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The Concircular Curvature Tensor On Contact Metric Generalized (k,μ)-Space Forms

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Abstract: In this paper, we study ξ-concircularly flat and pseudo-concircularly flat 3-dimensional contact metric generalized (k,μ)-space form and such a space form with concircular curvature tensor $\mathcal{C}$ satisfying the condition $\mathcal{C}(\xi,X)\cdot S=0$, where S denotes the Ricci curvature tensor.

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1. Introduction

In 1995, Blair, Koufogiorgos and Papantoniou [6] introduced the notion of contact metric manifolds with characteristic vector field ξ belonging to the (k,μ)-nullity distribution and such type of manifolds are called (k,μ)-contact metric manifolds. They obtained several results and examples of such a manifold. A full classification of this manifold has been given by Boeckx [8]. A contact metric manifold (M,φ,ξ,η,g) is said to be a generalized (k,μ)-space if its curvature tensor tensor satisfies the condition

$$R(X,Y)\xi = k\{\eta(Y)X-\eta(X)Y\}+\mu\{\eta(Y)hX-\eta(X)hY\}, \tag{1.1}$$

for some smooth functions k and μ on M independent choice of vector fields X and Y. If k and μ are constant, the manifold is called a (k,μ)-space. If a (k,μ)-space M has constant φ-sectional curvature c and dimension greater than 3, the curvature tensor of this (k,μ)-space form is given by [13]

$$R=\frac{c+3}{4}R_1+\frac{c-1}{4}R_2+\left(\frac{c+3}{4}-k\right)R_3+R_4+\frac{1}{2}R_5+(1-\mu)R_6, \tag{1.2}$$

where R_1, R_2, R_3, R_4, R_5, R_6 are the tensors defined by

$$R_1(X,Y)Z=g(Y,Z)X-g(X,Z)Y,$$
$$R_2(X,Y)Z=g(X,\varphi Z)\varphi Y-g(Y,\varphi Z)\varphi X+2g(X,\varphi Y)\varphi Z,$$
$$R_3(X,Y)Z=\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi,$$
$$R_4(X,Y)Z=g(Y,Z)hX-g(X,Z)hY+g(hY,Z)X-g(hX,Z)Y,$$
$$R_5(X,Y)Z=g(hY,Z)hX-g(hX,Z)hY+g(\varphi hX,Z)\varphi hY-g(\varphi hY,Z)\varphi hX,$$
$$R_6(X,Y)Z=\eta(X)\eta(Z)hY-\eta(Y)\eta(Z)hX+g(hX,Z)\eta(Y)\xi-g(hY,Z)\eta(X)\xi,$$

for all vector fields X,Y,Z on M, where $2h=L_\xi\varphi$ and L is the usual Lie derivative.

The notion of generalized Sasakian-space-form was introduced and studied by P. Alegre, D. E. Blair and A. Carrizo [1] with several examples. A generalized Sasakian-space-form is an almost contact metric manifold

$(M, \varphi, \xi, \eta, g)$ whose curvature tensor is given by

$$R(X,Y)Z = f_1 R_1 + f_2 R_2 + R_3 f_3,$$

where R_1, R_2, R_3 are the tensors defined above and f_1, f_2, f_3 are differentiable functions on M. In such case we will write the manifold as $M(f_1, f_2, f_3)$. Generalized Sasakian-space-forms have been studied by several authors, viz., ([2, 3, 4, 10, 11, 12, 14]).

By motivating the works on generalized Sasakian-space forms and (k,μ)-space forms, A. Carriazo, V. M. Molina and M. M. Tripathi [9] introduced the concept of generalized (k,μ)-space forms. A generalized (k,μ)-space form is an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ whose curvature tensor R is given by

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3 + f_4 R_4 + f_5 R_5 + f_6 R_6, \tag{1.3}$$

where $R_1, R_2, R_3, R_4, R_5, R_6$ are the tensors defined above and $f_1, f_2, f_3, f_4, f_5, f_6$ are differentiable functions on M. Further, in [15], the authors had been studied the properties of 3-dimensional contact metric generalized (k,μ)-space forms. Also, the recurrent generalized (k,μ)-space forms was studied in the paper [16].

In a 3-dimensional contact metric generalized (k,μ)-space form $M^3(f_1, \ldots, f_6)$, the concircular curvature tensor C is defined by

$$\mathrm{C}(X,Y)Z = R(X,Y)Z - \frac{r}{6}[g(Y,Z)X - g(X,Z)Y], \tag{1.4}$$

for all vector fields $X, Y, Z \in M$, where R is the Riemannian curvature tensor. In [7], authors classify concircular curvature tensor on a $N(k)$-contact metric manifold. On the concircular curvature tensor of a (k,μ)-manifolds was studied by Tripathi et. al., in the paper [17].

The object of the paper is to study 3-dimensional contact metric generalized (k,μ)-space forms with concircular curvature tensor. The paper is organized as follows. Section 2 deals with some preliminaries on contact metric manifolds and contact metric generalized (k,μ)-space forms. The study of ξ-cincircularly flat and pseudo-concircularly flat 3-dimensional contact metric generalized (k,μ)-space forms is carried out in section 3 and section 4 respectively. In section 5, we characterized 3-dimensional contact metric generalized (k,μ)-space form satisfying the condition $\mathrm{C}(\xi, X)\cdot S = 0$.

2. Contact metric generalized (k,μ)-space forms

A contact manifold is a $C^\infty-(2n+1)$ manifold M equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Given a contact form η it is well known that there exists a unique vector field ξ, called the characteristic vector field of η, such that $\eta(\xi) = 1$ and $d\eta(X,\xi) = 0$ for every vector field X on M. A Riemannian metric is said to be associated metric if there exists a tensor field φ of type (1, 1) such that $d\eta(X,Y) = g(X, \varphi Y)$, $\eta(X) = g(X,\xi)$, $\varphi^2 X = -X + \eta(X)\xi$, $\varphi\xi = 0$, $\eta(\varphi X) = 0$ and $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$, for all vector fields X, Y on M. Then the structure (φ, ξ, η, g) on M is called a contact metric structure and then manifold M equipped with such a structure is called a contact metric manifold [5].

Given a contact metric manifold $(M, \varphi, \xi, \eta, g)$ we define a (1, 1) tensor field h by $2h = L_\xi \varphi$. Then h is symmetric and satisfies the following relations

$$h\xi = 0, \quad h\varphi = -\varphi h, \quad trace\ (h) = trace\ (\varphi h) = 0, \quad \eta \cdot h = 0. \tag{2.1}$$

Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds:

$$\nabla_X \xi = -\varphi X - \varphi h X.$$

The vector field ξ is a Killing vector with respect to g if and only if $h = 0$. A contact metric manifold

$(M, \varphi, \xi, \eta, g)$ for which ξ is a Killing vector is said to be a K-contact manifold. Therefore, a generalized (k, μ)-space form with such a structure is actually a generalized Sasakian space form.

A generalized (k, μ)-space form is an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ whose curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\} + f_4\{g(Y,Z)hX - g(X,Z)hY \\ &+ g(hY,Z)X - g(hX,Z)Y\} + f_5\{g(hY,Z)hX - g(hX,Z)hY \\ &+ g(\varphi hX,Z)\varphi hY - g(\varphi hY,Z)\varphi hX\} + f_6\{\eta(X)\eta(Z)hY \\ &- \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi\}, \end{aligned} \tag{2.2}$$

for all vector fields X, Y, Z on TM. Where $f_1, f_2, f_3, f_4, f_5, f_6$ are differentiable functions on TM. In such case we denote the manifold as $M(f_1, ..., f_6)$.

Next, by using the definitions of the tensors $R_1, R_2, R_3, R_4, R_5, R_6$ and properties (2.1) of the tensor h in the formula (1.3), we obtain that the curvature tensor of a generalized (k, μ)-space form satisfies

$$R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} + (f_4 - f_6)\{\eta(Y)hX - \eta(X)hY\}, \tag{2.3}$$

for every vector field X, Y on TM.

If $M^3(f_1, ..., f_6)$ is a contact metric generalized (k, μ)-space form, then its Ricci tensor S and the scalar curvature r can be written as [10]:

$$S(X,Y) = (2f_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + f_3)\eta(X)\eta(Y) + (f_4 - f_6)g(hX,Y), \tag{2.4}$$

$$r = 2(3f_1 + 3f_2 - 2f_3). \tag{2.5}$$

From (2.4), we have

$$S(X,\xi) = 2(f_1 - f_3)\eta(X). \tag{2.6}$$

Also, from (2.2) we obtain

$$R(\xi,Y)Z = (f_1 - f_3)\{g(Y,Z)\xi - \eta(Z)Y\} + (f_4 - f_6)\{g(hY,Z)\xi - \eta(Z)hY\}. \tag{2.7}$$

Definition 1. *A contact metric manifold M is said to be an η-Einstein manifold* [5] *if it satisfies*

$$S = ag + b\eta \otimes \eta$$

for some smooth functions a and b. In particular, if $b = 0$, then M is called an Einstein manifold.

3. ξ-concircularly flat 3-dimensional contact metric generalized (k, μ)-space forms

Definition 2. *A 3-dimensional contact metric generalized (k, μ)-space form is said to be ξ-concircularly flat if it satisfies*

$$\mathsf{C}(X,Y)\xi = 0, \tag{3.1}$$

for all vector fields X, Y on TM.

Let us assume that $M^3(f_1, ..., f_6)$ is ξ-concircularly flat 3-dimensional contact metric generalized (k, μ)-space form. In view of (3.1) and (1.4), we get the following

$$R(X,Y)\xi - \frac{r}{6}(\eta(Y)X - \eta(X)Y) = 0. \tag{3.2}$$

By virtue of (2.3) and (3.2), we get

$$\left[(f_1 - f_3) - \frac{r}{6}\right](X - \eta(X)\xi) + (f_4 - f_6)hX = 0. \tag{3.3}$$

Changing X by hX in (3.3), we have

$$\left[(f_1 - f_3) - \frac{r}{6}\right]hX + (f_4 - f_6)h^2X = 0. \tag{3.4}$$

Taking the trace of h on both sides of (3.4), we obtain

$$(f_4 - f_6)trace\ (h^2) = 0. \tag{3.5}$$

Since $trace\ (h^2) \neq 0$, we conclude that $f_4 - f_6 = 0$. Hence, this leads the following theorem:

Theorem 2. *If a 3-dimensional contact metric generalized* (k,μ) *-space form is* ξ *-concircularly flat then* $f_4 - f_6 = 0$ *holds.*

Putting $f_4 - f_6 = 0$ in (3.3), we get

$$\left[(f_1 - f_3) - \frac{r}{6}\right](X - \eta(X)\xi) = 0, \tag{3.6}$$

which implies that $\left[(f_1 - f_3) - \frac{r}{6}\right] = 0$ or equivalently,

$$r = 6(f_4 - f_6). \tag{3.7}$$

Comparing (3.7) with (2.5), we get

$$3f_2 + f_3 = 0. \tag{3.8}$$

Thus we have the following:

Theorem 3. *In a* 3 *-dimensional* ξ *-concircularly flat contact metric generalized* (k,μ) *-space form with* $f_4 - f_6 = 0$ *satisfies* $3f_2 + f_3 = 0$.

4. Pseudo-concircularly flat 3-dimensional contact metric generalized (k,μ) -space forms

Definition 3. *A* 3 *-dimensional contact metric generalized* (k,μ) *-space form is said to be pseudo-concircularly flat if it satisfies*

$$g(\mathrm{C}(\varphi X, Y)Z, \varphi W) = 0, \tag{4.1}$$

for all vector fields X, Y, Z, W on TM .

In view of (1.4) and (4.1), we have

$$g(R(\varphi X, Y)Z, \varphi W) = \frac{r}{6}[g(Y,Z)g(\varphi X, \varphi W) - g(\varphi X, Z)g(Y, \varphi W)] \tag{4.2}$$

Let $\{e_i\}$, $i = 1, 2, 3$ be an orthonormal basis for tangent space at each point of the manifold. Putting $Y = Z = e_i$ in (4.2) and taking the summation over i, we get

$$S(\varphi X, \varphi W) = \frac{r}{3}g(\varphi X, \varphi W). \tag{4.3}$$

Replacing X by φX and W by φW in (4.3) and using $\varphi^2 X = -X + \eta(X)\xi$ and (2.6), we obtain

$$S(X,W)=\frac{r}{3}g(X,W)+\left[2(f_1-f_3)-\frac{r}{3}\right]\eta(X)\eta(W). \tag{4.4}$$

Again putting $X=W=e_i$ in (4.4) and taking the summation over $1\le i\le 3$, we get

$$r=6(f_1-f_3). \tag{4.5}$$

By virtue of (4.4) and (4.5), we get

$$S(X,W)=2(f_1-f_3)g(X,W). \tag{4.6}$$

Therefore, form (4.6) it is clear that $M^3(f_1,\ldots,f_6)$ is an Einstein manifold. Thus we state the following:

Theorem 4. *A* 3*-dimensional pseudo-concircularly flat contact metric generalized* (k,μ)*-space form* $M^3(f_1,\ldots,f_6)$ *is an Einstein manifold.*

Next, comparing (4.5) with (2.5), we have the following relation

$$3f_2+f_3=0. \tag{4.7}$$

From (4.7) we can state the following:

Corollary 1. *A* 3*-dimensional contact metric generalized* (k,μ)*-space form* $M^3(f_1,\ldots,f_6)$ *is pseudo-concircularly flat if* $3f_2+f_3=0$.

5. 3-dimensional contact metric generalized (k,μ)-space form satisfying $\mathrm{C}(\xi,X)\cdot S=0$

Let $M^3(f_1,\ldots,f_6)$ is a contact metric generalized (k,μ)-space form satisfying the condition $\mathrm{C}(\xi,X)\cdot S=0$.

Therefore, $(\mathrm{C}(\xi,X)\cdot S)(Y,W)=0$ implies that

$$S(\mathrm{C}(\xi,X)Y,W)+S(Y,\mathrm{C}(\xi,X)W)=0. \tag{5.1}$$

Putting $X=\xi$ in (1.4) and then using (2.7), we get

$$\mathrm{C}(\xi,Y)Z=\left[(f_1-f_3)-\frac{r}{6}\right][g(Y,Z)\xi-\eta(Z)Y]+(f_4-f_6)[g(hY,Z)\xi-\eta(Z)hY]. \tag{5.2}$$

In view of (5.1) and (5.2), we get

$$\begin{aligned}&\left[(f_1-f_3)-\frac{r}{6}\right][g(X,Y)S(W,\xi)+g(X,W)S(Y,\xi)\\&-S(X,Y)\eta(W)-S(X,W)\eta(Y)]\\&+(f_4-f_6)[g(hX,Y)S(W,\xi)+g(hX,W)S(Y,\xi)\\&-S(hX,Y)\eta(W)-S(hX,W)\eta(Y)]=0.\end{aligned} \tag{5.3}$$

Using (2.4), (2.5) and (2.6) in (5.3), we have

$$\begin{aligned}&\left(\frac{3f_2+f_3}{3}\right)[(3f_2+f_3)\{g(X,Y)\eta(W)+g(X,W)\eta(Y)-2\eta(X)\eta(Y)\eta(W)\}\\&+(f_4-f_6)\{g(hX,Y)\eta(W)+g(hX,W)\eta(Y)\}]\\&-(f_4-f_6)[(3f_2+f_3)\{g(hX,Y)\eta(W)+g(hX,W)\eta(Y)\}\\&+(f_4-f_6)\{g(h^2X,Y)\eta(W)+g(h^2X,W)\eta(Y)\}]=0.\end{aligned} \tag{5.4}$$

Taking $W=\xi$ in (5.4), we get

$$\left(\frac{3f_2+f_3}{3}\right)[(3f_2+f_3)\{g(X,Y)-\eta(X)\eta(Y)\}+(f_4-f_6)g(hX,Y)]$$

$$-(f_4-f_6)[(3f_2+f_3)g(hX,Y)+(f_4-f_6)g(h^2X,Y)]=0. \qquad (5.5)$$

Using $h^2X=(k-1)\varphi^2X$ in (5.5) and simple computation leads the following

$$\left[\frac{(3f_2+f_3)^2}{3}+(k-1)(f_4-f_6)^2\right]\{g(X,Y)-\eta(X)\eta(Y)\}$$

$$-\frac{2(f_4-f_6)(3f_2+f_3)}{3}g(hX,Y)=0, \qquad (5.6)$$

or equivalently,

$$\left[\frac{(3f_2+f_3)^2}{3}+(k-1)(f_4-f_6)^2\right](X-\eta(X)\xi)-\frac{2(f_4-f_6)(3f_2+f_3)}{3}hX=0. \qquad (5.7)$$

Replacing X by hX in (5.7), we get

$$\left[\frac{(3f_2+f_3)^2}{3}+(k-1)(f_4-f_6)^2\right]hX-\frac{2(f_4-f_6)(3f_2+f_3)}{3}h^2X=0. \qquad (5.8)$$

Taking the trace of h on both sides of the relation (5.8), we get

$$\frac{2(f_4-f_6)(3f_2+f_3)}{3}trace(h^2)=0. \qquad (5.9)$$

As $trace(h^2)\neq 0$, we conclude that either $3f_2+f_3=0$ or $f_4-f_6=0$. Hence we can state the following theorem:

Theorem 5. *If a* 3 *-dimensional contact metric generalized* (k,μ) *-space form* $M^3(f_1,...,f_6)$ *satisfying* $C(\xi,X)\cdot S=0$, *then either* $f_4-f_6=0$ *or* $3f_2+f_3=0$.

From the above Theorem 5, we can state the following corollary:

Corollary 2. *If a* 3 *-dimensional contact metric generalized* $N(k)$ *-space form* $M^3(f_1,...,f_6)$ *satisfying* $C(\xi,X)\cdot S=0$, *then* $3f_2+f_3=0$.

References

[1] P. Alegre, D. E. Blair and A. Carriazo, *Generalized Sasakian-space-forms*, Israel J. Math., **141** (2004), 157-183.

[2] P. Alegre and A. Carriazo, *Structures on generalized Sasakian-space-forms*, Differential Geom. and its application **26** (2008), 656-666.

[3] P. Alegre and A. Carriazo, *Submanifolds of generalized Sasakian-space-forms*, Taiwanese J. Math., **13** (2009), 923-941.

[4] P. Alegre and A. Carriazo, *Generalized Sasakian space forms and conformal change of metric*, Results Math. **59** (2011), 485-493.

[5] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math., **509**, Springer-Verlag, 1976.

[6] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math., **19** (1995), 189-214.

[7] D. E. Blair, J. -S. Kim and M. M. Tripathi, *On the concircular curvature tensor of a contact metric manifold*, J. Korean Math. Soc., **42** (2005), No. 5, pp. 883-892.

[8] E. Boeckx, *A full classification of contact metric* (k,μ) *-spaces*, Illinois J. Math., **44** (2000), 212-219.

[9] A. Carriazo, V .M. Molina and M. M. Tripathi, *Generalized* (k,μ) *-Space Forms*, Mediterr.J. Math., DOI 10.1007/s00009-012-0196-2.

[10] U. C. De and A. Sarkar, *Some results on generalized Sasakian-space-forms*, Thai J. Math. **8**(1) (2010), 1-10.
[11] U. C. De and A. Sarkar, *On the projective curvature tensor of generalized Sasakian-space-forms*, Quaestiones Mathematicae, **33**(2) (2010), 245-252.
[12] U. K. Kim, *Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms*, Note di matemetica **26** (2006), 55-67.
[13] T. Koufogiorgos, *Contact Riemannian manifolds with constant* φ *-sectional curvature*, Tokyo J. Math., **20** (1997), 55-67.
[14] D. G. Prakasha, *On generalized Sasakian-space-forms with Weyl-conformal curvature tensor*, Lobacheviskii J. Math., **33** (3) (2012), 223-228.
[15] D. G. Prakasha, S. K. Hui and K. K. Mirji, *On 3-Dimensional Contact Metric Generalized* (k,μ) *-Space Forms*, Int. J. Math. Math. Sci., Volume **2014**, Article ID 797162, 6 pages.
[16] C. R. Premalatha and H. G. Nagaraja, *Recurrent generalized* (k,μ) *–space forms*, Acta Universitatis Apluensis, No. **38**/2014, pp. 95-108.
[17] M. M. Tripathi and J. -S. Kim, *On the concircular curvature tensor of a* (k,μ) *-manifold*, Balkan J. Geom. Appl., **9**, No. 5, (2004), pp. 104-114.