D_0^{0.555}y(x) + e^{-2x} - [D_0^1 y(x)]^2$$

for $0 \le x \le 1$, equipped with the initial conditions $y(0) = 1$ and $y\,(0) = -1$, with the same algorithm. The exact solution of this problem is $y(x) = e^{-x}$. When applying our idea to this equation, we first need to calculate the order γ of the new system. In our case the result is $\gamma = 1/300$, and hence the dimension of the resulting system is $N = 1.355/\gamma$ rather large number. In a first attempt we have tried to solve the system with the Adams–Bash forth–Moulton scheme as:

Table 3.1The equation solved with Numerical Adams method

Size	solution	Error	Order convergence
0.5	5.1513147531	−0.151314	
0.25	5.0468410217	−0.046841	1.61
0.125	5.0160284121	−0.016129	1.56
0.0625	5.0056277043	−0.0056277	1.52

Example 3.2

$$D_0^n y(x) = \frac{40310}{\Gamma(9-n)}x^{7-n} - 2\frac{\Gamma(5+{}^{n}/_{3})}{\Gamma(5-{}^{n}/_{3})}x^{4-{}^{n}/_{2}} + \frac{9}{4}\Gamma(n+1)$$
$$+\left(\frac{3}{2}x^{\frac{n}{2}} - x^4\right)^3 - \left(y(x)\right)^{3/2}$$

for $x \in [0,1]$ with homogeneous initial conditions ($y(0) = 0,\ \ y'(0) = 0$), the latter only in case $n > 1$.
The exact solution of this initial value problem is:

$$y(x) = x^7 - 2x^{4+{}^{n}/_{2}} + \frac{9}{4}x^n,$$

and hence:

$$D_0^n y(x) = \frac{40310}{\Gamma(9-n)}x^{7-n} - 2\frac{\Gamma(5+{}^{n}/_{3})}{\Gamma(5-{}^{n}/_{3})}x^{4-{}^{n}/_{2}} + \frac{9}{4}\Gamma(n+1)$$

then $D_0^n y(x) \in C^2[0,1]$ if $n > n$, and thus the conditions are fulfilled. Moreover, if Lemma 2.3 holds, the results in Tables C.1 and Tables C.2 where the notation $-4.51(-3)$ stands for -4.51×10^{-3}. In each case, the left most column shows the step size used, the following column gives the error of our results [28,29].

Table 3.2 Errors for Exampl1 3.2 with $n = 0.5$, taken at $x = 1$

size	Adam's scheme	Extrapolated value			
$^1/_{10}$	$\mathbf{-4.51(-3)}$				
$\mathbf{^1/_{20}}$	$\mathbf{-1.34(-3)}$	$\mathbf{-1.79(-3)}$			
$\mathbf{^1/_{40}}$	$\mathbf{-3.32(-5)}$	$\mathbf{-3.60(-4)}$	$\mathbf{1.61(-5)}$		
$\mathbf{^1/_{80}}$	$\mathbf{-2.16(-5)}$	$\mathbf{-7.15(-5)}$	$\mathbf{1.89(-6)}$	$\mathbf{2.17(-7)}$	
$\mathbf{^1/_{160}}$	$\mathbf{-1.56(-6)}$	$\mathbf{-1.23(-6)}$	$\mathbf{2.32(-7)}$	$\mathbf{2.68(-8)}$	$\mathbf{1.45(-8)}$
$\mathbf{^1/_{320}}$	$\mathbf{-6.62(-6)}$	$\mathbf{-3.05(-7)}$	$\mathbf{2.58(-8)}$	$\mathbf{2.19(-9)}$	$\mathbf{5.24(-10)}$

$^{1}/_{640}$	$-2.91(-8)$	$-5.14(-8)$	$2.83(-9)$	$1.71(-10)$	$2.27(-11)$
EOC	**1.98**	**2.57**	**3.24**	$\mathbf{3.89}$	**8.62**

Table 3.3 Errors for Exampl1 3.2 with $n = 0.5$, taken at $x = 1$

size	Adams scheme	Extrapolated value			
$^{1}/_{10}$	$3.50(-3)$				
$^{1}/_{20}$	$1.81(-3)$	$-1.50(-1)$			
$^{1}/_{40}$	$4.33(-5)$	$-5.60(-3)$	$4.06(-2)$		
$^{1}/_{80}$	$2.93(-5)$	$-1.11(-4)$	$2.17(-3)$	$-7.14(-3)$	
$^{1}/_{160}$	$2.53(-6)$	$7.18(-5)$	$1.46(-4)$	$-3.73(-4)$	$1.33(-4)$
$^{1}/_{320}$	$7.62(-6)$	$2.48(-5)$	$1.88(-5)$	$-1.42(-5)$	$1.04(-5)$
$^{1}/_{640}$	$2.91(-8)$	$1.13(-5)$	$2.31(-6)$	$-8.60(-7)$	$5.01(-7)$
EOC	**1.92**	**1.61**	**2.53**	$\mathbf{4.08}$	**7.34**

Scheme at $x = 1$, and the columns after that give the extrapolated values. The bottom line states the experimentally determined order of convergence for each of the columns on the right of the table. According to our theoretical considerations, these values should be $1 + n, 2, 2 + n, 3 + n, 4, 4 + n, \ldots$ in the case $0 < n < 1$ and $2, 1 + n, 2 + n, 4, 3 + n, 4 + n, \ldots$ for $1 < n < 2$. The numerical data in the following tables show that these values are reproduced approximately at least for $n > 1$. In the case $0 < n < 1$, displayed in Table 3.3, the situation seems to be less obvious. Apparently, we need to use much smaller values for h than in the case $n > 1$ before we can see that the asymptotic behavior really sets in. Our belief in the truth of Lemma 2.2 is not only supported by the numerical results but also by the results of de Hoog and Weiss [30] who show that asymptotic expansions of this form hold if we use the fractional Adams–Moulton method and that a similar expansion can be derived for the fractional [18].

4. CONCLUSION

From this paper we extracted the investigation of a new fractal mathematical model that includes a non-singular derivative factor. The basic characteristics of the new model including non-negative and finite solutions and numerical simulation were presented, and some discussions of the mathematical aspect were presented. And the problem of optimal control of the new model was identified by introducing several new variables. And solving fractional order differential equations in an accurate, reliable and efficient manner than was more difficult than in the case of order of standard integers. We also review two of the most effective numerical methods for solving fractional order problems, namely the constant system and the solution of nonlinear systems included in the implicit. Methods. We have therefore presented a set of MATLAB procedures designed specifically to solve three sets of partial order problems: Partial Differential Equations (FDEs), and a number of examples are given to illustrate the use of the procedures.

REFERENCES

1. Ross, B.: The development of fractional calculus 1695–1900. Hist. Math. 4, 75–89 (1977).
2. Diethelm, K.: Multi-term fractional differential equations, multi-order fractional differential systems and their numerical solution. J. Eur. Syst. Autom. 42, 665–676 (2008).
3. Oldham, K.B., Spanier, J.: The Fractional Calculus. Academic, New York (1974).
4. Torvik, P.J., Bagley, R.L.: On the appearance of the fractional derivative in the behavior of real materials. J. Appl. Mech. 51, 294–298 (1984).
5. Caputo, M.: Linear models of dissipation whose *Q* is almost frequency independent – II. Geophys. J. Roy. Astron. Soc. 13, 529–539 (1967); reprinted in fractional. Calc. Appl. Anal. 11, 4–14 (2008).
6. Caputo, M., Mainardi, F.: A new dissipation model based on memory mechanism. Pure Appl. Geophys. 91, 134–147 (1971); reprinted in Fractional. Calc. Appl. Anal. 10, 310–323 (2007).

7. Caputo, M., Mainardi, F.: Linear models of dissipation in anelastic solids. Rivista del Nuovo Cimento 1, 161–198 (1971).
8. Diethelm, K.; Ford, J.M.; Ford, N.J.; Weilbeer, M. Pitfalls in fast numerical solvers for fractional differential equations. *J. Comput. Appl. Math.* 2006, *186*, 482–503.
9. Gorenflo, R. Mainardi, F. Fractional calculus: Integral and differential equations of fractional order. In Fractals and Fractional Calculus in Continuum Mechanics (Udine, 1996), Carpinteri, A, Mainardi, F, Eds, Springer: Vienna, Austria, 1997, Volume 378, pp. 223–276.
10. Galeone, L.; Garrappa, R. Explicit methods for fractional differential equations and their stability properties. J. Comput. Appl. Math. 2009, 228, 548–560.
11. Chen, W. Sun, H.; Zhang, X.; Korošak, D. Anomalous diffusion modeling by fractal and fractional derivatives. Comput. Math. Appl. 2010, *59*, 1754–1758.
12. Diethelm, K., Ford, N.J.: Numerical solution of the Bagley–Torvik equation. BIT 42, 490–507 (2002).
13. Diethelm, K., Ford, N.J., Freed A.D.: Detailed error analysis for a fractional Adams method. Numer. Algorithms 36, 31–52 (2004)
14. Lin, G.; Aucoin, D.; Giotto, M.; Canfield, A.; Wen, W.; Jones, A.A. Lattice model simulation of penetrant diffusion along hexagonally packed rods in a barrier matrix as determined by pulsed-field-gradient nuclear magnetic resonance. Macromolecules 2007, 40, 1521–1528.
15. Le Doussal, P.; Sen, P.N. Decay of nuclear magnetization by diffusion in a parabolic magnetic field: An exactly solvable model. Phys. Rev. *B* 1992, 46, 3465–3485.
16. Ford, N.J. Connolly, J.A. Comparison of numerical methods for fractional differential equations. Commun. Pure Appl. Anal. 5, 289–307 (2006).
17. Tavazoei, M.S.: Comments on stability analysis of a class of nonlinear fractional-order systems. IEEE Trans. Circ. Syst. II 56, 519–520 (2009).
18. Tavazoei, M.S., Haeri, M., Bolouki, S, Siami, M. Stability preservation analysis for frequency-based methods in numerical simulation of fractional-order systems. SIAM J. Numer. Anal. 47, 321–328 (2008)
19. Diethelm, K. *The Analysis of Fractional Differential Equations*; Lecture Notes in Mathematics; Springer: Berlin, Germany, 2010; Volume 2004, p. 247.
20. Garrappa, R. Numerical Solution of Fractional Differential Equations: A Survey and a Software Tutorial. *Mathematics* 2018, *6*, 16.
21. Diethelm, K., Freed, A.D.: On the solution of nonlinear fractional differential equations used in the modeling of viscoplasticity. In: Keil, F., Mackens, W., Voß, H., Werther, J. (eds.) Scientific Computing in Chemical Engineering II: Computational Fluid Dynamics, Reaction Engineering, and Molecular Properties, pp. 217–224. Springer, Heidelberg (1999).
22. Lin, G. Fractional differential, and fractional integral modified-Bloch equations for PFG anomalous diffusion and their general solutions. 2017.
23. B. Ross (editor), Fractional Calculus and Its Applications; Proceedings of the International Conference Held at the University of New Haven, June 1974, Springer Verlag, 1975.
24. Lin, G. The exact PFG signal attenuation expression based on a fractional integral modified-Bloch equation, 2017.
25. A. Mathai, H. Haubold, Special Functions for Applied Scientists, Springer, 2008.
26. A. Michel, C. Herget, Applied Algebra and Functional Analysis, Dover Publications,1993.
27. Wyss, W. The fractional diffusion equation. *J. Math. Phys.* 1986, *27*, 2782–2785.
28. W. Kelley, A. Peterson, Theory of Differential Equations: Classical and Qualitative, Upper Saddle River, 2004
29. Metzler, R.; Klafter, J. The random walk's guide to anomalous diffusion: A fractional dynamics approach. Phys. Rep. 2000, 339, 1–77.
30. Mainardi, F. Luchko, Y. Pagnini, G. The fundamental solution of the space-time-fractional diffusion equation. Fractional. Calc. Appl. Anal. 2001, 4, 153–192.