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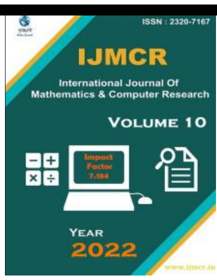

Pseudo-Hermitian Magnetic Curves in (*K*, *μ*) Manifold

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ARTICLE INFO	ABSTRACT
Published online: 02 September 2022 Corresponding Author: **Shweta Naik**	Object of the present paper is to study pseudo Hermitian magnetic curves in (κ,μ) manifold admitting Zamkovoy connection. We give main classification theorem for pseudo-Hermitian magnetic curve. Again we find the curvature and torsion of magnetic curves in a (κ,μ)-manifold admitting Zamkovoy connection.

KEYWORDS: (κ,μ) manifold, Pseudo-Hermitian magnetic curve, Zamkovoy connection, magnetic curve.
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1. INTRODUCTION

A differentiable curve α is said to be a magnetic curve for the magnetic field F if it is a solution of the Lorentz equation given by

$$\nabla_{\alpha 0(t)}\alpha^0(t) = \Phi(\alpha^0(t)). \qquad (1.1)$$

Where ∇ is the Riemannian connection on M. and Φ is the Lorentz force. These curves have constant speed and unit speed magnetic curves are called normal magnetic curves [2].

In [2], Druta-Romaniuc, Inoguchi, Munteanu and Nistor studied magnetic curves in a Sasakian manifold and the magnetic curves in Cosymplectic manifolds were studied in [3] by the same authors. Magnetic trajectories of an almost contact metric manifold were studied by Jleli, Munteanu and Nistor [5]. The classification of all uniform magnetic fields was obtained by Munteanu [6]. Guvenc and Ozgur studied slant magnetic curves in *S*manifold [4].

On an arbitrary oriented Riemannian manifold (M^{2n+1},g) canonically define a cross product X of two vector fields $X,Y \in \chi(M)$ as follows: $g(X \times Y,Z) = dv_g(X,Y,Z)$ for any $Z \in \chi(M)$. Where dv_g denotes the volume form defined by g, when M is an almost contact metric manifold. The cross product is given by the formula

$$X \times Y = g(\varphi X,Y)\xi - \eta(Y)\varphi X + \eta(X)\varphi Y .$$

Note that the unitary vector field X orthogonal to ξ, the basis $\{X,\varphi X,\xi\}$ is considered to be positively oriented.
Then we have $\xi \times Y = \varphi Y$ and $\xi \times \alpha^0 = \varphi\alpha^0$.

A Slant curve in an almost contact geometry arises as a generalization of curve of constant slope (also called a cylindrical helix) in an Euclidean space R^3. More precisely, a slant curve $\alpha^0(E_1)$ is defined by the condition that the scalar product $g(\alpha^0(E_1),\xi)$ of the tangent vector field $\alpha^0(E_1)$ and the Reeb vector field ξ to be constant. The Slant curves can also be viewed as a generalization of Legendre curves, in almost contact metric manifolds have been investigated intensively by many authors.

These studies motivated us to investigate pseudo-Hermitian magnetic curves in (κ,μ) manifold endowed with the Zamkovoy connection. In Section 2, we summarize the fundamental properties of (κ,μ) manifolds and the unique connection, namely the Zamkovoy connection. We find expression for curvature and torsion of magnetic curves in (κ,μ)manifolds admitting Zamkovoy connection. Finally we give classification theorem for pseudo-Hermitian magnetic curves.

2. PRELIMINARIES

Almost contact manifolds have odd-dimension. Let us denote the manifold that we study on by M. It carries two fields φ and ξ and a 1-form η. The field φ represents the endomorphism of the tangent spaces, the field ξ is called characteristic vector field and η is an 1-form such that

$$\varphi^2 = -I + \eta \otimes \xi, g(X,\xi) = \eta(X), \quad (2.1)$$

$$\eta(\xi) = 1, \varphi\xi = 0, \eta \cdot \varphi = 0, \quad (2.2)$$

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad (2.3)$$

$$g(\varphi X,Y) = -g(X,\varphi Y), g(X,\varphi Y) = d\eta(X,Y), \quad (2.4)$$

for any vector fields $X,Y \in \chi(M)$. The (κ,μ)-nullity distribution of a Riemannian manifold (M,g) for a real number κ and μ is a distribution

$$N(\kappa,\mu) : p \longmapsto N_p(\kappa,\mu) = \{Z \in \chi_p(M) : R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y) + \mu(g(Y,Z)hX - g(X,Z)hY)\} \quad (2.5)$$

for any $X,Y,Z \in \chi_p(M)$ and κ and μ being constants, where R denotes the Riemannian curvature tensor and $\chi_p(M)$ denotes the tangent vector space of M at any point $p \in M$.
If the characteristic vector field of a contact metric manifold belongs to the κ , μ nullity distribution, then the relation holds.

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \quad (2.6)$$

A contact metric manifold with $\xi \in N(\kappa,\mu)$ is called a (κ,μ)-contact metric manifold [1]. In a (κ,μ)-contact metric manifold M the following relations hold [1], [9]:

$$h^2 = (\kappa - 1)\varphi^2, \quad (2.7)$$

$$\nabla_X\xi = -\varphi X - \varphi hX, \quad (2.8)$$

$$(\nabla_X\varphi)Y = g(X + hX,Y)\xi - \eta(Y)(X + hX), \quad (2.9)$$

$$(\nabla_X\eta)Y = g(X + hX,\varphi Y), \quad (2.10)$$

The relation between Zamkovoy connection ∇^* and the Levi-Civita connection ∇ on M is given by [13],

$$\nabla^*_X Y = \nabla_X Y + \eta(Y)(\phi X + \phi hX) - g(X + hX, \phi Y)\xi + \eta(X)\phi Y. \quad (2.11)$$

$$\nabla^*_X\xi = 0. \quad (2.12)$$

3. MAGNETIC CURVE OF (κ,μ)-MANIFOLD WITH RESPECT TO ZAMKOVOY CONNECTION

Let $M(\varphi,\xi,\eta,g)$ be an $(2n + 1)$-dimensional (κ,μ)-contact metric manifold endowed with Zamkovoy connection ∇^* and $\alpha : I \longrightarrow M$ a curve parametrized by arc-length. If there exists a g-orthonormal vector fields $E_1, E_2,..., E_r$ along α such that

$$E_1 = \alpha^0,$$
$$\nabla*_{E1}E1 = k1*E2,$$
$$\nabla*_{E1}E1 = -k1*E1 + k2*E3, \quad (3.1)$$
$$..................$$
$$\nabla^*_{E_1}E_r = -k^*_{r-1}E_{r-1},$$

where α is called a Frenet curve for ∇^* of osculating order r, $(1 \le r \le 2n + 1)$. Here k^*_1, k^*_2,....., $^{k}{}_{r}{}^{*}{}_{-1}$ are called pseudo-Hermitian curvature functions of α and these functions are positive valued on I. A geodesic for ∇^* (or pseudo-Hermitian geodesic) is a Frenet curve of osculating order 1 for ∇^*. If $r = 2$ and ak^*_1 is a constant, then α is called a pseudo-Hermitian circle. A pseudo-Hermitian helix of order r, $r \ge 3$ is a Frenet curve for ∇^* of osculating order r with non-zero positive constant pseudo-Hermitian helix, we mean its osculating order is 3.

Let (M,φ,ξ,η,g) be a (κ,μ) manifold endowed with the Zamkovoy connection ∇^*.
Let us denote the fundamental 2-form of M by Ω. Then, we have

$$\Omega(X,Y) = g(X,\varphi Y). \quad (3.2)$$

From the fact that M is a (κ,μ) manifold, we have $\Omega = d\eta$. Hence $d\Omega = 0$, i.e., it is closed. Thus we can define a magnetic field F_q on M by

$$F_q(X,Y) = q\Omega(X,Y),$$

namely the contact magnetic field with strength q, where X, $Y \in \chi(M)$ and $q \in R$. The Lorentz force Φ associated to the contact magnetic field F_q can be written as $\Phi = -q\varphi$, so the Lorentz equation (1.1) is

$$\nabla_{E1}E_1 = -q\varphi E_1, \quad (3.3)$$

where $\alpha : I \Rightarrow M$ is a curve with arc length parameter, $E_1 = \alpha^0$ is a tangent vector field and ∇ is the Levi-civita connection. By using (2.11) and (3.2) we have

$$\nabla^*_{E_1} E_1 = [-q + (2 + \sqrt{1-\kappa})\eta(E_1)]\phi E_1. \quad (3.4)$$

Theorem 3.1. *Let α be a magnetic curve in $(2n+1)$-dimensional (κ,μ)-contact metric manifold admitting Zamkovoy connection with curvature and torsion $k^* = |q|sin\theta$ and $\tau^* = qcos\theta$ respectively, Moreover, its ratio is constant.*

Proof: Take the frame field $\{E_1,E_2,E_3\}$ along α. By definition $E_1 = \alpha^0$, then the magnetic equation is written as

$$\nabla^*_{\alpha'}\alpha' = k^* E_2. \quad (3.5)$$

Cosequently we get

$$k^{*2} = q^2[sin^2\theta]. \quad (3.6)$$

Thus α has constant curvature.

$$k^* = |q|sin\theta. \quad (3.7)$$

Assume that α is a non-geodesic normal magnetic curve, then from (4.1) we have

$$E_2 = \frac{q}{k^*}\phi\alpha'. \quad (3.8)$$

Next, the binormal vector field E_3 is obtained from the formula

$$E_3 = \frac{q}{k^*}(\xi - \alpha' cos\theta). \quad (3.9)$$

The covariant derivative of the binormal with respect to Zamkovoy connection is computed as

$$\nabla^*_{\alpha'}E_3 = -\frac{q^2}{k^*}\phi\alpha' cos\theta. \quad (3.10)$$

Now by using Frenet frame and (3.2) we obtain

$$\tau^* = qcos\theta. \quad (3.11)$$

Hence the proof.

Lemma 3.1. *Let α be a unit speed curve on a $(2n+1)$-dimensional (κ,μ)-manifold and E_1, E_2, E_3 be the tangent, principal normal and binormal of the curve α respectively. Then*

$$\eta(E_1)' = k_1^*\eta(E_2). \quad (3.12)$$

$$\eta(E_2)' = -k_1^*\eta(E_1) + k_2^*\eta(E_3). \quad (3.13)$$

$$\eta(E_3)' = -k_2^*\eta(E_2). \quad (3.14)$$

Proof: Let α be a unit speed curve on a $(2n+1)$- dimensional (κ,μ)-manifold admitting Zamkovoy connection. Differentiating $\eta(E_1)$, $\eta(E_2)$ and $\eta(E_3)$ along α with respect to Zamkovoy connection, we have

$$\eta(E_1)^0 = g(\nabla^*_{E_1}E_1,\xi) + g(E_1,\nabla^*_{E_1}\xi), \quad (3.15)$$

we get

$$\eta(E_1)' = k_1^*\eta(E_2). \quad (3.16)$$

Similarly we get,

$$\eta(E_2)' = -k_1^*\eta(E_1) + k_2^*\eta(E_3), \quad (3.17)$$

and

$$\eta(E_3)' = -k_2^*\eta(E_2), \quad (3.18)$$

Lemma 3.2. *For slant curve in (κ,μ)-manifold admitting Zamkovoy connection, we have $\eta(E_1)^0 = 0$.*

Proof: If a curve α in a (κ,μ)- manifold of dimensional $(2n+1)$ admitting Zamkovoy connection is a slant curve then we have $\eta(E_1) = cos\theta$ = constant. Which implies that $\eta(E_1)^0 = 0$.

Therefore from Lemma 3.1 and 3.2 we can state the following theorem:

Theorem 3.2. *A non Legendre Slant curve on a $(2n+1)$- dimensional (κ,μ)- manifold with respect to Zamkovoy connection is a geodesic.*

Theorem 3.3. *If α is a magnetic helix in $(2n+1)$-dimensional (κ,μ) manifold then*

$$\eta(E_2) = 0 \text{ and } \frac{\eta(E_1)}{\eta(E_3)} = \frac{k_2}{k_1^*}.$$

Proof: Let α be a magnetic helix curve in a $(2n+1)$-dimensional (κ,μ)- manifold with respect to ∇^*. Then

$$\nabla^*_{\alpha'}\alpha' = \phi\alpha', \quad (3.19)$$

where $\alpha^0 = E_1$(tangent vector). Using Frenet formula we have

$$k_1^* E_2 = \phi E_1. \quad (3.20)$$

Taking innerproduct of (3.20) with respect to ξ, we get

$$k_1^*\eta(E_2) = 0.$$

This implies that

$$\eta(E_2) = 0. \quad (3.21)$$

Differentiating (3.21) with respect to E_1, we have

$$\nabla^{*2}_{E_1}E_1 = \nabla^*_{E_1}\phi E_1. \quad (3.22)$$

It follows that

$$\nabla^*_{E_1}k_1^*E_2 = (\nabla^*_{E_1}\phi)E_1 + \phi\nabla^*_{E_1}E_1, \quad (3.23)$$

which gives

$$k_1^{*\prime}E_2 - k_1^{*2}E_1 + k_1^*k_2^*E_3 = k_1^*(-E_1 + \eta(E_1)\xi). \quad (3.24)$$

Taking innerproduct with ξ, we obtain

$$-k_1^{*2}\eta(E_1) - k_1^*k_2^*\eta(E_3) = 0. \quad (3.25)$$

Therefore we have

$$\frac{\eta(E_1)}{\eta(E_3)} = \frac{k_2^*}{k_1^*} \quad (3.26)$$

Hence the proof.

4. PSEUDO-HERMITIAN MAGNETIC CURVE IN (κ,μ)-MANIFOLD ADMITTING ZAMKOVOY CONNECTION

Definition 4.1. *Let $\alpha : I \Rightarrow M$ be a unit speed curve in (κ,μ) manifold (M,φ,ξ,η,g) endowed with the Zamkovoy connection ∇^*. Then it is called a normal magnetic curve with respect to Zamkovoy connection ∇^* (or shortly pseudo-Hermitian magnetic) if it satisfies* (3.4).

Lemma 4.1. *If α is a pseudo-Hermitian magnetic curve in (κ,μ) manifold admitting Zamkovoy connection then α is a slant curve.*

Proof: Let $\alpha : I \Rightarrow M$ be pseudo-Hermitian magnetic curve. Then we find

$$\frac{d}{dt}\eta(E_1) = \frac{d}{dt}g(E_1,\xi) = g(\nabla^*_{E_1}E_1,\xi) - g(E_1,\nabla^*_{E_1}\xi) = 0, \quad (4.1)$$

which implies that

$$\eta(E_1) = cos\theta = constant. \quad (4.2)$$

As a result , we can rewrite (3.4) as

$$\nabla^*_{E_1}E_1 = [-q + (2+\sqrt{1-\kappa})cos\theta]\phi E_1, \quad (4.3)$$

where θ is the contact angle of α.

Now, we are in the possition to prove our Main theorem:

Theorem 4.1. *Let (M,φ,ξ,η,g) be a (κ,μ) manifold admitting Zamkovoy connection ∇^*. Then a curve α in M is pseudo-Hermitian magnetic curve then it belongs to the following:*

a) Pseudo-Hermitian non-Legendre slant geodesics (including pseudo-Hermitian geodesics as integral curves of ξ)

b) Pseudo-Hermitian Legendre circles with $k_1^* = |q|$ and having the Frenet frame field (for ∇^*) $\{E_1, -sgn(q)\varphi E_1\}$

c) Pseudo-Hermitian slant helices with $k_1^* = |-q+(2+\sqrt{1-\kappa})cos\theta|sin\theta$,

$$k_2^* = |-q+(2+\sqrt{1-\kappa})cos\theta|sgn(cos\theta)cos\theta \text{ and having the Frenet frame field}$$

$$\text{(for } \nabla^*) \ \{E_1, -\frac{\delta}{sin\theta}\phi E_1, \frac{sgn(cos\theta)}{sin\theta}(\xi - cos\theta E_1)\},$$

$$\text{where } \delta = sgn(-q+(2+\sqrt{1-\kappa})cos\theta) \text{ and } cos\theta \neq \frac{q}{2}.$$

Proof: Let us assume that α be a normal magnetic curve with respect to ∇^*. Consequently, (4.3) must be validated. Let us assume that $k_1^* = 0$. Here, we have $cos\theta = \frac{q}{2+\sqrt{1-\kappa}}$ or $\phi E_1 = 0$.

If $cos\theta = \frac{q}{2+\sqrt{1-\kappa}}$, then α is a pseudo-Hermitian non-Legendre slant geodesic.

Otherwise $\varphi E_1 = 0$ which implies that $E_1 = \xi$.

So we have proved that α is pseudo-Hermitian non Legendre slant geodesic. (including Pseudo-Hermitian geodesics as integral curves of ξ).

Now let $k_1^* \neq 0$. From (4.1) and (4.3) , we find

$$\nabla^*_{E_1} E_1 = k_1^* E_2 = [-q + (2+\sqrt{1-\kappa})cos\theta]\phi E_1. \quad (4.4)$$

Since E_1 is unit, , from (2.1), (2.2) and (2.3) gives

$$g(\varphi E_1, \varphi E_1) = sin^2\theta. \quad (4.5)$$

By use of (4.3) and (4.4) we obtain

$$k_1^* = |-q + (2+\sqrt{1-\kappa})cos\theta| sin\theta, \quad (4.6)$$

which is constant. Let us denote by $\delta = sgn(-q + (2+\sqrt{1-\kappa})cos\theta)$.

$$\varphi E_1 = \delta sin\theta E_2. \quad (4.7)$$

Let us assume $k_2^* = 0$, i.e., $r = 2$ from the fact that k_1^* is a constant. Then α is pseudoHermitian circle. (4.7) gives us $\eta(\varphi E_1) = 0$ which implies that $\eta(E_2) = 0$ differentiating above equation with respect to ∇^*, we obtain $\nabla^*_{E1}\eta(E_2) = 0$

Since $r = 2$ and (2.12). we have $\eta(E_1) = 0$. Hence α is Legendre and $cos\theta = 0$, from (4.6) we have $k_1^* = |q|$.

In this case, we also obtain $\delta = -sgn(q)$, and $E_2 = -sgn(q)\varphi E_1$, we have proved that $k_1^* = |q|$ and having Frenet frame field $\{E_1, -sgn(q)\varphi E_1\}$.

Now let us assume $k_2^* \neq 0$. If we have $\varphi E_1 = 0$ we obtain

$$\nabla^*_{E_1}\phi E_1 = k_1^* \phi E_2. \quad (4.8)$$

From (2.1), (2.2) and (4.7), we find

$$\varphi^2 E_1 = \delta sin\theta \varphi E_2. \quad (4.9)$$

This gives us $\phi E_2 = \frac{\delta}{sin\theta}(-E_1 + cot\theta\xi)$ and so (4.8) becomes

$$\nabla^*_{E_1}\phi E_1 = k_1^* \frac{\delta}{sin\theta}(-E_1 + cot\theta\xi). \quad (4.10)$$

If we differentiate (4.7) with respect to ∇^*, we also have

$\nabla^*_{E1}\varphi E_1 = \delta sin\theta \nabla^*_{E1} E_2$. From (4.1) we obtain

$$\nabla^*_{E_1}\phi E_1 = \delta sin\theta(-k_1^* E_1 + k_2^* E_3). \quad (4.11)$$

By the use of (4.10) and (4.11), we obtain

$$k_2^* sin\theta E_3 = k_1^* cot\theta(\xi - cos\theta E_1), \quad (4.12)$$

also $g(\xi - cos\theta E_1, \xi - cos\theta E_1) = sin^2\theta$. From (4.12) we calculate

$$k_2^* = |-q + (2+\sqrt{1-\kappa})cos\theta| sgn(cos\theta) cos\theta \quad (4.13)$$

As a result we get

$$E_3 = \frac{sgn(cos\theta)}{sin\theta}(\xi - cos\theta E_1) \qquad (4.14)$$
$$E_2 = \frac{\delta}{sin\theta}\phi E_1$$

Differentiating (4.14) with respect to ∇^*, since $\varphi E_1 \| E_2$ we find that $k_3^* = 0$. Hence proof is completed.

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