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A Solution to Linear Black-Scholes Second-order Parabolic Equation in Sobolev Spaces

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ARTICLE INFO	ABSTRACT
Published Online: 29 October 2022 Corresponding Author: **Osu, B. O.**	In this paper, a set of functions were constructed that transforms Black-Scholes partial differential equation into weak formulations. The analytical solutions: existence, uniqueness and other estimates were also obtained in weak form with the use of boundary conditions to establish the effects of its financial implications in Sobolev spaces. The regularity conditions of the problem were considered which the coefficients, the boundary of the domain are all smooth functions. To this end, the definitions, assumptions which paved way to useful assertions are illustrated in this paper.

KEYWORDS: Weak solutions; Black-Scholes equations; Sobolev spaces; Option and Smooth functions
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1. INTRODUCTION

Mathematical analysis in finance is very important not only to mathematicians or investors but the generality of the masses. So understanding its dynamics will help economist, Government, opinion leaders to adequately plan their investments effectively well for future purposes. Hence, the special area of interest in this study is Mathematics of finance which has to deal with the evolution of option pricing. In other words, an underlying asset of an option is a business between parties who come together for agreement on either buying or selling an underlying asset at a particular strike price in the future for a fixed price. More so, the cost of fundamental asset which governs the growth of the option price used no-arbitrage argument to elucidate a Partial Differential Equation (PDE) with respect to the expiration. In financial applications, the Black-Scholes (BS) equation has been used extensively.

For instance, there are mammoth interest in financiers, mathematicians and statisticians over the partial differential equation derived by [1] to analyze the European option on a stock market that does not pay a dividend during the life of the option. Mathematically they restricted their analysis to conditions which made the problem simpler. In the dynamics of option pricing, describes the Black-Scholes PDE as a function of security index and of time to maturity of underlying assets. So due to the recent development in option pricing has resulted to many diverse mathematical and computational methods being in use, for example see [2-4] etc.

Therefore [5] studied implied volatility and implied risk free rate of return in solving systems of Black- Scholes equations. In their research they established that options prices provides important information for market participants for future expectations and market policies. In the same vain [6] analyzed BS formula for the valuation of European options; Hermite polynomials were applied. They concluded that BS formula can easily be achieved devoid of the use of partial differential equation. In another study of BS [7] considered the BS terminal value problem and observed that their proposed method is better, simple than the previous methods. In the work of [8] time varying factor were incorporated in the explicit formula for different aspect of options with the aim of providing exact solution for dividend paying equity of option. In considering the stability of stock market price of stochastic model, [9] applied Crank-Nicolson numerical scheme to BS model. The results showed stock prices being stable and its increasing rate of stock shares was obtained. Not quite long, [10] investigated the variation of stock market price using BS PDE. The convergence to equilibrium of growth rate and sufficient conditions for stability was achieved. However, [11] studied Black-Scholes model because of its biasness in mispricing options. They established a new technique of assessing pricing effects on the premise to reduce pricing bias. Details of more financial models can be seen in the works of [12-14] etc.

On the contrary, there are kin interests by mathematics scholars with research based on topological and analytical foundations; such solutions exist but are not trivial. Hence, there are so many algebraic families with the above concepts namely, Hilbert spaces, Banach spaces and Sobolev spaces etc. Here, we consider Sobolev spaces because of its redefinition of differentiability which starts with weak formulations to obtain weak solutions.

Nevertheless, good numbers of scholars have used PDEs in Sobolev spaces for different reasons and results obtained in different ways such as [15] considered the existence, uniqueness and stability analysis for Ordinary Differential Equation (ODE) with coefficients in Sobolev spaces. In their results method of renormalization solutions were used in the analysis of linear transport equations. [16] viewed at solution to nonlinear BS equations and concluded that the bounded domain of weak solution were extended to entire domain via diagonal processes. [17] considered a nonlinear Black Scholes Equation for incorporating transaction cost and portfolio risk as one of the financial models. These problems were solved in Sobolev space and they obtained a weak solution that has properties of Fourier transformation.

This study is aimed at solving Black-Scholes second order parabolic equation in Sobolev spaces on the basis of obtaining weak solutions which can be used in financial applications. This paper extends the work of [16] by considering BS PDE in such spaces. To the best of our knowledge this is the first study that has solved fully stochastic parabolic PDE with detailed proves, definitions and assumptions in Sobolev spaces.

The paper is arranged in the following ways: Section 2 Mathematical preliminaries of Black-Scholes, Section 3 presents Problem formulation of Black-Scholes equation in Sobolev Spaces, Main results of Black-Scholes equation in Sobolev spaces is seen in 4.This paper is concluded in Section 5.

2. MATHEMATICAL PRELIMINARIES OF BLACK-SCHOLES EQUATION

Supposing that $\rho \in [-1,1]$ is a bounded correlation coefficient which tends to ∞ within the processes W_t^1 and W_t^2 ; therefore the value $u(S,v,t)$ of discounted asset price which is at the rate r and are governed by the partial differential equations as follows, [18].

$$\frac{\partial u}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 u}{\partial S^2}+\rho\sigma vS\frac{\partial^2 u}{\partial S\partial v}+\frac{1}{2}v\sigma^2\frac{\partial^2 u}{\partial v^2}+rs\frac{\partial u}{\partial S}+\{K[\theta-v]-\lambda(S,v,t)\}\frac{\partial u}{\partial v}-ru=0,t>0 \quad .(1.1)$$

$$u(S,v,t)=f(S,v) \quad . \qquad (1.2)$$

In mathematical finance, a single asset for contingent claim of the generic PDE is of the form:

$$\frac{\partial u}{\partial t}+a(x,t)\frac{\partial^2 u}{\partial x^2}+b(x,t)\frac{\partial u}{\partial x}+c(x,t)u=0 \quad . \qquad (1.3)$$

Where t denotes time to maturity, x denotes the value of the underlying asset or functions of monotonic type (e.g log (S); log-spot) and u denotes the value of the claim which is a function of x and t the following terms a (.), b (.) and c(.) are diffusion, convection and reaction Coefficients.(1.3)can as well be written in the following manners :

$$\frac{\partial u}{\partial t}+a(x,t)\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)+b(x,t)\frac{\partial}{\partial x}\left(\beta(x,t)u\right)+c(x,t)u=0 \quad . \qquad (1.4)$$

The above PDE describes the dynamics of the transition density of stochastic variables or quantities, for example, value of a stock which is seen in the Fokkper-Planoke (Kolmogrorov forward) equation of probability measures. However our interest in this paper is the parabolic financial PDE which is governed with the dynamics of option pricing; hence we have the following:

$$\frac{\partial u}{\partial t}+\frac{1}{2}v\sigma^2\frac{\partial^2 u}{\partial v^2}+K[\theta-v_t]\frac{\partial u}{\partial v}-ru=0.\ t>0. \qquad (1.5)$$

The details of the above option model can be expressly be found in the following books: [19-21] etc.

3. PROBLEM FORMULATION OF BLACK-SCHOLES EQUATION IN SOBOLEV SPACES

Here, we investigate parabolic equations, which are Partial Differential Equations (PDEs). Let ϕ to be an open, bounded subset of $\mathbb{R}^N$;setting $\phi_T=\phi\times(0,T]$ in some certain fixed time, $T>0$. Hence, the initial/boundary value problem are written as:

$$\begin{cases} u_t + Lu = f \ \ in\ \phi_T, \\ u = 0 \ \ on\ \partial\phi \times [0,T], \\ u = g \ \ on\ \phi \times \{t=0\}. \end{cases} \quad (1.6)$$

Where $f:\phi_T \to \mathbb{R}$ and $g:\phi \to \mathbb{R}$ are given, and $u:\bar{\phi}_T \to \mathbb{R}$ is the unknown, $u = u(v,t)$. L is given for each time t a second-order partial differential operator with the divergence form as follows.

$$Lu = -\sum_{ij=1}^{N}\left((v,t)u_{(v,t)i}\right)(v,t)j + \sum_{ij=1}^{N}\left(\left(\frac{1}{2}v\sigma^2\right)^{ij}(v,t)u_{(v,t)i}\right)(v,t)\,j + \sum_{i=1}^{N}k\left[\theta - V_t\right]^i u_{(v.t)i} + r(v,t)u \quad ,(1.7)$$

Or also the non-divergence method.

$$Lu = -\sum_{ij=1}^{N}(v,t)u_{(v,t)i}\,(v,t)j + \sum_{ij=1}^{N}\left(\frac{1}{2}v\sigma^2\right)^{ij}(v,t)u_{(v,t)i}\,(v,t)\,j + \sum_{i=1}^{N}k\left[\theta - V_t\right]^i u_{(v.t)i} + r(v,t)u \quad .(1.8)$$

With the coefficients $\left(\frac{1}{2}v\sigma^2\right)^{ij}, k\left[\theta - V_t\right]^i, r\left(ij = 1,\ldots,N\right)$.

Definition 1. 1: Partial differential operator $\frac{\partial}{\partial t} + L$ is said to exists with a constant $\theta > 0$ and uniform parabolic properties such that,

$$\sum_{ij=1}^{N}\left(\frac{1}{2}v\sigma^2\right)^{ij}(v,t)\xi_i\xi_j \geq \theta\,|\,\xi\,|^2 \ . \quad (1.9)$$

For values of all $(v,t) \in \phi_T, \xi \in \mathbb{R}^N$.

Here a is the usual example, $\left(\frac{1}{2}v\sigma^2\right)^{1j} \equiv k\left(\theta - V_t\right)^i \equiv r \equiv f \equiv 0$,in which case $L = -\Delta$ and the PDE $\frac{\partial u}{\partial t} + Lu$ being the heat equation.

Weak solutions: Here ,We shall consider the case where L has the divergence form (1.8) and try to figure out an appropriate notion of weak solution for the initial /boundary –value problem (1.4).The assumptions of weak solution are:

$$\left(\frac{1}{2}v\sigma^2\right)^{ij}, \mathrm{k}\left(\theta - V_t\right), r \in L^\infty\left(\phi_T\right), ij = 1,\ldots,\mathrm{N} \ . \quad (1.10)$$

$$f \in L^2\left(\phi_T\right) \ . \quad (1.11)$$

$$g \in L^2\left(\phi\right) \ . \quad (1.12)$$

Assuming $\left(\frac{1}{2}v\sigma^2\right)^{ij} = \left(\frac{1}{2}v\sigma^2\right)^{ji}, ij = 1,\ldots,N$, introducing the concept of time dependent bilinear form as:

$$B[u,v;t] := \int_\phi \sum_{ij=1}^{N}(v,t)^{ij}(.,t)u_{(v,t)i}v_{(v,t)j} + \int_\phi \sum_{ij=1}^{N}\left(\frac{1}{2}v\sigma^2\right)^{ij}(.,t)u_{(v,t)i}v_{(v,t)j}$$
$$+\sum_{i=1}^{N}k\left[\theta - V_t\right]^i(.,t)u_{(s,t)i}v + r(.,t)uvd(v,t). \quad (1.13)$$

For $u,v \in H_0'(\phi)$ and almost everywhere $(a.e).\ 0 \leq t \leq T.$

4. MAIN RESULTS OF BLACK-SCHOLES EQUATION IN SOBOLEV SPACES

Definition 1. 2: Weak solution; in defining weak solution, we temporarily assume that $u = u(v,\mathrm{t})$ to be the parabolic problem of (1.7). Our point of focus via relating with u in a mapping such that:

$$u:[0,T]\to H_0'(\phi)\ ,$$

We define as follows $[u(t)](v):=u(v,t), v\in\phi, 0\le t\le T$ .In the forgoing, we shall consider u not as a function of v and t together, but as a mapping u of t in the space $H_0'(\phi)$ of function of v. Therefore, we make some clarifications in (1.4), and then we also define as well:

$$f:[0,T]\to L^2(\phi)\ ,$$

$$[f(t)](v):=f(v,t), v\in\phi, 0\le t\le T\ .$$

Now fixing a function $v\in H_0'(\phi)$ and multiplying the PDE $\frac{\partial u}{\partial t}+Lu=f\ by\ v$ and integrating by parts to obtain as follows:

$$(u',v)+B[\mathrm{u},\mathrm{v};\mathrm{t}]=(f,v)\ . \tag{1.14}$$

In each $0\le t\le T$, the pairing (,) denotes inner product in $L^2(\phi)$,We notice that

$$u_t=g^0+\sum_{j=1}^{N}g^j(v,t)j\ \ in\ \phi_T\ . \tag{1.15}$$

For $g^0:=f-\sum_{i=1}^{N}k[\theta-V_t]^i u_{(s,t)i}-ru\ and\ g^j:=\sum_{i=1}^{N}\left(\frac{1}{2}v\sigma^2\right)^{ij}u_{(v,t)}\ \ j=1,\ldots,N\ .$

As a result the of Right hand side of (1.16) lies in the Sobolev space $H^{-1}(\phi)$,with

$$\|u_t\|H^{-1}(\phi)\le\left(\sum_{j=0}^{N}\|g^j\|^2\ L^2(\phi)\right)^{1/2}\le C\left(\|u\|H_0'(\phi)+\|\mathrm{f}\|\mathrm{L}^2(\phi)\right)\ .$$

The estimate above is suggestive to find a weak solution with $u'\in H^{-1}(\phi)$ for a.e. , time $0\le t\le T$;in any case the first term in (1.15) is represented as $\langle u',v\rangle,\langle,\rangle$ being the paring of $H^{-1}(\phi)\ and\ H_0'(\phi)$.

Definition 1. 3: It suffices to say that a function $u\in L^2(0,T;H_0'(\phi))$ with $u'\in L^2(0,T;H^{-1}(\phi))$ is a weak solution of the parabolic initial/boundary-value problem (1.4) only if.

(i) $\langle u',v\rangle+B[\mathrm{u},\mathrm{v};\mathrm{t}]=(f,v)\ ,\ for\ each\ v\in H_0'(\phi)\ and\ a.e.\ time\ 0\le t\le T.$

(ii) $\mathrm{u}(0)=g$.

Definition 1.4: Existence of weak solutions, building a weak solution of the parabolic problem

$$\left.\begin{aligned}u_t+Lu=f\ \ in\ \phi_T,\\ u=0\ \ on\ \partial\phi\times[0,T],\\ u=g\ \ on\ \ \phi\times\{t=0\}.\end{aligned}\right\} \tag{1.16}$$

At first, we construct solutions of certain finite dimensional approximations to (1.17) and then taking the limits. This is known as Galerkin's method.

Setting a function w $w_k=w_k(v,t)$, k=1,…,as smooth such that,

$$\{w_k\}_{k=1}^{\infty}\ \text{is an orthogonal basis of}\ H_0'(\phi)\ . \tag{1.17}$$

$$\{w_k\}_{k=1}^{\infty}\ \text{is an orthonormal basis of}\ L^2(\phi)\ . \tag{1.18}$$

Taking $\{w_k\}_{k=1}^{\infty}$ to be the normalized Eigen-functions which are appropriately complete, set for $L=-\Delta\ in\ H_0'(\phi)$. In considering a function $u_m:[o,\mathrm{T}]\to\mathrm{H}_0'(\phi)$ we fix a positive integer m .

$$u_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k \ , \tag{1.19}$$

Which is where we pick the coefficients $d_m^k(t), 0 \le t \le T, k = 1,\dots,m,$ such that we have the following:

$$d_m^k(0) = (g, w_k),\ k = 1,\dots,m \ . \tag{1.20}$$

$$(u_m', w_k) + B[u_m, w_k; t] = (f, w_k),\ 0 \le t \le T, k = 1,\dots,m \ . \tag{1.21}$$

where $(,)$ is the inner product in $L^2(\phi)$ .Seeking a function u_m of the form(1.20) that satisfies the projection((1.22) of problem(1.17) on to the finite dimensional subspace spanned through $\{w_k\}_{k=1}^{m}$.

Theorem 1.1: (Constructing of approximate solutions). There exist a unique function u_m of the form (1.20) which satisfies (1.21) and (1.22) for each integer $m = 1, 2, \dots$.

Proof: let u_m be the mathematical structure (1.20), we remark first from (1.22) that,

$$(u_m'(t), w_k) = d_m^k(t) \ . \tag{1.22}$$

$$B[u_m, w_k; t] = \sum_{l=1}^{m} e^{kl}(t) d_m^l(t) \ . \tag{1.23}$$

For $e^{kl}(t) := B[\mathrm{wl}, \mathrm{w}_k; t]$, $\mathrm{k,l} = 1,\dots,\mathrm{m}$.

We write also,

$$f^k(t) := (f(t), w_k),\ k = 1,\dots,m \ . \tag{1.24}$$

Then (1.22) yields a linear system of ODE.

$$d_m^{k'}(t) + \sum_{l=1}^{m} e^{kl}(t) d_m^l(t) = f^k(t),\ k = 1,\dots,m \ . \tag{1.25}$$

With the following initial conditions of (1.21). According to the standard existence theory for ordinary differential equations, there exists a unique conditions of (1.17). Hence, we want certain uniform estimates.

Theorem 1. 2. (Energy estimates). There is a constant C , which depends only on ϕ, T and its coefficients is of L ,such that

$$\begin{aligned} &\max_{0 \le t \le T} \| \mathrm{u}_m(t) \| L^2(\phi) + \| \mathrm{u}_m \| \mathrm{L}^2(0, T; H_0'(\phi)) + \| u_m' \| L^2(0, T, H^{-1}(\phi)) \\ &\le C(\| f \| L^2(0, T; L^2(\phi)) + \| g \| L^2(\phi)), \end{aligned} \tag{1.26}$$

For $m = 1, 2, \dots,$.

Proof:

Multiplying (1.19) by $d_m^k(t)$ and sum for $k = 1,\dots,m$ and then note (1.21) to obtain the quantities:

$$(u_m', u_m) + B[u_m, u_m; t] = (f, u_m) \ . \tag{1.27}$$

For a.e . , $0 \le t \le T$. Thus we move to more to prove the existence of constants $\beta > 0, \gamma \ge 0$ such that the following condition holds:

$$\beta \| u_m \|^2 \ H_0'(\phi) \le B[u_m, u_m; t] + \gamma \| u_m \|^2 \ L^2(\phi) \ . \tag{1.28}$$

$$0 \le t \le T, m = 1, \dots \text{more so} \ |(f, u_m)| \le \frac{1}{2} \| f \|^2 \ L^2(\phi) + \frac{1}{2} \| \mathrm{u}_m \|^2 \ L^2(\phi), \text{ and } (u_m', u_m)$$

for all,

$$= \frac{d}{dt}\left(\frac{1}{2} \| u_m \|^2 \ L^2(\phi)\right) \ \textit{for a.e. } 0 \le t \le T.$$

Therefore (1.25) gives the inequality.

$$\frac{d}{dt}\left(\frac{1}{2} \| u_m \|^2 \ L^2(\phi)\right) + 2\beta \| u_m \| H_0'(\phi) \le C_1 \| \mathrm{u}_m \|^2 \ L^2(\phi) + C_2 \| \mathrm{f} \|^2 \ L^2(\phi) \ . \tag{1.29}$$

For a.e. $0 \le t \le T$ .where $C_1$and C_2 are constants. The following constants are defined:

$$\eta(t) := \| u_m(t) \|^2 \ L^2(\phi) \ , \tag{1.30}$$

$$\xi(t) := \| \mathrm{f}(t) \|^2 \ L^2(\phi) \ . \tag{1.31}$$

So (1.30) means $\eta'(t) \le C_1\eta(t) + C_2\xi(\mathrm{t})$ for a.e. , $0 \le \mathrm{t} \le \mathrm{T}$. hence, the differential form of Gronwall's inequality gives the estimate as.

$$\eta(t) \le e^{C_1 t}\left(\eta(0) + \mathrm{C}_2 \int_0^t \xi(s)ds\right), 0 \le t \le T \ . \tag{1.32}$$

Since $\eta(0) = \| u_m(0) \|^2 \ L^2(\phi) \le \| \mathrm{g} \|^2 \ L^2(\phi)$ by (1.18) gives (1.28)- (1.30) the estimates as.

$\max_{0 \le t \le T} \| \mathrm{u}_m(t) \|^2 \ L^2(\phi) \le \mathrm{C}\left(\| \mathrm{g} \|^2 \ \mathrm{L}^2(\phi) + \| f \|^2 \ L^2(0,T;\mathrm{L}^2(\phi)\right).$

From (1.30) and integrating from $0 \ to \ T$ and invoking the inequality above to obtain:

$\| u_m \|^2 \ L^2\left(0,T;H_0'(\phi)\right) = \int_0^T \| u_m \|^2 \ H_0'(\phi)dt \le C\left(\| g \|^2 \ L^2(\phi) + \| \mathrm{f} \|^2 \ L^2(0,T;L^2(\phi))\right)$.

For any $v \in H_0'(\phi), with \ \| v \| \ H_0'(\phi) \le 1$ and write $v = v' + v^2$ where $v' \in span\{w_k\}_{k=1}^m$ and $\left(v^2, w_k\right) = 0.k = 1,...,m.$

since the function $\{w_k\}_{k=0}^\infty$ are orthogonal in $H_0'(\phi), \| \mathrm{v}' \| \ \mathrm{H}_0'(\phi) \le \| \mathrm{v} \| \ \mathrm{H}_0'(\phi) \le 1.$ Applying (1.22) we reason for a.e. $0 \le t < T$ that

$\left(u_m', v'\right) + B[u_m, v'; t] = \left(f, v'\right)$,

Then (1.20) implies

$\langle u_m', v \rangle = \left(u_m', v\right) = \left(u_m', v'\right) = \left(f, v'\right) - B[u_m, v'; t],$

So

$| \langle u_m', v \rangle | \le C\left(\| \mathrm{f} \| \ \mathrm{L}^2(\phi) + \| \mathrm{u}_m \| \ H_0'(\phi)\right),$

Since $\| v' \| \ H_0'(\phi) \le 1,$

$\| u_m' \| \ H^{-1}(\phi) \le C\left(\| \ f \ \| \ L^2(\phi) + \| \mathrm{u}_m \| \ H_0'(\phi)\right),$

Hence,

$\int_0^T \| u_m' \|^2 \ H^{-1}(\phi)dt \le C\int_0^T \| \ f \ \|^2 L^2(\phi) + \| u_m \|^2 \ H_0'(\phi)dt \le C\left(\| g \|^2 \ L^2(\phi) + \| \mathrm{f} \|^2 \ L^2\left(0,T;L^2(\phi)\right)\right)$.

Existence and Uniqueness, we the take limits as $m \to \infty$ in order to build a weak solution of the initial/boundary-value problem (1.17).

Theorem 1.3. (Existence of weak solution). Following (1.17) there exists a weak solution.

Proof:

We want prove the existence and uniqueness of the energy estimates (1.27), following the sequence $\{u_m\}_{m=1}^\infty$ is bounded in $L^2\left(0,\mathrm{T};\mathrm{H}_0'(\phi)\right)$ and $\{u_m'\}_{m=1}^\infty$ is bounded in $L^2\left(0,T;H^{-1}(\phi)\right)$. Thus there exists a subsequence $\{u_{ml}\}_{l=1}^\infty \subset C\{u_m\}_{m=1}^\infty$ a function, $u \in L^2\left(0,T;H_0'(\phi)\right)$, with $u' \in L^2\left(0,T;H^{-1}{}_0(\phi)\right)$ such that we have (1.33).

$$\begin{cases} u_{ml} \to u \text{ weakly in } L^2\left(0,T;H_0'(\phi)\right), \\ u_{ml}' \to u' \text{ weakly in } L^2\left(0,T;H^{-1}(\phi)\right). \end{cases} \tag{1.33}$$

Fixing an integer N and choosing a function $v \in C'\left([0,T];H_0'(\phi)\right)$ which gives the form:

$$v(t) = \sum_{k=1}^N d^k(t)w_k \ . \tag{1.34}$$

Where $\left\{d^k\right\}_{k=1}^N$ are given smooth functions. We choosing $m \geq N$, and multiplying (1.22) by $d^k(t)$, sum $k=1,\ldots,N$, and then by integration with respect to t obtain:

$$\int_0^T \langle u'_m, v\rangle + B[u_m, v;t]dt = \int_0^T (f, v)dt \ . \tag{1.35}$$

Setting $m = ml$ and recall (1.31), to determine taking to weak limits such that,

$$\int_0^T \langle u', v\rangle + B[u, v;t]dt = \int_0^T (f, v)dt \ . \tag{1.36}$$

This type of equality holds for only functions $v \in L^2\left(0,T;H'_0(\phi)\right)$ as functions of the form (1.32) are dense in the in the space for functions of the form (1.32), in all we have:

$$\langle u', v\rangle + B[u, v;t] = (f, v) \ . \tag{1.37}$$

For each $v \in H'_0(\phi)$ and a.e. $, 0 \leq t \leq T$. According to theorem 1.3 we also observe that $u \in C\left([0,T], L^2(\phi)\right)$.

For the purpose of proving $u(0) = g$, we first note that from (1.34) that,

$$\int_0^T -\langle \mathrm{v}', \mathrm{u}\rangle + B[u, v;t]dt = \int_0^T (f, v)dt + \left(u(0), v(o)\right). \tag{1.38}$$

For each $v \in C'\left([0,T]; H'_0(\phi)\right)$ with $v(T) = 0$ .Similarly, from (1.33) we deduce,

$$\int_0^T -\langle \mathrm{v}', \mathrm{u}_m\rangle + B[u_m, v;t]dt = \int_0^T (f, v)dt + \left(u_m(0), v(o)\right). \tag{1.39}$$

Setting $m = ml$ and once all over again we invoke (1.34) to obtain,

$$\int_0^T -\langle \mathrm{v}', \mathrm{u}_m\rangle + B[u_m, v;t]dt = \int_0^T (f, v)dt + \left(\mathrm{g}, v(o)\right) \ . \tag{1.40}$$

Since $u_{ml(0)} \to g \, in \, L^2(\phi)$. As $v(0)$ is arbitrary, comparing (1.39) and (1.41) we can now say that $u(0) = g$.

Theorem 1. 4. (Uniqueness of weak solutions). There exists a weak solution of (1.17) using $f \equiv g \equiv 0$ is

$$u \equiv 0 \ . \tag{1.41}$$

***Proof*:**

In order to prove this point, we set $v = u$ in the identity of (1.38) $(\text{for} \equiv 0)$ we absorb, using theorem 3 such that,

$$\frac{d}{dt}\left(\frac{1}{2}\|u\|^2 \ L^2(\phi)\right) + B[\mathrm{u},\mathrm{u};\mathrm{t}] = \langle u', u\rangle + B[u, u;t] = 0 \ . \tag{1.42}$$

Since $B[u,u;t] \geq \beta \|u\|^2 \ H'_0(\phi) - \gamma \|\mathrm{u}\|^2 \ L^2(\phi) \geq -\gamma \|\mathrm{u}\|^2 \ L^2(\phi)$.

Using Gronwall's inequality and (1.43) implies (1.42). Hence, the details of Gronwall's inequality is seen in the following: [22-24] etc.

Regularity: Here describes the regularity of our weak solutions u to the Black-Scholes second-order parabolic equations. The primary objective is to prove that u is a smooth function as far as the coefficient of the PDE, the domain of the boundary is smooth. The interesting thing in the derivation of estimate is to increase some perceptions as to extend of regularity assertions could possibly be effective, temporarily let $u = u(v,t)$ is a smooth solution of this initial- value problem for the heat equation:

$$\begin{cases} u_t - \Delta u = f \ \ in \ \mathbb{R}^N \times (0,\mathrm{T}], \\ \quad u = g \ \ on \ \mathbb{R}^N \times \{t=0\}. \end{cases} \tag{1.43}$$

Assume also u tends to zero as $|v| \to \infty$ satisfactorily to explain the computations as follows: we calculate $0 \leq t \leq T$.

$$\int_{\mathbb{R}^N} f^2 dv = \int_{\mathbb{R}^N} \left(u_t - \Delta u\right)^2 dv - \int_{\mathbb{R}^N} u_t^2 - 2\Delta u u_t + \left(\Delta u\right)^2 = \int_{\mathbb{R}^N} u_t^2 + 2Du.Du_t + \left(\Delta u\right)^2 dv \ . \tag{1.44}$$

$2Du.Du_t = \frac{d}{dt}\left(|\mathrm{Du}|^2\right)$ and consequently $\int_0^t \int_{\mathbb{R}^N} 2Du.Du_t dvds = \int_{\mathbb{R}^N} |\mathrm{Du}|^2 dv \,|_{s=0}^{s=t}$

$$= \int_{\mathbb{R}^N} \left(\Delta u\right)^2 dv = \int_{\mathbb{R}^N} |\mathrm{D}^2\mathrm{u}|^2 dv.$$

Appling the above two equalities in (1.44) and integrating time to obtain,

$$\sup_{0\le t\le T}\int_{\mathbb{R}^N}|Du|^2dv+\int_0^T\int_{\mathbb{R}^N}u_t^2+|D^2u|^2\,dvdt\le C\left(\int_0^T\int_{\mathbb{R}^N}f^2dvdt+\int_{\mathbb{R}^N}|\mathrm{Dg}|^2\,dv\right). \qquad (1.45)$$

We therefore estimate the $L^2-norms\ of\ u_t$ and D^2u within $\mathbb{R}^N\times(0,T)$ in terms of the L^2 – norms of f on $\mathbb{R}^N\times(0,T)$ and *the* L^2 – norms of Dg *on* $\mathbb{R}^N$ .we can now differentiate the PDE with respect to t and setting $\bar{u}:=u_t$

$$\begin{cases}\bar{u}_t-\Delta\bar{u}=\bar{f}\ \ in\ \mathbb{R}^N\times(0,\mathrm{T}],\\ \bar{u}=\bar{g}\ \ on\ \mathbb{R}^N\times\{t=0\}.\end{cases} \qquad (1.46)$$

for $\bar{f}:=f_t,\bar{g}:=u_t(.,0)=f(.,0)+\Delta g$. Multiplying by $\bar{u}$,integrating by parts and invoking Gronwall's inequality, we infer that,

$$\sup_{0\le t\le T}\int_{\mathbb{R}^N}|u_t|^2dv+\int_0^T\int_{\mathbb{R}^N}|Du_t|^2dvdt\le C\left(\int_0^T\int_{\mathbb{R}^N}f^2dvdt+\int_{\mathbb{R}^N}|\mathrm{D^2g}|^2+f(.,0)^2\,dv\right). \qquad (1.47)$$

But then ,

$$\max_{0\le t\le T}\|f(.,t)\|L^2(\mathbb{R}^N)\le C\left(\|f\|L^2(\mathbb{R}^N\times(0,T))\right)+\left(\|f_t\|L^2(\mathbb{R}^N\times(0,T))\right). \qquad (1.48)$$

Following to theorem 1.2 and writing $-\Delta u=+-u_i$ gives:

$$\int_{\mathbb{R}^N}|D^2u|^2dv\le C\int_{\mathbb{R}^N}f^2+u_t^2dv\ . \qquad (1.49)$$

The Combinations of (1.48)-(1.50) leads us to the estimate of the following below:

$$\sup_{0\le t\le T}\int_{\mathbb{R}^N}|u_t|^2dv+|D^2u|^2\,dv+\int_0^T\int_{\mathbb{R}^N}|Du_t|^2dvdt\le C\left(\int_0^T\int_{\mathbb{R}^N}f^2dvdt+\int_{\mathbb{R}^N}|\mathrm{D^2g}|^2\,dv\right)\ . \qquad (1.50)$$

For some constant C .

Therefore, our earlier previous computations is suggestive to have estimates corresponding to (1.43) and (1.48) respectively for weak solution to Black-Scholes second order parabolic equation.

5. CONCLUSION

The analysis of Black-Scholes PDE in Sobolev spaces has been effortlessly established; hence derivatives are well understood in suitable weak sense to make the space complete. From the analysis, a set of functions were constructed that transforms Black-Scholes partial differential equation into weak formulations; which shows: existence, uniqueness and other estimates in weak form with the use of boundary conditions to establish the effects of its financial effects in Sobolev spaces. The regularity conditions of the problem were considered which the coefficients, the boundary of the domain are all smooth functions.

Generally, the analytical methods of solving PDEs are non-trivial; it becomes complex when solved in Sobolev spaces or any other algebraic spaces. This difficulty worsens when the analysis of the problem is being sought. This is the thrust of this paper; to solve these Black-Scholes second order parabolic equation in Sobolev spaces. The great challenge in analyzing this stochastic PDE is due to the definitions, assumptions, theorems and proofs which are not easy to understand as to apply appropriately.

However, in the next study, we shall be looking at the applications of these weak solutions and its implication in stock market price variations for capital investments.

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