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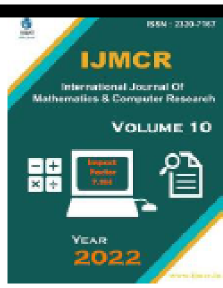

A Group Approach to Exact Solutions and Conservation laws of Burger's Equation

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Abstract

A classical Burger's equation is studied by symmetry analysis. The Lie point symmetries constructed are applied in symmetry reductions and the resulting reduced systems investigated for exact group-invariant solutions. We also construct solitons using symmetry span of space and time translations. Finally, we prove that Burgers equation is a conservation law by the multiplier technique.

Keywords: symmetry analysis; group-invariant solutions; stationary solutions; symmetry reductions; solitons.

1 Introduction

The Bateman-Burgers Equation(1.1), is one of the most imporant partial differential equations. First studied by Johannes Burgers [7] , the model often appears in vast areas of mathematics including; gas dynamic, fluid mechanics, traffic flow and nonlinear acoustics. In this manuscript we study a special case of the general Burgers equation

$$\Delta_0 \equiv \alpha u_t + \beta u u_x + \gamma u_{xx} = 0, \quad (\alpha, \beta, \gamma) \in \mathbb{R}^3 \setminus (0, 0, 0), \tag{1.1}$$

which shows up in th description of the movement of weak nonlinear waves in gases with sufficiently small dissipative effects that can be considered in first order approximation only. As dissipation vanishes, Equation (1.1) adequately describes waves traversing a non-viscous medium. The dependent variable u is a function of independent variables t and x . We study a special case of Equation (1.1), where

$$\alpha = -\beta = -\gamma = 1,$$

that is ,

$$\Delta \equiv u_t - u u_x - \gamma u_{xx} = 0. \tag{1.2}$$

2 Preliminaries

In this section, we outline preliminary concepts which are useful in the sequel.

Local Lie groups

In the Euclidean spaces $\mathbb{R}^n$ of $x = x^i$ independent variables and $\mathbb{R}^m$ of $u = u^\alpha$ dependent variables, we consider the transformations [22]

$$T_\epsilon : \quad \bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \tag{2.1}$$

involving the continuous parameter ϵ which ranges from a neighbourhood $\mathcal{N}' \subset \mathcal{N} \subset \mathbb{R}$ of $\epsilon = 0$ where the functions φ^i and ψ^α differentiable and analytic in the parameter ϵ.

Definition 2.1. The set $\mathcal{G}$ of transformations given by (2.1) is a local Lie group if it holds true that

1. (Closure) Given $T_{\epsilon_1}, T_{\epsilon_2} \in \mathcal{G}$, for $\epsilon_1, \epsilon_2 \in \mathcal{N}' \subset \mathcal{N}$, then $T_{\epsilon_1} T_{\epsilon_2} = T_{\epsilon_3} \in \mathcal{G}, \ \epsilon_3 = \phi(\epsilon_1, \epsilon_2) \in \mathcal{N}$.
2. (Identity) There exists a unique $T_0 \in \mathcal{G}$ if and only if $\epsilon = 0$ such that $T_\epsilon T_0 = T_0 T_\epsilon = T_\epsilon$.
3. (Inverse) There exists a unique $T_{\epsilon^{-1}} \in \mathcal{G}$ for every transformation $T_\epsilon \in \mathcal{G}$, where $\epsilon \in \mathcal{N}' \subset \mathcal{N}$ and $\epsilon^{-1} \in \mathcal{N}$ such that $T_\epsilon T_{\epsilon^{-1}} = T_{\epsilon^{-1}} T_\epsilon = T_0$.

Remark 2.2. Associativity of the group $\mathcal{G}$ in (2.1) follows from (1).

Prolongations

In the system,

$$\Delta_\alpha \left(x^i, u^\alpha, u_{(1)}, \ldots, u_{(\pi)}\right) = \Delta_\alpha = 0, \tag{2.2}$$

the variables u^α are dependent. The partial derivatives $u_{(1)} = \{u_i^\alpha\}$, $u_{(2)} = \{u_{ij}^\alpha\}, \ldots, u_{(\pi)} = \{u_{i_1 \ldots i_\pi}^\alpha\}$, are of the first, second, ..., up to the πth-orders.
Denoting

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \tag{2.3}$$

the total differentiation operator with respect to the variables x^i and δ_i^j, the Kronecker delta, we have

$$D_i(x^j) = \delta_i^j, \, ', \ u_i^\alpha = D_i(u^\alpha), \ \ u_{ij}^\alpha = D_j(D_i(u^\alpha)), \ \ \ldots, \tag{2.4}$$

where u_i^α defined in (2.4) are differential variables Ibragimov [12].

(i.) **Prolonged groups** Consider the local Lie group $\mathcal{G}$ given by the transformations

$$\bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \varphi^i\Big|_{\epsilon=0} = x^i, \ \ \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \tag{2.5}$$

where the symbol $\Big|_{\epsilon=0}$ means evaluated on $\epsilon = 0$.

Definition 2.3. The construction of the group $\mathcal{G}$ given by (2.5) is an equivalence of the computation of infinitesimal transformations

$$\begin{aligned} \bar{x}^i &\approx x^i + \xi^i(x^i, u^\alpha)\epsilon, \quad \varphi^i\Big|_{\epsilon=0} = x^i, \\ \bar{u}^\alpha &\approx u^\alpha + \eta^\alpha(x^i, u^\alpha)\epsilon, \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \end{aligned} \tag{2.6}$$

obtained from (2.1) by a Taylor series expansion of $\varphi^i(x^i, u^\alpha, \epsilon)$ and $\psi^i(x^i, u^\alpha, \epsilon)$ in ϵ about $\epsilon = 0$ and keeping only the terms linear in ϵ, where

$$\xi^i(x^i, u^\alpha) = \frac{\partial \varphi^i(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}, \quad \eta^\alpha(x^i, u^\alpha) = \frac{\partial \psi^\alpha(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}. \tag{2.7}$$

Remark 2.4. The symbol of infinitesimal transformations, X, is used to write (2.6) as

$$\bar{x}^i \approx (1 + X)x^i, \quad \bar{u}^\alpha \approx (1 + X)u^\alpha, \tag{2.8}$$

where

$$X = \xi^i(x^i, u^\alpha)\frac{\partial}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial}{\partial u^\alpha}, \tag{2.9}$$

is the generator of the group $\mathcal{G}$ given by (2.5).

Remark 2.5. To obtain transformed derivatives from (2.1), we use a change of variable formulae

$$D_i = D_i(\varphi^j)\bar{D}_j, \tag{2.10}$$

where $\bar{D}_j$ is the total differentiation in the variables $\bar{x}^i$. This means that

$$\bar{u}_i^\alpha = \bar{D}_i(\bar{u}^\alpha), \ \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha). \tag{2.11}$$

If we apply the change of variable formula given in (2.10) on $\mathcal{G}$ given by (2.5), we get

$$D_i(\psi^\alpha) = D_i(\varphi^j), \ \bar{D}_j(\bar{u}^\alpha) = \bar{u}_j^\alpha D_i(\varphi^j). \tag{2.12}$$

Expansion of (2.12) yields

$$\left(\frac{\partial \varphi^j}{\partial x^i} + u_i^\beta \frac{\partial \varphi^j}{\partial u^\beta}\right)\bar{u}_j^\beta = \frac{\partial \psi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \psi^\alpha}{\partial u^\beta}. \tag{2.13}$$

The variables $\bar{u}_i^\alpha$ can be written as functions of $x^i, u^\alpha, u_{(1)}$, that is

$$\bar{u}_i^\alpha = \Phi^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon), \quad \Phi^\alpha\Big|_{\epsilon=0} = u_i^\alpha. \tag{2.14}$$

Definition 2.6. The transformations in the space of the variables $x^i, u^\alpha, u_{(1)}$ given in (2.5) and (2.14) form the first prolongation group $\mathcal{G}^{[1]}$.

Definition 2.7. Infinitesimal transformation of the first derivatives is

$$\bar{u}_i^\alpha \approx u_i^\alpha + \zeta_i^\alpha \epsilon, \quad \text{where} \quad \zeta_i^\alpha = \zeta_i^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon). \tag{2.15}$$

Remark 2.8. In terms of infinitesimal transformations, the first prolongation group $\mathcal{G}^{[1]}$ is given by (2.6) and (2.15).

(ii.) **Prolonged generators**

Definition 2.9. By using the relation given in (2.12) on the first prolongation group $\mathcal{G}^{[1]}$ given by Definition 2.6, we obtain [8]

$$D_i(x^j + \xi^j\epsilon)(u_j^\alpha + \zeta_j^\alpha\epsilon) = D_i(u^\alpha + \eta^\alpha\epsilon), \quad \text{which gives} \tag{2.16}$$

$$u_i^\alpha + \zeta_j^\alpha\epsilon + u_j^\alpha\epsilon D_i\xi^j = u_i^\alpha + D_i\eta^\alpha\epsilon, \tag{2.17}$$

and thus

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \tag{2.18}$$

is the first prolongation formula.

Remark 2.10. Similarly, we get higher order prolongations [12],

$$\zeta^{\alpha}_{ij} = D_j(\zeta^{\alpha}_i) - u^{\alpha}_{i\kappa} D_j(\xi^{\kappa}), \ \ldots, \ \zeta^{\alpha}_{i_1,\ldots,i_\kappa} = D_{i_\kappa}(\zeta^{\alpha}_{i_1,\ldots,i_{\kappa-1}}) - u^{\alpha}_{i_1,i_2,\ldots,i_{\kappa-1}j} D_{i_\kappa}(\xi^j). \tag{2.19}$$

Remark 2.11. The prolonged generators of the prolongations $\mathcal{G}^{[1]}, \ldots, \mathcal{G}^{[\kappa]}$ of the group $\mathcal{G}$ are

$$X^{[1]} = X + \zeta^{\alpha}_i \frac{\partial}{\partial u^{\alpha}_i}, \ \ldots, X^{[\kappa]} = X^{[\kappa-1]} + \zeta^{\alpha}_{i_1,\ldots,i_\kappa} \frac{\partial}{\partial \zeta^{\alpha}_{i_1,\ldots,i_\kappa}}, \ \kappa \geq 1, \tag{2.20}$$

where X is the group generator given by (2.9).

Group invariants

Definition 2.12. A function $\Gamma(x^i, u^\alpha)$ is called an invariant of the group $\mathcal{G}$ of transformations given by (2.1) if

$$\Gamma(\bar{x}^i, \bar{u}^\alpha) = \Gamma(x^i, u^\alpha). \tag{2.21}$$

Theorem 2.13. *A function $\Gamma(x^i, u^\alpha)$ is an invariant of the group $\mathcal{G}$ given by (2.1) if and only if it solves the following first-order linear PDE: [8]*

$$X\Gamma = \xi^i(x^i, u^\alpha)\frac{\partial \Gamma}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial \Gamma}{\partial u^\alpha} = 0. \tag{2.22}$$

From Theorem (2.13), we have the following result.

Theorem 2.14. *The local Lie group $\mathcal{G}$ of transformations in $\mathbb{R}^n$ given by (2.1) [12] has precisely $n-1$ functionally independent invariants. One can take, as the basic invariants, the left-hand sides of the first integrals*

$$\psi_1(x^i, u^\alpha) = c_1, \ldots, \psi_{n-1}(x^i, u^\alpha) = c_{n-1}, \tag{2.23}$$

of the characteristic equations for (2.22):

$$\frac{\mathrm{d}x^i}{\xi^i(x^i, u^\alpha)} = \frac{\mathrm{d}u^\alpha}{\eta^\alpha(x^i, u^\alpha)}. \tag{2.24}$$

Symmetry groups

Definition 2.15. The vector field X (2.9) is a Lie point symmetry of the PDE system (2.2) if the determining equations

$$X^{[\pi]}\Delta_\alpha\Big|_{\Delta_\alpha=0} = 0, \quad \alpha = 1, \ldots, m, \quad \pi \geq 1, \tag{2.25}$$

are satisfied, where $\Big|_{\Delta_\alpha=0}$ means evaluated on $\Delta_\alpha = 0$ and $X^{[\pi]}$ is the π-th prolongation of X.

Definition 2.16. The Lie group $\mathcal{G}$ is a symmetry group of the PDE system given in (2.2) if the PDE system (2.2) is form-invariant, that is

$$\Delta_\alpha\left(\bar{x}^i, \bar{u}^\alpha, \bar{u}_{(1)}, \ldots, \bar{u}_{(\pi)}\right) = 0. \tag{2.26}$$

Theorem 2.17. *Given the infinitesimal transformations in (2.5), the Lie group $\mathcal{G}$ in (2.1) is found by integrating the Lie equations*

$$\frac{\mathrm{d}\bar{x}^i}{\mathrm{d}\epsilon} = \xi^i(\bar{x}^i, \bar{u}^\alpha), \quad \bar{x}^i\Big|_{\epsilon=0} = x^i, \quad \frac{\mathrm{d}\bar{u}^\alpha}{\mathrm{d}\epsilon} = \eta^\alpha(\bar{x}^i, \bar{u}^\alpha), \quad \bar{u}^\alpha\Big|_{\epsilon=0} = u^\alpha. \tag{2.27}$$

Lie algebras

Definition 2.18. A vector space $\mathcal{V}_r$ of operators [22] X (2.9) is a Lie algebra if for any two operators, $X_i, X_j \in \mathcal{V}_r$, their commutator

$$[X_i, X_j] = X_i X_j - X_j X_i, \tag{2.28}$$

is in $\mathcal{V}_r$ for all $i, j = 1, \ldots, r$.

Remark 2.19. The commutator satisfies the properties of bilinearity, skew symmetry and the Jacobi identity [23].

Theorem 2.20. *The set of solutions of the determining equation given by (2.25) forms a Lie algebra[8].*

Conservation laws

Let a system of πth-order PDEs be given by (2.2).

Definition 2.21. The Euler-Lagrange operator $\delta/\delta u^\alpha$ is

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{\kappa \geq 1} (-1)^\kappa D_{i_1}, \ldots, D_{i_\kappa} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{2.29}$$

and the Lie- Bäcklund operator in abbreviated form [8] is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots. \tag{2.30}$$

Remark 2.22. The Lie- Bäcklund operator (2.30) in its prolonged form is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{\kappa \geq 1} \zeta_{i_1 \ldots i_\kappa} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{2.31}$$

where

$$\zeta^\alpha_i = D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \qquad \ldots, \zeta^\alpha_{i_1 \ldots i_\kappa} = D_{i_1 \ldots i_\kappa}(W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_\kappa}, \quad j = 1, \ldots, n. \tag{2.32}$$

and the Lie characteristic function is

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{2.33}$$

Remark 2.23. The characteristic form of Lie- Bäcklund operator (2.31) is

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + D_{i_1 \ldots i_\kappa}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}. \tag{2.34}$$

The method of multipliers

Definition 2.24. A function $\Lambda^\alpha \left(x^i, u^\alpha, u_{(1)}, \ldots\right) = \Lambda^\alpha$, is a multiplier of the PDE system given by (2.2) if it satisfies the condition that [19]

$$\Lambda^\alpha \Delta_\alpha = D_i T^i, \tag{2.35}$$

where $D_i T^i$ is a divergence expression.

Definition 2.25. To find the multipliers Λ^α, one solves the determining equations (2.36) [4],

$$\frac{\delta}{\delta u^\alpha}(\Lambda^\alpha \Delta_\alpha) = 0. \tag{2.36}$$

Notation 2.26. We will use $C_i,\ i \in \mathbb{N}$ as constants of integration and $C_i(x_1, x_2, \ldots),\ i \in \mathbb{N}$ as arbitrary function of $x_1, x_2, \ldots$.

3 Main results

3.1 Lie point symmetries of (1.2)

We start first by computing Lie point symmetries of the Burgers Equation (1.2)which admits the continuous Lie group of transformations infinitesimally generated by

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{3.1}$$

if and only if

$$X^{[2]}\Delta\Big|_{\Delta=0} = 0. \tag{3.2}$$

Using the definition of $X^{[2]}$ in (2.20), we have

$$(T\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{22}\frac{\partial}{\partial u_{xx}})\big(u_t - uu_x - u_{xx}\big)\Big|_{u_t-uu_x-u_{xx}=0} = 0 \tag{3.3}$$

which gives

$$-\eta u_x + \zeta_1 + -\zeta_2 u + -\zeta_{22}\big|_{u_{xx}=u_t-uu_x} = 0, \tag{3.4}$$

where ζ_1, ζ_2 and ζ_{22} are

$$\zeta_1 = \eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_tu_x\xi_u, \tag{3.5}$$
$$\zeta_2 = \eta_x + u_x\eta_u - u_t\tau_x - u_tu_x\tau_u - u_x\xi_x - u_x^2\xi_u, \tag{3.6}$$

$$\zeta_{22} = \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu} - 2u_{xx}\xi_x - u_x\xi_{xx} - 2u_x^2\xi_{xu} - 3u_xu_{xx}\xi_u \tag{3.7}$$
$$-u_x^3\xi_{uu} - 2u_{tx}\tau_x - u_t\tau_{xx} - 2u_tu_x\tau_{xu} - (u_tu_{xx} + 2u_xu_{tx})\tau_u - u_tu_x^2\tau_{uu}, \tag{3.8}$$

Substituting the values of ζ_1, ζ_2 and ζ_{22} in (3.4) we obtain the following determining equation:

$$\begin{aligned} &-\eta u_x + \big(\eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_tu_x\xi_u\big) + \\ &-\big(\eta_x + u_x\eta_u - u_t\tau_x - u_tu_x\tau_u - u_x\xi_x - u_x^2\xi_u\big)u + \\ &-\big(\eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu} - 2u_{xx}\xi_x - u_x\xi_{xx} - 2u_x^2\xi_{xu} \\ &-3u_xu_{xx}\xi_u - u_x^3\xi_{uu} - 2u_{tx}\tau_x - u_t\tau_{xx} \\ &-2u_tu_x\tau_{xu} - (u_tu_{xx} + 2u_xu_{tx})\tau_u - u_tu_x^2\tau_{uu}\big)\big|_{u_{xx}=u_t-uu_x} = 0 \end{aligned} \tag{3.9}$$

Now replacing u_{xx} by $u_{xx} = u_t - uu_x$ in the above equation we obtain

$$
\begin{aligned}
&-\eta u_x + \big(\eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_tu_x\xi_u\big) + \\
&-\big(\eta_x + u_x\eta_u - u_t\tau_x - u_tu_x\tau_u - u_x\xi_x - u_x^2\xi_u\big)u + \\
&-\big(\eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu} - 2\{u_t - uu_x\}\xi_x - u_x\xi_{xx} - 2u_x^2\xi_{xu} \\
&-3u_x\{-u_t - uu_x\}\xi_u - u_x^3\xi_{uu} - 2u_{tx}\tau_x - u_t\tau_{xx} \\
&-2u_tu_x\tau_{xu} - (u_t\{u_t - uu_x\} + 2u_xu_{tx})\tau_u - u_tu_x^2\tau_{uu}\big) = 0
\end{aligned}
\tag{3.10}
$$

or

$$
\begin{aligned}
&-\eta u_x + \big(\eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_tu_x\xi_u\big) + \\
&-\big(\eta_x + u_x\eta_u - u_t\tau_x - u_tu_x\tau_u - u_x\xi_x - u_x^2\xi_u\big)u + \\
&-\big(\eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu} + 2\{-u_t\xi_x + 2uu_x\}\xi_x - u_x\xi_{xx} - 2u_x^2\xi_{xu} \\
&+3u_x\{-u_t\xi_u + uu_x\}\xi_u - u_x^3\xi_{uu} - 2u_{tx}\tau_x - u_t\tau_{xx} \\
&-2u_tu_x\tau_{xu} + \big(\{u_t^2 + uu_tu_x\} - 2u_xu_{tx}\big)\tau_u - u_tu_x^2\tau_{uu}\big) = 0
\end{aligned}
\tag{3.11}
$$

By definition, τ, ξ and η are functions of t, x and u only. For that reason, we can then split Equation (3.11) on the derivatives of u (without losing any information) and obtain

$$\xi_{xx} - 2\eta_{xu} - \xi_x u - \xi_t - \eta = 0 \tag{3.12}$$

$$\eta_t - \eta_x u - \eta_{xx} = 0 \tag{3.13}$$

$$2\xi_x - \tau_t = 0, \tag{3.14}$$

$$\xi_u = \tau_u = \tau_x = \eta_{uu} = 0 \tag{3.15}$$

The observations from Equations in (3.15) imply that

$$\tau = \tau(t), \quad \xi = \xi(t, x), \quad \text{and} \quad \eta = A(t, x)u + B(t, x). \tag{3.16}$$

By using the expressions in (3.16) in Equation (3.12), we obtain

$$\xi_{xx}(t, x) - 2A_x(t, x) - \xi_x(t, x)u - \xi_t - A(t, x)u - B(t, x) = 0 \tag{3.17}$$

By splitting the terms in (3.17) on powers of u (this does no harm since none of the terms is a function of u), we obtain

$$u: \xi_x(t, x) + A(t, x) = 0 \tag{3.18}$$

$$u^0: \xi_{xx}(t, x) - 2A_x(t, x) - \xi_t - B(t, x) = 0. \tag{3.19}$$

Now using the expression for η in Equation (3.13), one gets that

$$A_t(t, x)u + B_t(t, x) - u(A_x(t, x)u + B_x(t, x)) - (A_{xx}(t, x)u + B_{xx}(t, x)) = 0 \tag{3.20}$$

which splits on powers of u to yield

$$u^2: A_x(t, x) = 0 \tag{3.21}$$

$$u: A_t(t, x) - B_x(t, x) - A_{xx}(t, x) = 0 \tag{3.22}$$

$$u^0: B_t(t, x) - B_{xx}(t, x) = 0 \tag{3.23}$$

Equation is sufficient for

$$A(t,x) = A(t), \quad A_{xx}(t,x) = 0 \quad \text{thus} \quad A_t(t,x) = B_x(t,x) \Longrightarrow B_{xx}(t,x) = 0. \tag{3.24}$$

The last conclusion is informed by the condition that $A(t,x) = A(t)$. Furthermore, Equation (3.23) forces $B(t,x) = B(x)$. The integration of $B_{xx}(t,x) = 0$ with respect to x twice yields

$$B(t,x) = C_1 x + C_2. \tag{3.25}$$

By the value for $B(t,x)$ in (3.25), and the relation in (3.24), we have

$$A_t(t,x) = C_1 \Longrightarrow A(t,x) = C_1 t + C_3. \tag{3.26}$$

By Equation (3.18),

$$\xi_x(t,x) = -C_1 t - C_3 \Longrightarrow \xi(t,x) = -C_1 tx - C_3 x + D(t). \tag{3.27}$$

From Equation (3.19), we have that

$$-C_1 x + D_t(t) = -C_1 x - C_2 \Longrightarrow D(t) = -C_2 t + C_4. \tag{3.28}$$

Finally, using the value of $\xi(t,x) = -C_1 tx - C_2 t - C_3 x + C_4$ in Equation (3.14), we have that

$$\tau_t(t,x) = -2C_1 t - 2C_3 \Longrightarrow \tau(t,x) = -C_1 t^2 - 2tC_3 + C_5. \tag{3.29}$$

Thus our desired functions are

$$\tau = -C_1 t^2 - 2tC_3 + C_5 \tag{3.30}$$
$$\xi = -C_1 tx - C_2 t - C_3 x + C_4 \tag{3.31}$$
$$\eta = C_1(tu + x) + C_2 + C_3 u. \tag{3.32}$$

A Lie algebra spanned by the following symmetries is thus obtained:

$$X_1 = -t^2\frac{\partial}{\partial t} - tx\frac{\partial}{\partial x} + (x + tu)\frac{\partial}{\partial u} \tag{3.33}$$
$$X_2 = -t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \tag{3.34}$$
$$X_3 = -2t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \tag{3.35}$$
$$X_4 = \frac{\partial}{\partial x} \tag{3.36}$$
$$X_5 = \frac{\partial}{\partial t}. \tag{3.37}$$

3.2 Commutator Table for Symmetries

We evaluate the commutation relations for the symmetry generators. By definition of Lie bracket [23], for example, we have that

$$[X_4, X_5] = X_4X_5 - X_5X_4 = \left(\frac{\partial}{\partial x}\frac{\partial}{\partial t}\right) - \left(\frac{\partial}{\partial t}\frac{\partial}{\partial x}\right) = 0. \tag{3.38}$$

Remark 3.1. The remaining commutation relations are obtained analogously. We present all commutation relations in table (1) below.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$2X_1$	$-X_2$	$-X_3$
X_2	0	0	X_2	0	X_4
X_3	-2X_1	-X_2	0	X_4	$2X_5$
X_4	X_2	0	-X_4	0	0
X_5	X_3	-X_4	-2X_5	0	0

Table 1: A commutator table for the Lie algebra spanned by the symmetries of Burger's equation.

3.3 Group Transformations

The corresponding one-parameter group of transformations can be determined by solving the Lie equations [24]. Let T_{ϵ_i} be the group of transformations for each $X_i, i = 1, 2, 3, 4$. We display how to obtain T_{ϵ_i} from X_i by finding one-parameter group for the infinitesimal generator X_1, namely,

$$X_4 = \frac{\partial}{\partial x}. \tag{3.39}$$

In particular, we have the Lie equations

$$\begin{aligned} \frac{\mathrm{d}\bar{t}}{\mathrm{d}\epsilon} =& 0, \quad \bar{t}\Big|_{\epsilon=0} = t, \\ \frac{\mathrm{d}\bar{x}}{\mathrm{d}\epsilon} =& 1, \quad \bar{x}\Big|_{\epsilon=0} = x, \\ \frac{\mathrm{d}\bar{u}}{\mathrm{d}\epsilon} =& 0, \quad \bar{u}\Big|_{\epsilon=0} = u. \end{aligned} \tag{3.40}$$

Solving the system (3.40) one obtains,

$$\bar{t} = t, \quad \bar{x} = x + \epsilon, \quad \bar{u} = u, \tag{3.41}$$

and hence the one-parameter group T_{ϵ_4} corresponding to the operator X_4 is

$$T_{\epsilon_4} : \quad \bar{t} = t, \quad \bar{x} = x + \epsilon_4, \quad \bar{u} = u. \tag{3.42}$$

All the five one-parameter groups are presented below :

$$\begin{aligned} T_{\epsilon_1} : &\quad \bar{t} = \frac{t}{1 + \epsilon_1 t}, \quad \bar{x} = \frac{x}{e^{\epsilon_1 t}}, \quad \bar{u} = \frac{(tu + x)e^{\epsilon_1 t} - x}{t}, \\ T_{\epsilon_2} : &\quad \bar{t} = t, \quad \bar{x} = x - t\epsilon_2, \quad \bar{u} = u + \epsilon_2, \\ T_{\epsilon_3} : &\quad \bar{t} = te^{-2\epsilon_3}, \quad \bar{x} = xe^{-\epsilon_3}, \quad \bar{u} = ue^{\epsilon_3}, \\ T_{\epsilon_4} : &\quad \bar{t} = t, \quad \bar{x} = x + \epsilon_4, \quad \bar{u} = u, \\ T_{\epsilon_5} : &\quad \bar{t} = t + \epsilon_5, \quad \bar{x} = x, \quad \bar{u} = u. \end{aligned} \tag{3.43}$$

3.4 Construction of Group-Invariant Solutions

Now we compute the group invariant solutions of Burger's equation.

(i) **The Case for X_1 .**
We consider the operator

$$X_1 = -t^2 \frac{\partial}{\partial t} - tx \frac{\partial}{\partial x} + (x + tu) \frac{\partial}{\partial u} \tag{3.44}$$

The associated Lagrangian equations to (3.44) yield two invariants, $J_1 = \frac{x}{t}$ and $J_2 = x+ut$. Thus, the group-invariant solution is $u(t,x) = \frac{1}{t}\varphi(\frac{x}{t}) - \frac{x}{t}$. Substitution reduces Equation (1.2) into

$$\varphi'' + \varphi\varphi' = 0 \tag{3.45}$$

which is admitted by

$$\varphi(\frac{x}{t}) = C_3 \coth\left\{\frac{C_3 x}{2t} + C_4\right\} \tag{3.46}$$

where C_3 and $C_4$are arbitrary constants. Hence the group-invariant solutions for 1.2 under the X_1 take the form

$$u(t,x) = \frac{C_3}{t}\coth\left\{\frac{C_3 x}{2t} + C_4\right\} - \frac{x}{t}. \tag{3.47}$$

(ii) **Galilean-invariant solutions.**
Consider the Galilean boost operator

$$X_2 = -t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \tag{3.48}$$

Characteristic equations associated to the operator (3.48) yields$J_1 = t$ and $J_2 = \frac{x}{t} + u$. As a result, the group-invariant solution of (1.2) for this case is $J_2 = \phi(J_1)$, for ϕ an arbitrary function. That is,

$$u(t,x) = \phi(t) - \frac{x}{t}. \tag{3.49}$$

Substitution of the value of u from equation (3.49) into equation (1.2) yields a first order ordinary differential equation $\phi'(t) + \frac{\phi(t)}{t} = 0$, whose general solution is $\phi(t) = \frac{\delta}{t}$ with δ an arbitrary constant of integration. Hence, the group-invariant solution under X_2 is

$$u(t,x) = \frac{\delta - x}{t}, \quad t \neq 0. \tag{3.50}$$

(iii) **Scale-invariant solutions**.
We consider the scaling operator

$$X_3 = -2t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} \tag{3.51}$$

The associated Lagrangian equations to (3.51) yield two invariants, $J_1 = \frac{x}{\sqrt{t}}$ and $J_2 = u\sqrt{t}$. Thus, the group-invariant solution is $u(t,x) = \frac{1}{\sqrt{t}}f(\lambda), \quad \lambda = \frac{x}{\sqrt{t}}$.
Substitute this solution in (1.2) to obtain

$$q'' + qq' + \frac{\lambda q' + q}{2} = 0 \tag{3.52}$$

which is expressible in the form

$$(q')' + \frac{1}{2}(q^2)' + \frac{(\lambda q)'}{2} = 0 \tag{3.53}$$

and integrating once yields

$$q' + \frac{1}{2}q^2 + \frac{\lambda q}{2} = C_1, \tag{3.54}$$

where C_1 is an arbitrary constant. By letting $C_1 = 0$, the Equation (3.54) reduces to a Bernoulli’s differential equation for q , whose solution is

$$q(\lambda) = \frac{2}{\sqrt{\pi}}\left[\frac{e^{-\frac{\lambda^2}{4}}}{C_2 + \operatorname{erf}(\frac{\lambda}{2})}\right] \tag{3.55}$$

for some arbitrary constant C_2 and

$$\operatorname{erf}(\omega) = \frac{2}{\sqrt{\pi}}\int_0^\omega e^{-j^2}\mathrm{d}j \tag{3.56}$$

is the error function. So the scale-invariant solutions for 1.2 take the form

$$u(t,x) = \frac{2}{\sqrt{\pi t}}\left\{\frac{e^{-\frac{x^2}{4t}}}{C_2 + \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)}\right\} \tag{3.57}$$

(iv) **Translationally-invariant solutions**
We consider the space translation operator

$$X_4 = \frac{\partial}{\partial x}. \tag{3.58}$$

Characteristic equations associated with the operator (3.58) are

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{0}, \tag{3.59}$$

which give two invariants $J_1 = t$ and $J_2 = u$. Therefore, $u = \psi(t)$ is the group-invariant solution for some arbitrary function ψ. Substitution of $u = \psi(t)$ into (1.2) yields

$$\psi'(t) = 0, \tag{3.60}$$

whose solution is

$$\psi(t) = C_1, \tag{3.61}$$

for C_1 an arbitrary constant. Hence the group-invariant solution of (1.2) under the space translation operator (3.58) is

$$u(t,x) = C_1. \tag{3.62}$$

(v) **Stationary solutions**
Consider the time translation operator

$$X_5 = \frac{\partial}{\partial t}. \tag{3.63}$$

The Lagrangian system associated with the operator (3.63) is

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{0}, \tag{3.64}$$

whose invariants are $J_1 = x$ and $J_2 = u$. So, $u = \psi(x)$ is the group-invariant solution. Substituting of $u = \psi(x)$ into (1.2) yields

$$\psi'(x)\psi(x) - \psi''(x) = 0. \tag{3.65}$$

which is satisfied by the function

$$\psi(x) = C_1 \coth\left(\frac{C_4}{2}x + C_5\right). \tag{3.66}$$

3.5 Soliton solutions

We obtain traveling wave solutions of Burgers equation by considering a linear combination of the symmetries X_4 and X_5, namely, [22]

$$X = cX_4 + X_5 = c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad \text{for some constant } c. \tag{3.67}$$

The characteristic equations are

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{c} = \frac{\mathrm{d}u}{0} \tag{3.68}$$

We get two invariants, $J_1 = x - ct$ and $J_2 = u$. So the group-invariant solution is

$$u(t, x) = \phi(x - ct), \tag{3.69}$$

for some arbitrary function ϕ and c the velocity of the wave.

Substitution of u into (1.2) yields a second order ordinary differential equation

$$c\varphi' + \varphi\varphi' + \varphi'' = 0. \tag{3.70}$$

Integration of equation (3.70) with respect to φ yields

$$2c\varphi + \varphi^2 + 2\varphi' = 0, \tag{3.71}$$

where we take 0 as a constant of integration.
Now we have

$$\varphi' = -c\varphi - \frac{\varphi^2}{2} \tag{3.72}$$

or

$$\frac{\mathrm{d}\varphi}{2c\varphi + \varphi^2} = -\frac{\mathrm{d}\xi}{2}, \tag{3.73}$$

where $\xi = x - ct$.
By resolving the left hand side into partial fractions, we obtain a one-soliton solution,

$$\varphi(x, t) = \frac{2cAe^{-c(x-ct)}}{1 - Ae^{-c(x-ct)}}, \quad A \text{ is an arbitrary constant.} \tag{3.74}$$

4 Conservation laws of the Burgers equation

We now construct conserved vectors for Equation (1.2) via the technique of multipliers.

4.1 The multipliers

We make use of the Euler-Lagrange operator defined as defined in [24] to look for a zeroth order multiplier $\Lambda = \Lambda(t, x, u)$. The resulting determining equation for computing Λ is

$$\frac{\delta}{\delta u}\left[\Lambda\{u_t - uu_x - u_{xx}\}\right] = 0. \tag{4.1}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} + \dots \tag{4.2}$$

Expansion of equation (4.1) yields

$$\Lambda_u(u_t - uu_x - u_{xx}) - u_x\Lambda - D_t(\Lambda) + D_x(\Lambda u) - D_x^2(\Lambda) = 0. \tag{4.3}$$

Invoking the total derivatives

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots, \tag{4.4}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \dots. \tag{4.5}$$

on equation (4.3) produces

$$\Lambda_t - u\Lambda_x + \Lambda_{xx} + (2\Lambda_{xu})u_x + (2\Lambda_u)u_{xx} + \Lambda_{uu}u_x^2 = 0 \tag{4.6}$$

Splitting equation (4.6) on derivatives of u produces an overdetermined system of four partial differential
equations, namely,

$$u_x : \Lambda_{xu} = 0, \tag{4.7}$$

$$u_x^2 : \Lambda_{uu} = 0, \tag{4.8}$$

$$u_{xx} : \Lambda_u = 0, \tag{4.9}$$

$$rest : \Lambda_t - u\Lambda_x + \Lambda_{xx} = 0. \tag{4.10}$$

Note that equations (4.7) and (4.8) are trivially satisfied by equation (4.9).

Again from Equation (4.9) we deduce that $\Lambda = \Lambda(t, x)$ and upon substitution in Equation (4.10) gives

$$\Lambda_t - u\Lambda_x + \Lambda_{xx} = 0. \tag{4.11}$$

Split Equation (4.11) on powers on u (no harm since Λ is not a function of u) to obtain

$$u : \Lambda_x = 0, \tag{4.12}$$

$$rest : \Lambda_t + \Lambda_{xx} = 0. \tag{4.13}$$

Now Equation (4.12) forces

$$\Lambda = \Lambda(t) \quad \text{and} \quad \Lambda_{xx} = 0 \Longrightarrow \Lambda_t = 0. \tag{4.14}$$

Hence we have

$$\Lambda = C_1, \tag{4.15}$$

which gives the nontrivial multiplier

$$\Lambda(t, x, u) = 1. \tag{4.16}$$

Remark 4.1. Recall that a multiplier Λ for equation(1.2) has the property that for the density $T^t = T^t(t, x, u, u_x)$ and flux $T^x = T^x(t, x, u, u_x, u_{xx})$,

$$\Lambda\left(u_t - uu_x - u_{xx}\right) = D_t T^t + D_x T^x. \tag{4.17}$$

We derive conservation law corresponding to each of the multipliers.
Conservation law for the multiplier $\Lambda(t, x, u) = 1$.

Expansion of equation (4.17) gives

$$u_t - uu_x - u_{xx} = T^t_t + u_t T^t_u + u_{tx} T^t_{u_x} + T^x_x + u_x T^x_u + u_{xx} T^x_{u_x}. \tag{4.18}$$

Splitting equation (4.18) on second derivatives of u yields

$$u_{tx} \quad : \quad T^t_{u_x} = 0, \tag{4.19}$$

$$u_{xx} \quad : \quad T^x_{u_x} = -1, \tag{4.20}$$

$$\text{Rest} \quad : \quad u_t - uu_x = T^t_t + T^t_u u_t + T^x_x + T^x_u u_x. \tag{4.21}$$

Equation (4.19) implies that

$$T^t_{=} T^t(t, x, u) \tag{4.22}$$

and integrating equation (4.20) with respect to u_x gives

$$T^x = -u_x + A(t, x, u), \tag{4.23}$$

for A an arbitrary function of its arguments.

Substituting the expression of T^x from (4.23) into equation (4.21) we get

$$u_t - uu_x = T^t_t + T^t_u u_t + A_x + A_u u_x. \tag{4.24}$$

which splits on first derivatives of u, to give

$$u_x \quad : \quad A_u = -u, \tag{4.25}$$

$$u_t \quad : \quad T^t_u = 1, \tag{4.26}$$

$$\text{Rest} \quad : \quad 0 = T^t_t + A_x. \tag{4.27}$$

Integrating equations (4.25) and (4.26) with respect to u manifests that

$$T^t = u + P(t, x), \tag{4.28}$$

$$A = -\frac{u^2}{2} + Q(t, x). \tag{4.29}$$

By substituting the obtained functions into Equation (4.27), we have

$$P_t(t, x) + Q_x(t, x) = 0. \tag{4.30}$$

Since C and D contribute to the trivial part of the conservation law, we take $P = Q = 0$ and obtain the conserved quantities

$$T^t = u, \tag{4.31}$$

$$T^x = -\frac{u^2}{2} - u_x \tag{4.32}$$

from which the conservation law corresponding to the multiplier $\Lambda_1 = 1$ is given by

$$D_t\left(u\right) - D_x\left(\frac{u^2}{2} + u_x\right) = 0. \tag{4.33}$$

Remark 4.2. The fact that $\Lambda = 1$ is multiplier is proof that Burgers Equation (1.2) is itself a conservation law.

5 Conclusion

In this paper, a five-dimesional Lie algebra of symmetries has been applied to study a simple variant of Burger's equation. Some of the basis vectors of the determined Lie algebra include; Galilean boost, scaling, space and time translations. Each symmetry has further been used in reductions and construction of invariant solutions. These describe the different states of the Burger's equation model. Such exact solutions are applicable as a benchmark against numerical computer simulations. More conservation laws need to be constructed using Ibragimov's theorem (or by taking higher order multiplies) and further give more exact solutions.

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Author's contribution

The author wrote the article as a scholarly duty and passion to disseminate mathematical research and hereby declares no conflict of interest.

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