International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 10 Issue 11 November 2022, Page no. – 2961-2968

Index Copernicus ICV: 57.55, Impact Factor: 7.362

DOI: 10.47191/ijmcr/v10i11.03

A Classification of Surds of Non-Arithmetical Groups

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ARTICLE INFO

Published Online:
12 November 2022

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ABSTRACT

Definitions and classifications of reduced surds for non-arithmetical groups on the hyperbolic plane is proposed, as well as those of some congruence subgroups. The considered groups are the non-arithmetical desymmetrized group, its Γ0 congruence subgroup, and a generalized Hecke group according the choice of the commutator subgroup of the desymmetrized group. The definitions of the reduced surds are studied according to the ordering of the generators of the conjugacy subclasses of the considered groups. The construction of tori is defined as obtained after the tessellation of the Upper Hyperbolic Half Plane and discussed. Cutting trajectories and cutting-trajectories sequences are uniquely defined after the new definition of reduced surds.

KEYWORDS: PACS numbers: 02.20.-a Group theory mathematics; 05.45.-a Nonlinear dynamics and chaos; Algebra and Geometry.

I. INTRODUCTION

The relevance of studying the invariant measures for Hamiltonian systems is exposed in [25]. The outcomes of the study of geodesics flows, maps and invariant measures of Hamiltonian systems is explained in [26]. In particular, the symbolic dynamics allows one to encode the geodesics flow into the conjugacy subclasses of the chosen group (on the upper Poincar´e half-plane(UPHP)).

In the present paper, reduced surds of periodic geodesics are investigated for non-arithmetical groups; the results are compared with the definitions of reduced surds of arithmetical groups. The tori constructions are compared. The definition of cutting geodesics and that of cutting-trajectories sequences of a sequence of trajectories on (punctured) tori is defined uniquely.

The degeneracy of the length spectrum has long been investigated. The degeneracy of the length spectrum is proven to be logarithmic. The study of the degeneracy of the length spectrum is important in the classification of geodesics, and, in particular, in the classification of reduced geodesics.

More generally, there is a difference in the grow rates of the logarithm of the number of lengths and the number of periodic orbits. The degeneracy is therefore exponential, and such behavior is difficultly matched with the Randol-Horowitz theorem [15].

The definition of reduced surds crucially depends on the uniqueness of the definition of the solutions of the LaplaceBeltrami operator. The study of the uniqueness of geodesics can be juxtaposed with the study of the uniqueness of the geodesics flow generated after the Laplace-Beltrami operator on the UPHP. The Laplace-Beltrami operator does not admit non-trivial isospectral deformations [18]. In particular, it is possible to prove that the upper hyperbolic plane is spectrally rigid, i.e. in particular with respect to the Laplace-Beltrami operator, producing geodesics, see the references [18], [19]. Furthermore, in [19], the uniqueness of the eigenfuctions of the Laplace-Beltrami operator has been investigated, which results apply on the UPHP; such latter investigation is useful in the implementation of the quantum regime of the system as well as in that of the semiclassical one.

Pseudomodular reflection groups have been studied in [8].

Commutator Subgroups of Generalized Hecke groups and Extended Generalized Hecke groups have been studied in [9] and further generalized in [10]. In particular, the choices of the reflection on the non-degenerate circumference will be discussed and compared with the choice adopted in the present paper.

There exist finite-coarea Fuchsian subgroups of $PSL2(Q)$ which are not arithmetic and whose cusp set equals those of

the modular group [16], and are therefore called pseudomodular groups.
The definition of (punctured) tori and that of the pertinent cutting trajectories and that of the corresponding sequences of cutting trajectories has been indicated in [28] and [29] as a successful method in the investigation of the properties of surds. Tori for pseudomodular reflection groups have been studied in [23]. Fricke groups are defined as one-cusp tori. Infinitely many Fricke groups are not pseudomodular [23].
The issue to associate cutting trajectories with continued fractions was hinted in [6]. Cutting trajectories are not uniquely defined in [28] and [29] as ones starting a cutting sequence of a tori within the analysis of the tessellation of the hyperbolic plane.
After the pertinent analyses reported in the above, the construction of punctured tori can be envisaged. In [24], peculiar constructions of punctured tori for non-arithmetical groups are analyzed, and drawn in Fig (4.2) and Fig. (5.1) of [24].
After the definition of (punctured) tori, the analysis of cutting trajectories and those of sequences of cutting trajectories for tori can be looked for as indicated in [28] and [29]. The uniqueness of the definition of cutting trajectories is associate with the study of the surds. One of the aims of the present paper is to prove that the uniqueness of the definition of reduced surds leads to the definition of uniqueness of cutting trajectories and to that of sequences of cutting trajectories.
In the present paper, reduced surds are studied for non-arithmetical groups in hyperbolic space on the Upper Poincar´e Half Plane.
The study of geodesics for non-arithmetical groups requires one to code the dynamics as a sequence of (hyperbolic) reflections. Reduced surds are defined as those starting particular sequences of reflections, as it will be dealt with throughout all the paper. Non arithmetical groups, a particular subgroup of non- arithmetical groups and particular generalized Hecke groups will be studied as far as their dynamics is concerned.
The definitions of reduced surds for non-arithmetical groups are usefully compared with the definition of reduced surds for arithmetical groups [12] in [11].
In particular, the Hecke groups analyzed in this paper are those obtained after the particular of the choice of the reflection $z \to z' = 1/\bar{z}$. More in detail, this choice is apt for the purpose of discussion of the construction(s) of tori, cutting trajectories on the tori and cutting-trajectories sequences on the tori. The choice $z \to z' = 1/\bar{z}$ can be compared with the constructions in [1], [2], [3], [4], [5].
Furthermore, The construction of tori will be envisaged as the approrpiate tessellation(s) of the UPHP according to the non-arithmetical domains.
Cutting trajectories are defined in [28] and [29] as the starting geodesics of a cutting sequence of trajectories of a (punctured) torus; within the analysis of [28] and [29], cutting geodesics are not uniquely defined.
The definition of reduced surds, which here extends the definition in [11] of non-arithmetical groups, implies a unique definition of cutting trajectories also for non-arithmetical groups, as well as for the definition of cutting sequences of trajectories..
The paper is organized as follows.
In Section II, the Upper Poincar´e Half Plane is introduced. non-arithmetical desymmetrized triangle groups are introduced. The construction of one particular triangle congruence subgroup is accomplished. A peculiar implementation of the generalized Hecke groups is presented.
The definition and construction of periodic geodesics on the desymmetrized non-arithmetical triangular domain is reviewed in Subsection IIA.
In Section (III), desymmetrized non-arithmetical domain, its congruence subgroup Γ_0 and the generalized Hecke groupare constructed.
Complex-reflections-Hecke-groups constructions have been recapitulated in [20] (and the references therein), and compared with reflection groups, with braid groups, and with finite reductive groups.
In Subsection IIB, the definition of reduced surds for the desymmetrized non-arithmetical group is provided with. In Section VI, reduced surds for the non-arithmetical desymmetrized domain are defined.
In Section VII, reduced surds for the Γ_0 congruence subgroup of the non-arithmetical desymmetrized domain are defined.
In Section VIII, reduced surds for the generalized Hecke groups are defined.
In Section IX, the construction of tori is discussed; in particular, the production of punctured tori is explained.
In Section X, the definitions of cutting trajectories and that of cutting sequences of trajectories are given.
In Section XI, outlooks and perspectives are presented.
An appendix is devoted to recalling the properties of periodic geodesics in non-arithmetical billiards.

II. BASIC STATEMENTS

The UPHP is a Riemann space of constant negative curvature. Geodesics on the UPHP are semicircumferences centered on the abscissa axis u, and can be parameterized by means of the radius r and of the center u_0, i.e.

$$(u - u_0)^2 + v^2 = r^2, \qquad (1)$$

as well as by means of the endpoints. Oriented geodesics can be parameterized according to the oriented endpoints u^+ and u^- defined after the radius r and the center u_0 as

$$r = \frac{|u^+ - u^-|}{2} \qquad (2)$$

and

$$u_0 = \frac{u^+ + u^-}{2} \qquad (3)$$

. Degenerate geodesics are semicircumferences with one endpoint at infinity, i.e. u = *const* (’vertical’) lines on the UPHP.

Groups can be described on the UPHP after the choice of a domain on it, and reflections on the boundaries of the domain; reflections write as (geometric) reflections with respect to geodesics (and, in particular, also with respect to degenerate geodesics) on the UPHP which define the boundaries of the chosen domain. (Bigger) congruence (sub)groups can be constructed as well.

A. The desymmetrized non-arithmetical domain

The desymmetrized fundamental domain of a non-arithmetical group is chosen on the hyperbolic plane with a desymmetrized domain described by the sides

$$a: \quad u = 0, \tag{4a}$$
$$b: \quad u = -\cos\alpha, \tag{4b}$$
$$c: \quad u^2 + v^2 = 1, \tag{4c}$$

whose elements are generated after the reflections on the domain boundaries as

$$A \equiv T_1: \quad z \to -\bar{z}, \tag{5a}$$
$$B \equiv T_2: \quad z \to -\bar{z} + 2\cos\alpha, \tag{5b}$$
$$C \equiv T_3: \quad z \to -\frac{1}{\bar{z}}. \tag{5c}$$

Here, α is the angle between the a side and the goniometric circumference, $-\cos\alpha \equiv y$, with $y < 0$ (which is here set for future purposes). The non-arithmetic group is this way understood as the group of reflections on two degenerate geodesics (vertical lines) (4a) and (4b) and one reflection with respect to the (goniometric) circumference (4c) and is characterized by the generic angle α.

The desymmetrized domain of a non-arithmetical group is depicted in Fig. (1); it is also compared with the desymmetrized domain of the modular group. An oriented geodesics for the non-arithmetical desymmetrized domain is drawn in Fig. 2.

The desymmetrized non-arithmetical-groups domain is apt for constructing congruence subgroups.

B. Brief description of periodic orbits of the non-arithmetical desymmetrized group

Periodic orbits of the non-arithmetical desymmetrized group are described in [13] and the related Literature. Given the generators of the non-arithmetical group defined on the non-arithmetical triangle desymmetrized domain Eq.’s (4), the periodic orbits obey the relations

$A^2 = B^2 = C^2 = l$; $(AC)^2 = -l$, $(BC)^{k+2} = -l$.

Periodic orbits of non-arithmetical triangles are classified by the conjugacy subclasses of the generators of the group

There exists is no direct relationship between the number of reflections in the conjugacy subclass and the length of the periodic orbits.

According to the Huber’s law, the number of orbits shorter than or corresponding to a chosen length is known asymptotically. Further details are recalled in the Appendix.

III. NON-ARITHMETICAL SUBGROUPS

A. A congruence subgroup of the desymmetrized non-arithmetical group

The non-arithmetic desymmetrized domain on the Poincar´e Plane is described by the domain constitutes by the sides of a generic non-arithmetic triangle Eq.’s (4), and the non-arithmetical Γ_0 subgroup construction is defined after the reflections Eq.’s (5).

The Γ_0 congruence subgroup of the desymmetrized non-arithmetical group can be constructed on the domain

$$a: \quad u = 0, \tag{7a}$$
$$b: \quad u = -2\cos\alpha, \tag{7b}$$
$$c: \quad u^2 + v^2 = 1, \tag{7c}$$
$$d: \quad (u - cos\alpha)^2 + v^2 = 1. \tag{7d}$$

and its elements are the reflections with respect to the domain sides.

The Γ0 congruence subgroup Γ0 of the non-arithmetical desymmetrized domain is depicted in Fig. 3.

B. Generalized Hecke groups

Generalized Hecke groups are defined on the symmetric domain

In the present work, generalized Hecke groups endowed with the reflection $z \to -\frac{1}{z}$ will be considered.

The matrix elements of Hecke groups are algebraic integer numbers in the totally real field Q = $2\cos(\pi/n)$. The isomorphisms φk of this field are written as α such that

$$\phi_k : \alpha \to \alpha_k = 2\cos(\frac{\pi k}{n}) \qquad (9)$$

$\forall k$ odd integers, s.t. $k < n$, coprime with n.

For Hecke triangles α and $-\alpha$ are considered as corresponding to the same group. Differently, the number of such isomorphisms is equal to the degree of the defining polynomial; the transformation can be considered to lay in the group of isomorphisms (for n even), if it does not modify the lengths of the periodic orbit: the dimension of the group of isomorphisms of periodic orbit lengths, q, is $q = N$ for n even, and $q = N/2$ for n odd.

The traces of group matrices for Hecke triangular groups (as well as for other groups) are defined as integers of an algebraic field of finite degree $V\check{}$; to each group matrix M there correspond not only the length [13], but also a number q of different lengths corresponding to q different isomorphisms $V\check{}M$ of the basis field acting on the matrix. In the asymptotical limit,

$$lk = 2\ln|tr\varphi k(M)|, \qquad (10)$$

where lk corresponds to the considered transformation. Indeed, l_1 corresponds to the identity transformation and is the length of a periodic orbit solution to the Hamiltonian problem; $\forall lk$ s.t. $k \geq 2$ are named transformed lengths and are redundant with respect to the eigenfunctions of the Laplace-Beltrami operator in the associated Hamiltonian problem on such a space [27], [19].

IV. CLASSIFICATION OF GEODESICS FOR NON-ARITHMETICAL GROUPS

Different possibilities are available for the classification of reduced geodesics, whose oriented specification is the solution of the Hamiltonian problem associated to the domain of the non-arithmetical reflection group, and consisting of free Hamiltonian on the domain, while an infinite potential defines the expression of the Hamiltonian on the boundaries of the domain. The choice adopted in the present approach is the only possible one, which allows one to apply the Gauss-Kuzmin version associate to the related map as far as the time-evolution of the solution fr the Hamiltonian problem is concerned. In particular, the classification chosen applies to all the convergence subclasses of composition of the generators of the group, which allows one to express the time evolution of the oriented geodesics solutions to the associated Hamiltonian problem. It is interesting to outline that a classification of reduced surds is possible for non-arithmetical domains, such that the Gauss-Kuzmin theorem can be applied to one oriented endpoints, respectively.

V. CLASSIFICATION OF REDUCED SURDS OF NON-ARITHMETICAL GROUPS

Reduced surds are particular oriented geodesics. The orientation of the geodesics depends on the Hamiltonian flow of the Hamiltonian system considered.

For time-reversal-invariant systems, two different periodic orbits can have the same lengths, for example, if they are one the time-reversal-applied oriented geodesics of the other ($\bar{g} = 2$) [13].

Def.: A reduced surd of non-arithmetical groups is defined as the oriented trajectory corresponding to the first 2- letter word corresponding to the first two reflections within the two degenerate geodesics (delimiting the group domain) in the corresponding conjugacy subclass.

VI. REDUCED SURDS OF THE DESYMMETRIZED NON-ARITHMETICAL GROUP

Periodic orbits are therefore described through the classified reduced surds Eq.’s (5) as

$$z' = \prod_i (T_1 T_2)^{F(n_i)} T_3 z. \tag{11}$$

According to the known results, the expression (11) defines the generators order in the conjugacy subclasses (11) , and is is chosen for the comparison with the conjugacy subclasses (6) and for the comparison of the classification of geodesics with those of arithmetical groups (where the latter was performed in [11]).

Def.: Reduced surds of non-arithmetical groups are defined as the trajectories starting according to the first 2- letter word T_1T_2 (i.e. $F(ni) \equiv 1$) in the conjugacy subclass Eq. (11).

Remark It is important to remark that the definition of reduced surds for the desymmetrized non-arithmetical group correspond to the first oriented trajectory defined after the first 2-letter word *AB* in the conjugacy subclass (6).

Oriented geodesics of non-arithmetical groups can be parameterized by the oriented endpoints u^+ and u^-, i.e. by the semicircumferences Eq.’s 1, 3, 2.

Two subcases can be discussed, i.e. according to the choices of *i*) $-1/2 < -2\cos\alpha < 0$, and *ii*) $-1 < -2\cos\alpha < -1/2$.

In both cases, **Def:** reduced surds of the non-arithmetical desymmetrized triangular group are described as

$$2y < u^- < y, \tag{12a}$$

$$-\frac{1}{2y} < u^+ < \infty, \tag{12b}$$

with $0 > y \equiv -\cos\alpha$.

VII. REDUCED SURDS FOR A NON-ARITHMETICAL SYMMETRIZED CONGRUENCE SUBGROUP

It is possible to chose a non-arithmetical congruence subgroup Γ_2 of the desymmetrized group with a domain

$$a: \quad u = 0, \tag{13a}$$

$$b: \quad u = -2\cos\alpha, \tag{13b}$$

$$c: \quad u^2 + v^2 = 1, \tag{13c}$$

$$d: \quad v = \sqrt{2uu_A - u_A^2}. \tag{13d}$$

The Γ_2 congruence subgroup of the non-arithmetical desymmetrized domain is depicted in Fig. 4 and compared with the desymmetrized domain of the modular group.

Def: A reduced surd of the non-arithmetical symmetrized Γ_2 congruence subgroup of the non-arithmetical desymmetrized triangular group is defined as the oriented trajectory corresponding to the first 2-letter word corresponding to the first oriented trajectory defined between the generators corresponding to the reflections with respect to the two degenerate geodesics Eq. (13a) and Eq. (13b) (delimiting to the group domain) in Eq.’s (13) in the corresponding conjugacy subclass.

Def.: Reduced surd of the non-arithmetical symmetrized Γ_2 congruence subgroup of the non-arithmetical desymmetrized triangular group are described as

$$2y < u^- < y. \tag{14a}$$

$$-2y < u^+ < \infty. \tag{14b}$$

VIII. REDUCED SURDS OF GENERALIZED HECKE GROUPS

Non-arithmetical symmetrized groups can described on a symmetrized domain

$$a: \quad u = \cos\alpha, \tag{15a}$$

$$b: \quad u = -\cos\alpha, \tag{15b}$$

$$c: \quad u^2 + v^2 = 1 \tag{15c}$$

Def.: a reduced surd of non-arithmetical generalized Hecke groups is the geodesics corresponding to the oriented trajectory connecting the two degenerate (’vertical’) geodesics delimiting the group domain (i.e. the side Eq. (15a) and the side Eq. (15b) of Eq.’s (15)) which are the first 2-

letter word of the conjugacy subclass. **Def.**: reduced surds of non-arithmetical generalized Hecke groups are described as

$$2y < u^- < y \tag{16a}$$

$$-2y < u^+ < \infty \tag{16b}$$

IX. RESULTS ABOUT TORI

The construction of tori by means of a suitable congruence subgroup of (5) can also be discussed, and this procedure depends on the value *uA* in (4b) in (4). according to the symmetry

$$v^2 = (u_0 - 2u_A) - (u - u_0)^2, \quad u_0 = \frac{3u_A^2 - v_A^2}{2u_A} \tag{17a}$$

$$v^2 = (u_0 - 2u_A) - (u - u_0')^2, \quad u_0' = \frac{1}{2u_A}. \tag{17b}$$

Theorem 1 The tiling of any trajectory in the case *i*) describes a punctured torus.

Proof of Theorem 1 By construction.

Tori obtained for the case *i*) are depicted in Fig. 5 and compared with the tiling of the UPHP of the desymmetrized modular group.

Theorem 2 For the case *ii*), the construction of a Γ_2 subgroup already leads to a generalized torus, according to the symmetry lines indicated by the asymmetric domain, i.e. by the inversion of (5c) with respect to (17a). The folding of a trajectory leads to the definition of a torus.

Proof of Theorem 2 By construction.

The tori obtained for the case *ii*) are depicted in Fig. 6 and compared with the tiling of the UPHP of the desymmetrized modular group.

X. CUTTING TRAJECTORIES

According to the present study, it is possible to define cutting trajectories of non-arithmetic triangle groups domains. In particular, the problem raised in [28] and [29] is the determination of cutting sequences, as they are searched for in [28] and [29] as trajectories originating (i.e. starting) cutting sequences of the (tori) tessellation; within the presentation, these cutting trajectories are not uniquely defined.

A. Cutting trajectories- uniqueness

Within the present definitions of reduced surds, cutting trajectories are uniquely defined.

Def.: Cutting trajectories for these tori are allowed as initial conditions of the Hamiltonian problem associated, and are defined according to the value of the oriented endpoints on the intervals of the domain containing the absolute.

Cutting trajectories may be defined as those periodic orbits non-uniquely identified [28], for which the same tessellation of the hyperbolic plane follows [29], [30].

Within the present definition of reduced surds, differently, cutting trajectories are uniquely determined.

B. Cutting sequences of trajectories- uniqueness

Accordingly to the identification of reduced surds, the definition of cutting sequences of trajectories is given.

Def.: A cutting sequences of trajectories is defined as one whose starting trajectories is a reduced surd, i.e. is started by a reduced surd which is defined after the initial conditions associated of the Hamiltonian flow associated with the Laplace-Beltrami operator of the UPHP, which is spectrally rigid.

A cutting trajectory associated with its sequence of cutting trajectories therefore tessellates the UPHP uniquely, according to the defined (punctured) torus.

XI. OUTLOOK AND PERSPECTIVES

The aim of the paper is to define reduced surds of periodic orbits of particular non-arithmetical groups. More in detail, the desymmetrized non-arithmetical domain, the Γ_2 congruence subgroup and generalized Hecke groups have been analyzed, for which reduced surds have been defined.

The choice of the definition of the commutator subgroups of the generalized Hecke groups in [9] can be compared with the generalized Hecke groups analyzed in the present paper. In particular, the commutator subgroups of the generalized Hecke groups chosen in [9] is one containing the reflection *c* of Eq.'s (6).

In [10], larger domains for the generalized Hecke groups are studied. Domains consisting of an appropriate tiling of the UPHP according to the symmetric non-arithmetic domain are drawn in Fig. 5.

Reduced surds are defined as the geodesic corresponding to one particular tile of the tiling of the UPHP made of nonarithmetical groups. According to the definition chosen in the present paper, in [13], the particular tiling corresponds to the first 2-letter word *AB* in (6) of a particular congruence subgroup of the non-arithmetical groups. In the present paper, for the non-arithmetical groups here analyzed, a reduced surd is defined as the first oriented trajectory corresponding to the first 2-letter word corresponding to the reflections between the two degenerate geodesics ('vertical' lines on the UPHP) in the chosen conjugacy subclass; these definition are apt for comparison with the description in[13] and with the definitions in [11].

The relevance in defining the reduced surds of periodic orbits of non-arithmetical groups relies on the possibility to define uniquely the sequence of congruence subgroups needed for the tiling of the UPHP according to the periodic geodesic.

Three non-arithmetical groups are considered: the desymmetrized non-arithmetical group, its Γ_2 congruence subgroup, and its generalized Hecke group. The choice of the reflection $z \to z' = 1/\bar{z}$ is compared with the Demir groups [1]-[5] as a particular realization of the commutator (sub)groups.

Results about tori constructed after non-arithmetical groups and for non-arithmetical subgroups have been analyzed. More specifically, the tori originated after the tessellation of the UPHP according to the non-arithmetical desymmetrized triangular domain have been defined.

Cutting trajectories and cutting-trajectories sequences have been therefore uniquely defined, as interrogated from [28] and [29].

Appendix. The multiplicities of periodic orbits of the non-arithmetical desymmetrized group

The multiplicities of periodic orbits of the non-arithmetical desymmetrized group can be studied.

For the study of the non-asymptotic behavior, it is possible that a certain number of periodic orbits in the non-asymptotic scale are counted only once, being the law formulated for the asymptotic behavior.

A sequence of orbits is proven to be finite if and only if the fractional part of u^+ is rational.

A sequence is proven to be periodic if and only if the fractional part of u^+ is a quadratic irrational, i.e. the root of a quadratic equation with integer coefficients.

In [13], periodic geodesics orbits are classified according to the congruence subgruppal structures which originates by the generators needed for the reflections a geodesics to undergo the description of the periodic orbit in non-arithmetical triangles.

In [13], the multiplicities of periodic orbits of Hecke triangles is studied. In the case of the desymmetrized domain of a non-arithmetical triangle, (i.e. half (with respect to the v axis) the generalized Hecke triangle), it is demonstrated that the multiplicities are the same for each periodic orbit in the non-arithmetical desymmetrized triangle; to do so, for each sequence of generators defining the periodic orbit in the Hecke triangle, a reflection transformation can be inserted where needed within the sequence of transformations defining the orbit.

Numerical calculations are illustrated in [14].

The mean multiplicity of periodic orbits $\bar{g}(l)$ with a given (hyperbolic) length l, appear to be fittable as increasing exponentially according to the law

$$\bar{g}(l) \sim e^{\lambda l} \qquad (18)$$

with λ s.t. $\lambda < 1/2$.

A criterion is established for the definition of a lower bound of multiplicities, which applies analytically only in specific cases, as a confirmation of the exponential growth of multiplicities.

Acknowledgments

OML is grateful to Hermann Nicolai for having pointed out the relevance of Ref. [14].

OML acknowledges the Programme Education in Russian Federation for Foreign Nationals of the Ministry of Science and Higher Education of the Russian Federation.

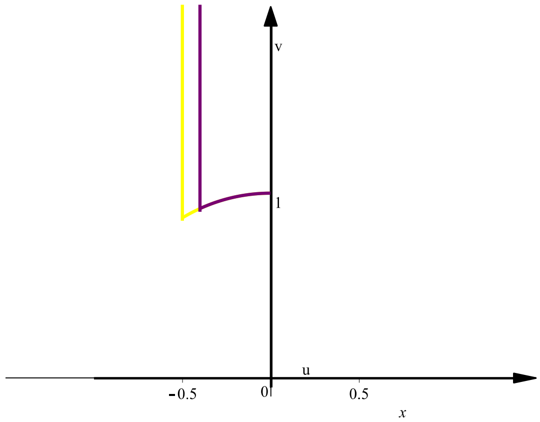

FIG. 1: The fundamental desymmetrized domain of the non-arithmetical group on the Upper Poincar´e Half Plane (purple, dashed) line is compared with the fundamental desymmetrized domain of the modular group (yellow, solid) line. The fundamental desymmetrized domain of the non-arithmetical Hecke group n the UPHP is defined after the degenerate geodesic $u = 0$, the degenerate geodesic $u = -2\cos\alpha$, and the geodesic $u^2 + v^2 = 1$.

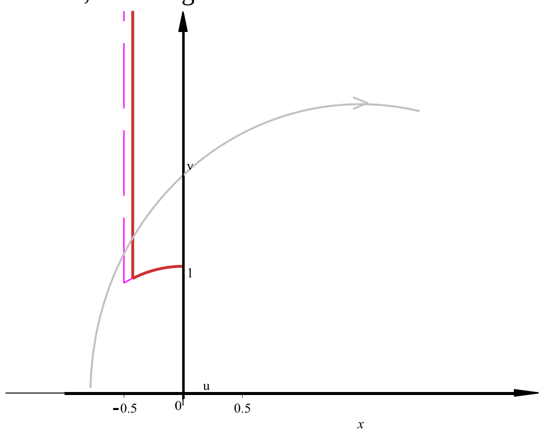

FIG. 2: An oriented geodesics (grey, thin line) of the non-arithmetical desymmetrized domain. The role of the fundamental desymmetrized domain of the non-arithmetical group (orange, solid line) on the Upper Poincar´e Half Plane is compared with that of the fundamental desymmetrized domain of the modular group (magenta, dashed line). The oriented geodesic here depicted is one corresponding to the *BA* codification of 6; in particular, it is a reduced surd.

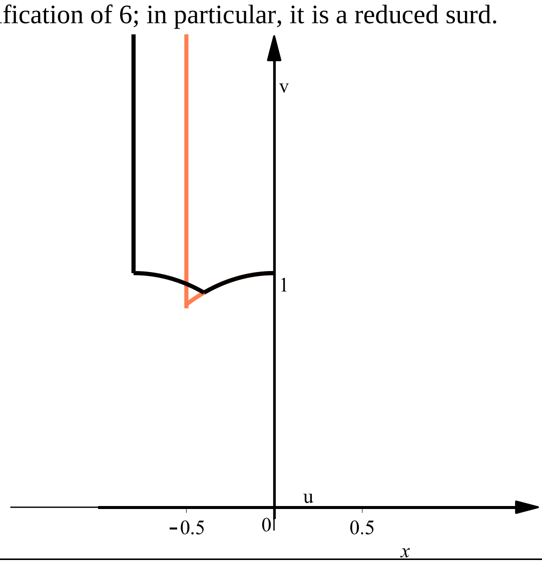

FIG. 3: The (symmetric) Γ0 congruence subgroup of a non-arithmetical desymmetrized domain (black line) is compared with the fundamental desymmetrized domain of the modular group (orange (gray) line). The (symmetric) Γ0 congruence subgroup of a non-arithmetical desymmetrized domain is defined after the degenerate geodesic $u = 0$, the degenerate geodesic $u = -4\cos\alpha$, the geodesic $u^2 + v^2 = 1$, and the geodesic $(u - \cos\alpha)^2 + v^2 = 1$.

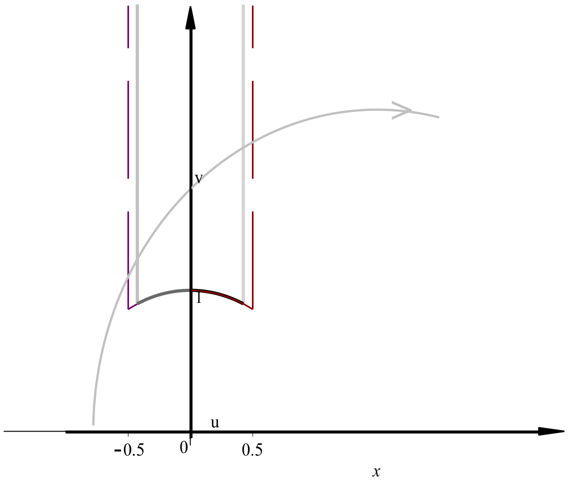

FIG. 4: The (symmetric) generalized Hecke group (solid line) is compared with the PSL2 subgroup of the desymmetrized fundamental domain of the modular group (dashed line). A geodesics of the type *BA* of Eq.’s (6) is depicted; in particular, it is a reduced surd.

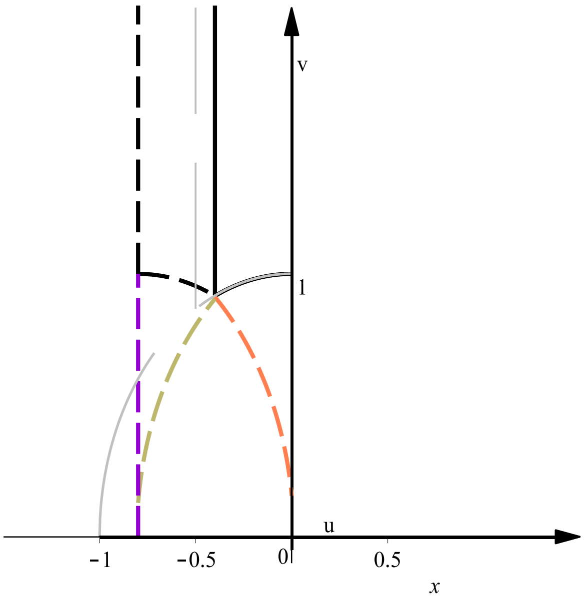

FIG. 5: A punctured torus constructed after the tiling for the case *i*). It is compared with the desymmetrized domain of the modular group, and with the geodesic $u^2 + v^2 = 1$ (gray, dashed lines).

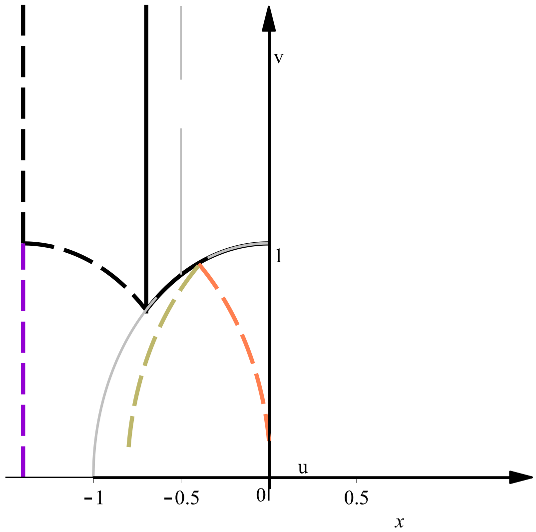

FIG. 6: A generalized torus constructed after the tiling for the case *ii*). It is compared with the desymmetrized domain of the modular group, and with the geodesic $u^2 + v^2 = 1$ (gray, dashed lines).

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