[Volume 4 issue 05 May 2016]
Page No. 1387-1400 ISSN: 2320-7167

ANALYSIS OF EXPLICIT DIFFERENCE APPROXIMATE SOLUTION FOR TIME FRACTIONAL SOIL MOISTURE DIFFUSION EQUATION

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Abstract

In this paper, we give a detailed analysis for the explicit finite difference approximation for time fractional soil moisture diffusion equation (TFSMDE). Furthermore, the stability and convergence of the scheme in a bounded domain are discussed. As an application of the scheme we solve some test problems and their solutions are represented graphically by a powerful software Mathematica.

Keywords and phrases: time fractional, soil moisture diffusion equation, finite difference scheme, stability analysis, convergence analysis, fractional derivatives, software Mathematica.

1 INTRODUCTION

Fractional calculus is mere generalization of full integer order integral and differential calculus to real or even complex order. But, its complexity prevented it from being used in practice until only very recently. A list of mathematicians, who have provided important contributions up to the middle of our century, includes P.S. Laplace(1812), J.B.J. Fourier(1822), N.H. Abel(1823-1826), J. Liouville(1832-1873), B. Riemann(1847), H. Holmgren(1865-67), A.K. Grunwald(1867-1872), A.V. Letnikov(1868-1872), H. Laurent(1884), P.A. Nekrassov(1888). Fractional differential equations have been used in modeling many physical phenomena in different branches of science and engineering like biology, physics, chemistry, visco elasticity, control systems, thermo dynamics, statistics, finance etc. The study of fractional partial differential equations has increased in recent years. But, due to the complex geometries of most of the fractional differential equations, they lack exact solution. Henceforth, the numerical techniques are widely preferred because of their accuracy and high computational efficiency. Few of them are Finite Difference Methods [7, 14, 18, 21], Adomian's Decomposition Method, He's Variational Iteration Method, Homotopy Perturbation Method, Collocation Method etc. Several papers have recently been published on finite difference methods for solving the anomalous diffusion equation. The fractional diffusion equations were first studied by Wyss[22, 23] and Schneider[22]. Liu et al considered the time fractional advection dispersion equation. Gao and Sun [4] have presented a high order finite difference scheme for the fractional sub-diffusion equations. The literal interpretation of fractional derivative is that it represents the degree of memory in the diffusing material. A generalised diffusion equation is obtained from a fractional Fick Law[1] that describes transport processes with long memory. In practise, we have two generic forms of fractionalisation of the derivatives with respect to time and space. The time fractional differential equations is a tool to tackle problems involving non-markovian random walk, generally utilised to treat sub-diffusive flow process in which the net motion of particles happens more slowly than the Brownian motion[1, 5].

In this paper we use an effective explicit finite difference scheme for developing the discrete model for time fractional soil moisture diffusion equation which is suitable for simulating random variables whose spatial probability density evolves in time according to this fractional diffusion equation.

The solution of the linear partial differential equation of flow was first proposed by Casagrande, through the use of the graphical flowent method [8]. The method is based on the assumptions that water flows region must be defined in terms of head or non-head flow. The flowent solutions proposed by Casagrande were for simple unconfined flow cases without flux boundary conditions. First experimental study on the movement of water in the soil was done by Henry Darcy (1856). Edgar Buckinghm (1907) described the water flow in unsaturated porous media modifying the equation of Darcy. Richard's (1931) combined the equations of Darcy and Buckingham with the equation of continuity to establish an over all relationship. Klute (1972) described several methods for estimating the hydraulic conductivity and diffusivity for unsaturated soils [2, 7, 8, 11]. To understand such phenomenon, soil scientists have made some models for the flow of water into soil. Furthermore, many researchers developed different types of equations that models the water flow into soil. We consider the general diffusion equation of unsaturated flow of soil moisture as follows

$$\frac{\partial}{\partial x}(D\frac{\partial U}{\partial x}) + \frac{\partial}{\partial y}(D\frac{\partial U}{\partial y}) + \frac{\partial}{\partial z}(D\frac{\partial U}{\partial z}) + \frac{\partial K}{\partial z} = \frac{\partial U}{\partial t} \tag{1.1}$$

where,

$$\begin{aligned} U(x,y,z,t) &= \text{the volumetric soil moisture content,} \\ D &= \text{the diffusivity of soil moisture,} \\ K &= \text{ the capillary or hydraulic conductivity of soil moisture.} \end{aligned}$$

If for equation (1.1), the flow takes place in the Z direction, as for infiltration of water into the soil, then the equation (1.1) becomes one-dimensional flow equation, which is given below

$$\frac{\partial}{\partial z}(D\frac{\partial U}{\partial z}) + \frac{\partial K}{\partial z} = \frac{\partial U}{\partial t} \tag{1.2}$$

where

$$\begin{aligned} D &= K\frac{\partial h_t}{\partial U} \\ h_t &= \text{the tension head and} \\ K &= \text{the capillary conductivity.} \end{aligned}$$

If the flow is considered in x direction (taken horizontal) then equation (1.1) becomes

$$\frac{\partial}{\partial x}(D\frac{\partial U}{\partial x}) = \frac{\partial U}{\partial t} \tag{1.3}$$

Now we assume that D is a constant then the one-dimensional diffusion equation is

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2} \tag{1.4}$$

which is exactly the diffusion heat flow equation [8] and it is well studied by Richard's [4] for water flow instead of heat flow. The model problem for the moisture flow in horizontal tube is given by

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2}, t > 0, x \geq 0 \tag{1.5}$$

To solve a particular model problem of moisture flow into a horizontal tube, we have to impose proper initial and boundary conditions. So, we consider an initial uniform moisture percentage of U as U_0 (U_0 is constant) at time $t = 0$, which becomes the initial condition and is mathematically expressed as

$$U(x,t) = U_0,\ t = 0,\ x \geq 0 \tag{1.6}$$

Now for left boundary condition, there is source of water applied and placed at $x = 0$ so as to maintain at all times after t=0 as U_L, and which is mathematically expressed as

$$U(x,t) = U_L,\ x = 0,\ t \geq 0 \tag{1.7}$$

Now for right boundary condition, there is source of water applied and placed at semi-infinite plane so as to maintain at all times after $t = 0$ is U_R, which is mathematically expressed as

$$U_x(x,t) = U_R,\ x \to \infty,\ t \geq 0 \tag{1.8}$$

Therefore, the model initial boundary value problem (IBVP) for soil moisture flow is given as follows

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2}, t > 0, x \geq 0 \tag{1.9}$$

subject to the initial and boundary conditions

$$U(x,t) = U_0,\ t = 0,\ x \geq 0 \tag{1.10}$$

$$U(x,t) = U_L,\ x = 0,\ t \geq 0,\ U_x(x,t) = U_R,\ x \to \infty,\ t \geq 0 \tag{1.11}$$

for $U(x,t)$ is volumetric water content and D is the diffusivity constant of soil moisture.

We consider the following definitions for further developments.

Definition 1.1 *The Caputo time fractional derivative of order* α, $(0 < \alpha \leq 1)$ *is defined as follows*

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma 1-\alpha} \int_0^t \frac{\partial U(x,t)}{\partial \xi} \frac{d\xi}{(t-\xi)^\alpha} \ , & 0 < \alpha < 1 \\ \frac{\partial U(x,t)}{\partial t}, & \alpha = 1 \end{cases}$$

where $\Gamma(.)$ is a Gamma function.

Definition 1.2 *The symmetric second order difference quotient in space at time level* $t = t_k$ *is given as follows*

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{U(x_{i-1},t_k) - 2U(x_i,t_k) + U(x_{i+1},t_k)}{h^2}$$

We organize the paper as follows: In section 2 , we develop the explicit fractional order finite difference scheme for time fractional soil moisture diffusion equation. The stability of the solution is proved in section 3 and section 4 deals with convergence of the scheme. The numerical solution of time fractional soil moisture diffusion equation is obtained using Mathematica software in the last section.

2 APPROXIMATE FINITE DIFFERENCE SCHEME

We consider the following time fractional soil moisture diffusion equation (TFSMDE) with initial and boundary conditions

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = D\frac{\partial^2 U(x,t)}{\partial x^2}; \ (x,t) \in \Omega : [0,L] * [0,T] \tag{2.1}$$

$$Initial\ condition: \ U(x,0) = U_0, \ 0 \leq x \leq L \tag{2.2}$$

$$Boundary\ conditions: \ U(0,t) = U_L, \ U_x(L,t) = 0, \ x \to \infty, \ t \geq 0 \tag{2.3}$$

where $0 < \alpha \leq 1$ and D: diffusivity constant.
Note that for $\alpha = 1$, we recover in the limit the well known diffusion equation of Markovian process

$$\frac{\partial U(x,t)}{\partial t} = D\frac{\partial^2 U(x,t)}{\partial x^2}; \ x \epsilon R; \ t \geq 0$$

For the explicit numerical approximation scheme, we define $h = \frac{(x_R - x_L)}{N} = \frac{L}{N}$ and $\tau = \frac{T}{N}$ the space and time steps respectively, such that $t_k = k\tau$; k = 0,1,...,N be the integration time $0 \leq t_k \leq T$ and $x_i = x_L + ih$ for i = 0,1, ..., N. Define $U_i^k = U(x_i,t_k)$ and let U_i^k denote the numerical approximation to the exact solution $U(x_i,t_k)$.
In the differential equation (2.1), the time fractional derivative term is approximated

by the following scheme

$$\frac{\partial^\alpha U(x_i,t_{k+1})}{\partial t^\alpha} \approx \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{k+1}} \frac{1}{(t_{k+1}-\xi)^\alpha}\frac{\partial U(x_i,\xi)}{\partial \xi}d\xi$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{U(x_i,t_{j+1})-U(x_i,t_j)}{\Delta t}\int_{j\tau}^{(j+1)\tau}\frac{d\xi}{(t_{k+1}-\xi)^\alpha}$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{U(x_i,t_{j+1})-U(x_i,t_j)}{\tau}\int_{(k-j)\tau}^{(k+1-j)\tau}\frac{d\eta}{\eta^\alpha}$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{U(x_i,t_{k+1-j})-U(x_i,t_{k-j})}{\tau}\int_{(j)\tau}^{(j+1)\tau}\frac{d\eta}{\eta^\alpha}$$

$$= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{k}\frac{U(x_i,t_{k+1-j})-U(x_i,t_{k-j})}{\tau}\times [(j+1)^{1-\alpha}-j^{1-\alpha}]$$

$$\frac{\partial^\alpha U(x_i,t_{k+1})}{\partial t^\alpha} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[U(x_i,t_{k+1})-U(x_i,t_k)]+ \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k} b_j[U(x_i,t_{k+1-j})-U(x_i,t_{k-j})]$$

where $b_j=(j+1)^{1-\alpha}-j^{1-\alpha},\ j=0,1,2,...,k.$
For approximating the second order space derivative, we adopt a symmetric second order difference quotient in space at time level $t=t_k$

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{U(x_{i-1},t_k)-2U(x_i,t_k)+U(x_{i+1},t_k)}{h^2}$$

Therefore, the fractional approximated equation is

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[U_i^{k+1}-U_i^k]+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k} b_j[U_i^{k-j+1}-U_i^{k-j})] = D\frac{[U(x_{i-1},t_k)-2U(x_i,t_k)+U(x_{i+1},t_k)]}{h^2} \tag{2.4}$$

After simplification , we get

$$U_i^{k+1} = rU_{i-1}^k+(1-2r-b_1)U_i^k+rU_{i+1}^k+\sum_{j=1}^{k-1}(b_j-b_{j+1})U_i^{k-j}+b_kU_i^0,\ i=0,1,...,N,\ k=0,1,... \tag{2.5}$$

where $r=\frac{D\,\tau^\alpha\Gamma(2-\alpha)}{h^2}$ and $b_j=(j+1)^{1-\alpha}-j^{1-\alpha}$

The initial condition is approximated as $U_i^0 = U_0$, i = 1, 2, ... , N.
The left boundary condition is approximated as $U_0^k = U_L,\ k = 0, 1, 2, ..., N.$
Now, using central difference the right boundary condition is approximated as follows

$$\frac{U_{N+1}^k - U_{N-1}^k}{2h} = 0,\ \ k = 0, 1, 2, ..., N$$

Therefore, the complete fractional approximated IBVP is

$$U_i^1 = rU_{i-1}^0 + (1-2r)U_i^0 + rU_{i+1}^0 \ for\ k = 0 \tag{2.6}$$

$$U_i^{k+1} = rU_{i-1}^k + (1-2r-b_1)U_i^k + rU_{i+1}^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})U_i^{k-j} + b_k U_i^0,\ for\ k \geq 1 \tag{2.7}$$

$$Initial\ condition:\ U_i^0 = U_0, i = 0, 1, 2, ..., N. \tag{2.8}$$

$$Boundary\ conditions:\ U_0^k = U_L\ and\ U_{N+1}^k = U_{N-1}^k \tag{2.9}$$

where $r = \frac{D\ \tau^\alpha \Gamma(2-\alpha)}{h^2}$ and $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}, j = 0, 1, 2, ..., k.$
Therefore, the fractional approximated IBVP (2.6) − (2.9) can be written in the following matrix equation form

$$U^1 = BU^0 + S,\ for\ k = 0 \tag{2.10}$$

$$U^{k+1} = AU^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})U^{k-j} + b_k U^0 + S\ for\ k \geq 1 \tag{2.11}$$

where $U^k = (U_1^k, U_2^k, ..., U_N^k)^T,\ k = 0, 1, 2..., N$
A and B are tri-diagonal matrices of order N given by

$$A = \begin{pmatrix} (1-2r-b_1) & r & 0 & 0\cdots & 0 & 0 & 0 \\ r & (1-2r-b_1) & r & 0\cdots & 0 & 0 & 0 \\ 0 & r & (1-2r-b_1) & r\cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots\ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0\cdots & r & (1-2r-b_1) & r \\ 0 & 0 & \cdots & 0\cdots & 0 & 2r & (1-2r-b_1) \end{pmatrix}$$

$$B = \begin{pmatrix} (1-2r) & r & 0 & 0\cdots & 0 & 0 & 0 \\ r & (1-2r) & r & 0\cdots & 0 & 0 & 0 \\ 0 & r & (1-2r) & r\cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots\ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0\cdots & r & (1-2r) & r \\ 0 & 0 & \cdots & 0\cdots & 0 & 2r & (1-2r) \end{pmatrix}$$

and S is a constant column matrix of order N given by

$$S = (rU_L,\ 0,\ 0\cdots, 0,\ 0)^T.$$

The above system of algebraic equations is solved by using Mathematica software in section 5.
In the next section we discuss the stability of the solution of fractional explicit finite difference scheme (2.6) – (2.9) for the time fractional soil moisture diffusion equation (TFSMDE) (2.1) – (2.3)

3 STABILITY

Lemma: The eigenvalues of the NXN tri-diagonal matrix

$$\begin{pmatrix} a & b & & & & & \\ c & a & b & & & & \\ & c & a & b & & & \\ & & \cdots & \cdots & & & \\ & & & \cdots & \cdots & & \\ & & & & c & a & b \\ & & & & & c & a \end{pmatrix}$$

are given as

$$\lambda_s = a + 2\sqrt{bc}\ cos\frac{s\pi}{N+1},\ \ s = 1, 2, \cdots N$$

where a,b and c may be real or complex.

Theorem 3.1 *The solution of the explicit finite difference scheme* (2.6) – (2.9) *for the time fractional soil moisture diffusion equation* (2.1) – (2.3) *is stable, when*

$$r \leq \min\left\{\ \tfrac{1}{2}, \tfrac{2-b_1}{4}, \tfrac{b_1}{2(\sqrt{2}-1)}, \tfrac{2-b_1}{2(1+\sqrt{2})}\ \right\}$$

Proof: We shall use the mathematical induction to analyse the stability.
For $k = 0$ and $1 \leq i \leq N-1$ the eigen values of B are given by

$$\begin{aligned}
\lambda_s &= 1 - 2r + 2rcos\frac{s\pi}{N};\ \ s = 1, 2, \cdots, N-1 \\
&\leq 1 \\
\lambda_s &= 1 - 2r + 2rcos\frac{s\pi}{N};\ \ s = 1, 2, \cdots, N-1 \\
&\geq 1 - 2r - 2r = 1 - 4r \\
&\geq -1\ when\ 1 - 4r \geq -1 \Rightarrow r \leq \frac{1}{2} \\
&\Rightarrow |\lambda_s| \leq 1\ when\ r \leq \frac{1}{2}
\end{aligned}$$

For $k=0$ and $i=N$

$$\begin{aligned}\lambda_s &= 1-2r+2\sqrt{2}rcos\frac{s\pi}{N+1};\ s=1,2,\cdots,N\\ &\leq 1-2r+2\sqrt{2}r = 1-2r(1-\sqrt{2})\\ &\leq 1\\ \lambda_s &= 1-2r+2\sqrt{2}rcos\frac{s\pi}{N+1};\ s=1,2,\cdots,N\\ &\geq 1-2r-2\sqrt{2}r = 1-2r(1+\sqrt{2})\\ &\geq -1\ when\ 1-2r(1+\sqrt{2})\geq -1 \Rightarrow r\leq \frac{1}{1+\sqrt{2}}\\ &\Rightarrow |\lambda_s|\leq 1\ when\ r\leq \frac{1}{1+\sqrt{2}}\end{aligned}$$

Hence, for $k=0$ and $1\leq i\leq N$ we have $|\lambda_s|\leq 1\ when\ r\leq \min\{\frac{1}{2},\frac{1}{1+\sqrt{2}}\}$

$$\Rightarrow |\lambda_s|\leq 1\ when\ r\leq \frac{1}{2}$$

Therefore,

$$\begin{aligned}\|B\|_2 &= \max_{1\leq i\leq N}|\lambda_s|\leq 1\\ \Rightarrow \|B\|_2 &\leq 1\\ \|U^1\|_2 &= \|BU^0\|_2\\ &\leq \|B\|_2\|U^0\|_2\\ &\leq \|U^0\|_2\end{aligned}$$

That is

$$\|U^1\|_2\leq \|U^0\|_2\ true for\ n=1$$

We assume $\|U^k\|_2\leq \|U^0\|_2$ for $n\leq k$ is true
We prove that $\|U^{k+1}\|_2\leq \|U^0\|_2$ for $n=k+1$
For $\|A\|_2$ we have, for $1\leq i\leq N-1$, the eigen values of A are given by

$$\begin{aligned}\lambda_s &= 1-2r-b_1+2rcos\frac{s\pi}{N};\ s=1,2,\cdots,N-1\\ &\leq 1-2r-b_1+2r\\ &\leq 1-b_1\ for\ b_1>0\\ \lambda_s &= 1-2r-b_1+2rcos\frac{s\pi}{N};\ s=1,2,\cdots,N-1\\ &\geq 1-2r-b_1-2r = 1-4r-b_1\\ &\geq -1\ when\ 1-4r-b_1\geq -1\Rightarrow r\leq \frac{2-b_1}{4}\\ &\Rightarrow |\lambda_s|\leq 1\ when\ r\leq \frac{2-b_1}{4}\ for\ 1\leq i\leq N-1\end{aligned}$$

For $i = N$

$$\begin{aligned}
\lambda_s &= 1 - 2r - b_1 + 2\sqrt{2}rcos\frac{s\pi}{N+1};\ s = 1, 2, \cdots, N \\
&\leq 1 - 2r - b_1 + 2\sqrt{2}r = 1 - b_1 - 2r(1 - \sqrt{2}) \\
&\leq 1\ when\ 1 - b_1 - 2r(1 - \sqrt{2}) \leq 1\ \Rightarrow r \leq \frac{b_1}{2(\sqrt{2} - 1)} \\
\lambda_s &= 1 - 2r - b_1 + 2\sqrt{2}rcos\frac{s\pi}{N+1};\ s = 1, 2, \cdots, N \\
&\geq 1 - 2r - b_1 - 2\sqrt{2}r = 1 - b_1 - 2r(1 + \sqrt{2}) \\
&\geq -1\ when\ 1 - b_1 - 2r(1 + \sqrt{2}) \geq -1 \Rightarrow r \leq \frac{2 - b_1}{2(1 + \sqrt{2})} \\
&\Rightarrow |\lambda_s| \leq 1\ when\ r \leq \min\{\frac{b_1}{2(\sqrt{2} - 1)}, \frac{2 - b_1}{2(1 + \sqrt{2})}\}
\end{aligned}$$

Hence, for $1 \leq i \leq N$ we have

$$|\lambda_s| \leq 1\ when\ r \leq \min\{\frac{2 - b_1}{4}, \frac{b_1}{2(\sqrt{2} - 1)}, \frac{2 - b_1}{2(1 + \sqrt{2})}\}$$

Therefore, $\|A\|_2 \leq 1$
Hence

$$\begin{aligned}
\|U^{k+1}\|_2 &= \|AU^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})U^{k-j} + b_kU^0 + S\|_2 \\
&\leq \|A\|_2\|U^k\|_2 + (b_1 - b_2 + b_2 - b_3 + \cdots + b_{k-1} - b_k)\|U^{k-j}\|_2 + b_k\|U^0\|_2 \\
&\leq (1 - b_1)\|U^0\|_2 + (b_1 - b_k)\|U^0\|_2 + b_k\|U^0\|_2 \\
&\leq (1 - b_1 + b_1 - b_k + b_k)\|U^0\|_2
\end{aligned}$$

$$\Rightarrow \|U^{k+1}\|_2 \leq \|U^0\|_2$$

That is, result true for $n = k + 1$
Hence, by induction

$$\|U^k\|_2 \leq \|U^0\|_2$$

Therefore, this shows that the scheme is stable when

$$r \leq \min\left\{ \tfrac{1}{2}, \tfrac{2-b_1}{4}, \tfrac{b_1}{2(\sqrt{2}-1)}, \tfrac{2-b_1}{2(1+\sqrt{2})} \right\}$$

The next section is devoted for convergence of the finite difference scheme.

4 CONVERGENCE

Theorem 4.1 *Let $\bar{U}^k$ be the vector of exact solution and U^k be the vector of approximate solution of the time fractional soil moisture diffusion equation TFSMDE*

$(2.1)-(2.3)$ *then* U^k *converges to* $\bar{U}^k$ *as* $(h,\tau)\to(0,0)$ *when,*

$$r \le \min\left\{ \tfrac{1}{2}, \tfrac{2-b_1}{4}, \tfrac{b_1}{2(\sqrt{2}-1)}, \tfrac{2-b_1}{2(1+\sqrt{2})} \right\}$$

Proof: Let

$$U^k = [u_1, u_2, \cdots, u_N]^T$$
$$\bar{U}^k = [\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_N]^T$$

Then,

$$E^k = \bar{U}^k - U^k$$

Let us assume that

$$|e_l^k| = \max_{1\le i\le N} |e_i^k| = \|E^k\|_\infty, \; for\; l = 1, 2, ...$$

and

$$T_l^k = \max_{1\le i\le N} |T_i^k| = h^2 O(\tau + h^2), \; for\; l = 1, 2, ...$$

For $k = 0$, from equation (2.6) we have

$$\begin{aligned}
|e_l^1| &= |re_{i-1}^0 + (1-2r)e_i^0 + re_{i+1}^0| + r|T_i^1| \\
&\le |re_{i-1}^0| + |(1-2r)e_i^0| + |re_{i+1}^0| + |rT_i^1| \\
&\le r|e_l^0| + (1-2r)|e_l^0| + r|e_l^0| + r|T_l^1| \\
&\le (r + 1 - 2r + r)|e_l^0| + r|T_l^1| \\
&\le |e_l^0| + r|T_l^1| = |e_l^0| + rh^2 O(\tau + h^2) \\
\Rightarrow \|E^1\|_\infty &\le \|E^0\|_\infty + \tau^\alpha \Gamma[2-\alpha] O(\tau + h^2)
\end{aligned}$$

That is, the result holds for $n = 1$.
For $n = k$, we assume

$$\|E^k\|_\infty \le \|E^0\|_\infty + k\tau^\alpha \Gamma[2-\alpha] O(\tau + h^2)$$

For $n = k+1$, we prove that

$$\|E^{k+1}\|_\infty \le \|E^0\|_\infty + (k+1)\tau^\alpha \Gamma[2-\alpha] O(\tau + h^2)$$

Now, from equation (2.7) we have

$$\begin{aligned}
|E_l^{k+1}| &= |re_{i-1}^k + (1-2r-b_1)e_i^k + re_{i+1}^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})e_i^{k-j} + b_k e_i^0| + r|T_l^k| \\
&\le r|e_{i-1}^k| + (1-2r-b_1)|e_i^k| + r|e_{i+1}^k| + \sum_{j=1}^{k-1}(b_j - b_{j+1})|e_i^{k-j}| + b_k|e_i^0| + r|T_l^k| \\
&\le r|e_l^k| + (1-2r-b_1)|e_l^k| + r|e_l^k| + (b_1 - b_2 + b_2 \cdots + b_{k-1} - b_k)|e_l^k| + b_k|e_l^k| + r|T_l^k| \\
&\le (r + 1 - 2r - b_1 + r + b_1 - b_k + b_k)|e_l^k| + r|T_l^k| \\
&\le |e_l^k| + r|T_l^k| \\
&\le \|E^k\|_\infty + r|T_l^k| \\
&\le \{\|E^0\|_\infty + k\tau^\alpha \Gamma[2-\alpha] O(\tau + h^2)\} + \tau^\alpha \Gamma[2-\alpha] O(\tau + h^2) \\
&\le \|E^0\|_\infty + (k+1)\tau^\alpha \Gamma[2-\alpha] O(\tau + h^2)
\end{aligned}$$

Therefore, we conclude that if we assume $r \le \min\left\{ \frac{1}{2}, \frac{2-b_1}{4}, \frac{b_1}{2(\sqrt{2}-1)}, \frac{2-b_1}{2(1+\sqrt{2})} \right\}$ then $\|E^k\|_\infty \to 0$ as $h \to 0$ and $\tau \to 0$ which results in the convergence of U_i^k to $U(x_i, t_k)$. Hence, the proof is completed.

5 NUMERICAL SOLUTIONS

In this section, we obtain the approximated solution of time fractional soil moisture diffusion equation with initial and boundary conditions. To obtain the numerical solution of the time fractional soil moisture diffusion equation (TFSMDE) by the finite difference scheme, it is important to use some analytical model. Therefore, we present an example to demonstrate that TFSMDE can be applied to simulate behavior of a fractional diffusion equation by using Mathematica Software.
We consider the following one-dimensional time fractional soil moisture diffusion equation with initial and boundary conditions

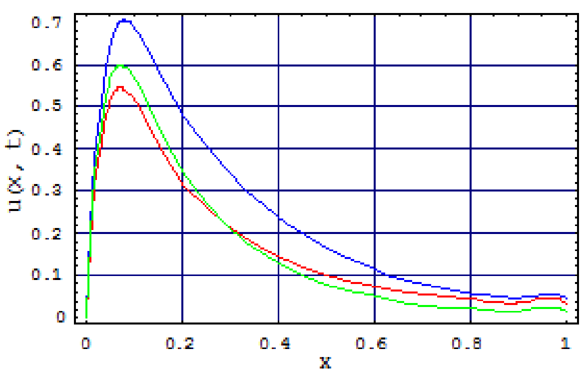
$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = \frac{\partial^2 U(x,t)}{\partial x^2}\, 0 < x < 1,\, 0 < \alpha \le 1,\; t > 0$$

$$initial\ condition : U(x,0) = 0,\; 0 \le x \le 1$$

$$boundary\ conditions : U(0,t) = 1,$$

$$U_x(x,t) = 0,\; as\ x \to \infty,,\; t > 0$$

with the diffusion coefficient $D = 1$.
The numerical solutions are obtained at $t = 0.004$ by considering the parameters $\tau = 0.0004,\; h = 0.1,\; \alpha = 0.7,\; \alpha = 0.8,\; and\ \alpha = 0.9$, is simulated in the following figure.



$Fig.5.1$: The soil moisture diffusion profile with $t = 0.05,\; h = 0.1,$

$\alpha = 0.7\ \beta = 1.7(red),\; \alpha = 0.8, \beta = 1.9(blue)\ and\ \alpha = 0.9,\; \beta = 1.8(green)$

CONCLUSIONS

The proposed explicit difference approximation for time fractional soil moisture diffusion equation can be reliably applied to solve any fractional order dynamical systems and controllers, minding the conditions for stability and convergence of the scheme. The numerical results are also compatible with theoretical analysis, hence showing the numerical stability of the finite difference scheme.

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