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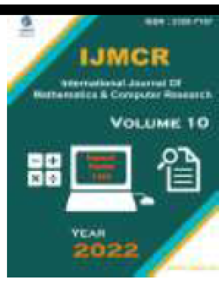

Selected topics in Cohomology of non-arithmetical groups (algebras) and open questions

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Abstract
Selected topic in the cohomology of non-arithmetical groups algebras and grouppal structures are chosen and reviewed. The basics formalisms are recalled. The comparisons between the different approaches are proposed, from which new open questions arise; in particular, the application of the method of satellites to the study of non-arithmetical grouppal structures is evidentiated to request the definition of new structures in the calculation of the irreducible characters.

I. INTRODUCTION

Hecke grouppal structure constitute a very broad domain of investigation, whose peculiar features have left room for examination.
Among the possible guidelines of investigation, congruence relations of similar types [1] constitute a robust starting point for the analyses here proposed.
Automorphic forms are a very useful tool for the analysis of the symmetry groups of operators (i.e. corresponding to a (sub-)grouppal structure) on a suitably-chosen domain the Hyperbolic Plane, and, in particular, of the Upper Poincaré Half Plane (UPHP), as their definition is in one-to one correspondence to the definition of geodesics through the analysis of the corresponding Hamiltonian problem (defined as a free motion on the domain, delimited by infinite Hamiltonian potential walls) via the eigenfunction of the Laplacian operator (automorphic forms), which ensure the the well-posed-ness of the Hamiltonian problem with respect to the associated grouppal structures.

It is sometimes customary to replace the automorphic-form associated structures with the corresponding co-homology groups. The corresponding procedure implies a loss of information about the cople structure and about the ring structure of the automorphic forms. Via the algebra of the algebras associated to the chosen operators, it is possible to retrieve some interesting types of information.
The interest of this investigation is that originated by (desymmmetrized) non-arithmetic triangles.
Two grouppal structures can be associated to desymmetrized non-arithmetic triangles are the Hecke operators corresponding to one symmetrization method of the desymmetrized non-arithmetical triangular domains. The other kind of symmetrization of the desymmetrized Hamiltonian domains is the construction of the domain related constructed according to the techniques proper of the *reflection* group $PGL(2, \mathbb{C})$, (as an extension to the group $PGL(2, \mathbb{Z})$, as far as the content of reflections is concerned).
The difference between the two symmetrization procedures for the domains is the number of cusps of the corresponding symmetrized domain (and the possible congruence 'bigger' subgroups).
The rigidity of the Laplace-Beltrami operator on the Upper Poincaré half plane provides one with a useful tool to investigate about the geodesics.
The free Hamiltonian on the UPHP admits as solution the geodesics of the UPHP; automorphic forms are the eigenfunctions of the Laplace-Beltrami operator. The consideration of domains with cusps on which the Hamiltonian and the Palace-Beltrami operator are applied leads one to the discovery of new structures. The application of the

methods of satellites to the grouppal structures used for the calculation of the irreducible characters is evidentiated to raise new problems.
The paper is organized as follows.
In Section II, some features of the Hecke grouppal structures are recapitulated.
In Section III, congruence conditions of non-arithmetic triangles and the Fourier coefficients are recalled.
In Section IV, the peculiarities of triangle groups useful for the investigation are reviewed.
In Section V, the definitions of traces are recollected.
In Section VI, the method of satellites are summarized.
Outlook and Perspectives VII comprehend the steps in which the consideration of the methods of satellites introduce open questions about the steps in defining the traces.
Appendix A is aimed at laying down the notation.
Appendix B is devoted to particular features of the analysis of the integrability properties of triangle groups.
Appendix C is dedicated to reporting the steps which do not hold any more in the definition of traces after the consideration of the methods of satellites.

II. SOME FEATURES OF THE HECKE GROUPPAL STRUCTURES

Stable classes can be interpreted as fixed points for an action of the Hecke algebra.
Hecke operators act on certain kinds of cohomology groups. Canonical isomorphisms among the cohomology groups can be constructed; the compatibilities of the procedures are discussed in [2].
The action of the Hecke operators can be interpreted as transfer maps of the cohomology of groups.
Fixed points can therefore be interpreted as common eignenvectors [3].
The eigenfunction of non-arithmetical groups have been investigated, i.e. in [4] and the Reference therein, and, in particular, those of non-arithmetic triangle groups with one cusp.

III. CONGRUENCE CONDITIONS OF NON-ARITHMETIC TRIANGLES AND THE FOURIER COEFFICIENTS

The problem of non-arithmetic triangles with a cusp has been analysed, among other papers, in [4].
The problem is approached after the viewpoint of congruence condition as expressed form the order of the generators of the group. As an example, a particular Hecke triangle group is analyzed.
Two main difficulties arise within this construction:

i) The lack of Hecke theory for them, i.e. also as far as the rationality of coefficients is concerned
by considering the modular group $PSL(2,\mathbb{Z}) \equiv \Gamma(1)$, among the modular forms of odd functions for $\Gamma(1)$, $\exists$ a natural infinite class of Fuchsian triangular groups; and
ii) whether the integrability has distinguished enumerative properties.

In [5], several problems in mapping groups are revised, which nicely complete this overview, even though they are not among the focuses of the investigation.
As hinted in [6] about the open questions about mapping class groups as far as congruence subgroups are concerned within the Hecke groups algebras investigations, the approach of [7] can be followed. In particular, as far as the investigation of Hecke structures is concerned, it is possible to show that [7] the homology corresponds to the kernel of the boundary map.

Characteristic classes of surface bundles can be defined, which are smooth fibre bundles whose fibres are some closed orientable surface [8] (for further developments, see [9]).
In [10], the properties of $GL(n,k)$ are investigated, which would be used in the understanding of the further development of the technical nomenclature in the below.

IV. TRIANGLE GROUPS AND THE HAUPTMODUL

The expansion of automorphic forms of triangular groups can be obtained by considering particular classes of primes [4]. Because of the lack of Hecke theory, it is possible to study the hypergeometric functions F, $F\ln(z)+G$ up to some Γ factors within the automorphic forms for Fuchsian groups (Eisenstein-Theta series)
with the period of

$$C_z^{a,b,c} \quad s.t. \quad y = x^a(x-1)^b(x-z)^c. \tag{1}$$

It is relevant to analyse that he action of Hecke operators on hypergeometric functions has nevertheless been studied in [11]; more in detail, as a result, the action of Hecke operators on the set of all hypergeometric functions that vanish to the appropriate order at the origin has been defined. Furthermore, 'quasi'-automorphic forms for triangle groups have been given a generalization in [12].
In [4], triangle groups with one cusp are investigated. The investigation in [4] and [12] is based on the study of automorphic forms for Fuchsian groups; many items of analysis which hold in the case of arithmetical groups do not hold in the case of non-arithmetical groups (and, in particular the explicit q-expansion).
The details of the investigations in [12] and [4] are reported in Appendix B.

The algebra of automorphic forms for the group Γ_t s.t. $m_1 < m_2 < \infty$ is generated by [12], [4]

$$E_{2k}^{(1)}, \quad 3 \leq k \leq m_1, \tag{2a}$$

$$E_{2k}^{(2)}, \quad 1 \leq k \leq m_2; \tag{2b}$$

such that the system Eq.'s (B1) becomes

$$E_{2k}^{(1)} = (t_1 - t_2)(t_3 - t_2)^{k-1} \in 1 + q\mathbb{Q}[[q]], \tag{3a}$$

$$E_{2k}^{(2)} = (t_1 - t_2)^{k-1}(t_3 - t_2) \in 1 + q\mathbb{Q}[[q]]; \tag{3b}$$

the further two casesare investigated in [12].

a. Fuchsian theta-series (Eisenstein series) Fuchsian theta-series correspond to q-expansions for hyperbolic triangle groups.
For the modular grouppal structure $PSL(2,\mathbb{Z})$, among the modular forms and the functions for $\Gamma(1)$ there exists a natural infinite class of Fuchsian groups for which automorphic forms can be determined explicitly [12].

As from [12], [4] a Hecke absolute invariant J_q can be rewritten as

$$J_q = \frac{t_3 - t_2}{t_3 - t1} \tag{4}$$

For $J = 1$, $z(k \cdot q)$, with $k = -m_1^2 m_2^2$, being $z(q)$ the inverse of q (from J_q) as a function of z.
$J_q(z)$ can be defined as a generator.
As from [13], one poses

$$\rho \equiv -exp^{-\sqrt{-1}\pi/q}; \tag{5}$$

$J_q(z)$ is specified for ρ as

$$J_q(\rho) = 0, \tag{6a}$$

$$J_q(\sqrt{-1}) = 1, \tag{6b}$$

$$J_q(\sqrt{-1}\infty) = \infty. \tag{6c}$$

such that $J_q(z)$ is the Hecke absolute invariant.
The following Fourier expansion [14] holds

$$J_q(z) = \sum_{n \geq -1} a_q(n) r_q^n exp^{\sqrt{-1}\pi n z/\lambda_q} \tag{7}$$

with $\lambda_q = 2\cos(\pi/q)$, $a_q(n) \in \mathbb{Q}$, and $r_q \in \mathbb{R}$ is determined up to a rational factor. The coefficient a_q can be written as a rational function of q [13].
An explicit analytic invariant can be built after the use of r_q [15].

b. A theorem about the Hauptmoduls Non-arithmetical triangle groups have been proposed [4] to be classified by classifying all the prime qualifying the Hauptmodul J.
Theorem 1: Be $m_1 \leq m_2 \in N$ and be p be a prime s.t. $p > 2m_1m_2$. The Hauptmodul J is $J \equiv \frac{t_3-t_2}{t_3-t_1}$, for the triangle group of type $(m1, m2, \infty)$ is p-integral if and only if for $m_1 < m_2 \in \mathbb{N}$, p prime, $p > m_1m_2$ with $\epsilon = \pm 1$, $\epsilon' = \pm 1$,

$$p \equiv {}^{2m_1}\epsilon, \quad p \equiv {}^{2m_2}\epsilon'\epsilon, \tag{8a}$$

$$p \equiv {}^{2m_1}m_1 + \epsilon, \quad p \equiv {}^{2m_2}m_2 + \epsilon'\epsilon. \tag{8b}$$

□

.

c. Different examples Different examples as far the number of cusps of the grouppal structure is concerned are here reported for the sake of comparison with the one-cusp triangle structures.
As a different example, for the triangle group (m, ∞, ∞) and $p > 2m$ the Hauptmodul J is p-integral if and only if

$$p \equiv {}^{2m} \pm 1. \tag{9}$$

The condition $p > 2m_1m_2$ implies the 'if' statement of the theorem, while the 'only if' part is implied only after p to not divide $2m_1m_2$.
As a corollary, it is interesting to consider the following case. Be $p > 2m_1m_2$ a prime number; then, the algebra of automorphic forms of the triangle (group) of type (m_1, m_2, ∞) is p-integral iff p conditions of Theorem 1 hold.
The study of the integrality problem for the coefficients of modular forms of noncongruence subgroups of $\Gamma(1) = SL(2, \mathbb{Z})(= \gamma(2, 3, \infty))$ was performed in [16].

d. More about the Fourier coefficients In [16], the existence is proven of positive integers d and N such that $d^n a_n \in \mathbb{O}_F[1/N]$, with a_n the n-th Fourier coefficient of a modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ from some subgroup of $\Gamma(1)$, and F a number field.The case $N = 1$ is conjectured if and only if the subgroup contains a congruence subgroup in [17]. The result of [16] implies that at most finitely many distinct primes can appear in the denominators of modular forms for a noncongruence subgroup of $\Gamma(1)$. On the other hand, when the group is not commensurable with $\Gamma(1)$, one would expect infinitely many primes in the denominators. This prediction is compatible with the result in the case of hyberpolic triangle groups.

Differential equations associated with nonarithmetic Fuchsian groups are described in [18]; globally nihilpotent differential operators of rank 2 [18] defined over a number field whose monodromy group is a nonarithmetic Fuchsian group are described. These are counterexamples to conjectures by Chudnovsky-Chudnovsky and Dwork.

V. TRACES

The weights of Markov traces for Hecke algebras have been formulated in [19] by the methods of reduction and by the methods of inclusion.
In particular,
i) by defining an algebra of type B on a reduced algebra of the A type; and
ii) for D algebra, by inclusion of an algebra of the type D in an algebra of the type B.
The need of definition of algebras of type B for Hecke groups is that it is semisimple for any non-rational factor $Q = -qk, k \in [0, \pm 1, ..., \pm(n-1)]$, and q is not a root of unity (as for symmetric non-arithmetical groups).

In [20], the Farrell-Tate cohomology is applied to discrete groups; it expresses the cohomology only with the group structure with a non-trivial bundle.

In particular, also for $PSL(\mathbb{O})$, the cohomology is determined by the numbers of conjugacy subclasses rather than by the geometry of the underlying manifold. The formulae for the numbers of conjugacy classes of finite subgroups will not be applied here, as the fiber is assumed Ricci-flat[1].

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[1] Nevertheless, in the case of an A_n^1 Hecke algebra, in [7] a topological construction of representations of the series of the Hecke algebra is given in terms of the monodromy representation procured from a vector bundle on which there a naturally-flat connection is defined.

a. Remark It is nevertheless demonstrated [16] that there exists subgroups, which are not congruence subgroups of $SL(2,\mathbb{Z})$, for which the Hecke operators do not exist.
It is therefore necessary to investigate the problem from different guidelines.
It is nevertheless possible to construct series of the Hecke algebras, as demonstrated only in one particular case in [7], in terms of peculiar monodromy representations.
The application of some cohomogy groups to asymmetric groups can be achieved after the methods of satellites [21].

VI. THE SATELLITES

As in [21], more in detail, the considered cohomology groups are $Hn(G;M)$, $n \in \mathbb{Z}$, for an arbitrary group G and G-module M, and can be applied to asymmetric groups after using the methods of satellites. The considered cohomology groups are able to generalize the Farrell-Tate groups for groups of 'finite virtual cohomological dimension'; they are able to form a 'connected sequence of functors', which are specified after a natural universal property, i.e. Property: the family of H^0 is able to form a sequence of functors based on the concept of completion wrt to the projective modules. For Abelian groups, it is possible to demonstrate that the consecutive application of to homomorphism is is zero, for which a complete family of functors is formed. For arbitrary groups, the connected sequence of additive functors if any sequence associated with any sequence of functors is exact.

a. Irreducible characters The method of satellites can be therefore applied to the so-called Markov traces and knot invariants [22] related to Iwahori-Hecke algebras of type B.
Indeed, studies have been conducted for the cases in which reduced expressions of the representatives of minimal length in the conjugacy classes involve some of the inverses.

It is worth recalling the following Definition and the following Theorem.
Definition 1: Be $z \in A$ and $\tau : H \to$ an A-linear map. τ is then named a Markov trace (with parameter z) if the following conditions are satisfied: 1) τ is a trace function on H;
2) τ is normalized as $\tau(1) = 1$;
3) $\tau(hg_n) = z\tau(h) \forall n \geq 1$ and $h \in H_n$. □
All the generators g_i for $i = 1, 2, ...$ are conjugate in H. More in detail, all of the trace functions on H must have the same value on the considered elements: for this, the parameter z is does not depend on n in 3) in the above.
Theorem 2: Be $z, y_1, y_2, ... \in A$. Therefore $\exists!$ a Markov trace τ on H whose parameter is parameter z s.t.
$\tau(t'_0 t'_1 t'_2 ... t'_{k-1}) = y_k \ \forall k \geq 1$. □
It is possible to use the implications of the Theorem 2 and of the Definition 1 in the above to provide with a full classification of Markov traces of Iwahori-Hecke algebras of the type D.
After these implementations, it is possible to define the wanted traces when the generators obey particular quadratic relations [23].

From a different viewpoint, it is possible to analyse the group without any straightforward link to the matrix realization [24]. Indeed, it is possible to point out that the Hecke groups are groups generated after simple reflections. More in particular, it is possible to consider W a Weyl group and S the set of simple reflections.
The trace of the Hecke algebra defines the irreducible characters.
The values of irreducible characters ϕ_j is constants on the basis elements T_w (w denoting the conjugacy subclass). By means of this expression, it is possible to achieve the computation on another basis.
To do so, it is necessary to spell out the subgrouppal structure underlying the procedure; for this, it is computed by construction that either only the following composition equalities can hold:

$$T_s^{-1} T_w T_s = T_{sws}, \tag{10a}$$
$$T_s T_w T_s^{-1} = T_{sws}, \tag{10b}$$
$$T_s T_w T_s^{-1} = q_s T_{sws} + (q_{s-1}) T_{ws}. \tag{10c}$$

After the opportune considerations, it is possible to see that, given C the conjugacy classes of W, there exist polynomials $f_{w,C} \in \mathbb{Z}$ such that

$$\phi_j(T_w) = \sum_C f_{w,C}(T_{w_C}), \quad \forall j \tag{11}$$

with w_C some element in C_{min}.
The following theorem holds (and helps one to see this):
Theorem 3: Given C a conjugacy class of W,
1) for each w in C $\exists$ $w' \in C_{min}$ such that $w \to w'$; and
2) if $w, w' \in C_{min}$, then $w \sim w'$. $\square$
With the tools of the irreducible characters, i.e. the traces, it is possible to further analyse the properties of the Hecke algebra of type B.
In particular, it is possible to demonstrate [19] that $\exists$ an homeomorphism between the Hecke algebra of type B and the Hecke algebra of type A.

b. Remark It is therefore necessary to control how the procedures in [21] affect the generators.
Indeed, after applying the analysis of [21], one has to control what happens to the following aspects.
From the analysis of [22], one had that he generators g_i are (also) invertible in H.
From [22], the following quadratic relations hold among the generators s_i, t in $B_{1,n}$, with the fixed parameter Q from the ground ring over which the algebra is defined:

$$t^2 = (Q-1)t + Q \cdot 1, \tag{12a}$$

$$g_i^2 = (q-1)g_i + q \cdot 1, \tag{12b}$$

where the last equality is for g_i the images of the s_i in H, and the corresponding parameter q.
The algebra H_n is a finite-dimensional one, endowed with the basis $\{g_w\}$, which is labelled after the elementsof W_n.
The elements for studying the trace functions τ are this way set as $\tau(hg_n) = z\tau(h)$, with $h \in H_n$, and z the corresponding fixed parameter.

Indeed, in [23], from the quadratic relations, one had the following equality, as the fact that q was invertible in A implied that the generators g_i are also invertible in H, as

$$g_i^{-1} = q^{-1}g_i + (q^{-1} - 1) \cdot 1 \in H_n. \tag{13}$$

Using the inversion formulae, one had the linear combinations

$$g_w = q^{n_w} r'_1 ... r'_n + l.c. \tag{14}$$

$l.c$ being a linear combination of elements g_v with $l(v) < l(w)$.
The further details are spelled in Appendix C, which should be reconsidered after the Perspectives VII.

VII. OUTLOOK AND PERSPECTIVES

a. Outlook Two main questions arise:
1) as already pointed out, [16] there exists subgroups, which are not congruence subgroups of $SL(2, \mathbb{Z})$, for which the Hecke operators do not exist (except for particular cases as one in [7]), after which the methods of satellites is of relevance in the investigation of the Hecke grouppal structures;
2) after the application of the methods of satellites, the generators g_i in [22] must be controlled to be (also) invertible in H: the elements Eq. (C1) will not form a basis any more. More in detail, the steps which follow Proposition (3.2) from [22] which where used for the definition of traces will not hold any more; they are recalled in the Appendix C.

b. Perspectives The Eisenstein series of $GL(3, \mathbb{Z})$ have been investigated in [25].
Non-arithmetical groups have been approached also as in [26]. The construction of non-arithmetical polyciclic groups is indicated as the semidirect product of a free Abelian group of rank $n > 2$ and an infinite cyclic group.
Polycyclic groups are demonstrated to operate on polynomials which contain also simple traslations [27].
As far as cohomology hypergroup manifold and
bounded classes in the cohomology of manifolds are concerned,
as a hint of investigation, it is useful to consider the action of Hecke groups on Tate cohomology.
For this, the results hold, that [28] the Sato-Tate distribution and the Gaussian distribution are descriptions of the distributions of the Hecke eigenvalues. In [29], the distribution of the eigenvalues of the Hecke operators on Hecke eigenforms in the interval $[-2, 2]$ is studied.
In [30] In [32], selected peculiarities of Hecke eigenforms have been pointed out.
In [31], numerical investigation of the value distribution and distribution of Fourier coefficient for arithmetical Fuchsian groups and non-arithmetical ones is presented.
Further applications of the material in the above can be found after the asymmetric twin representation [33], in which the Hecke grouppal structures are involved.

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Appendix A: SPELLING OF THE INTRODUCTORY TECHNICALITIES

In the present section, some technical analyses and definitions are recalled, for the purpose of the further analyses. Most definitions are worked out from [34] and [35].

The analyses of Hecke operators offer a pertinent way to analyze the analytic continuations of L functions associated to modular forms by the method of θ functions [34][2]. Theta operators can be defined after the analytical continuation of th Eisenstein series. In particular, Hecke operators will be framed within the $GL(2,\mathbb{Z})$ classification. Scalar matrices A cancel from $[Y[A]^0]$ in the $m-th$ Θ operator T_m defined as for generalized Θ operators in the analysis of the automorphic forms for $GL(n,\mathbb{Z})$ in the harmonic analysis on $\mathcal{P}_n/GL(n,\mathbb{Z})$ $(Y\in\mathcal{P}_n)$.
A possible definition for the Hecke operators in the non-holomorphic case is one defined as a Maass waveform.
Definition A1: For a Maass waveform $f\in\mathbb{N}(SL(2,\mathbb{Z}), r(r-1))$, $n=1,2,3,...$ the Hecke operator T_n in the non-holomorphic case is defined as

$$T_n f(z)\equiv n^{-1/2}\sum_{ad=n,\quad d>0,\quad b\ mod\quad d} f\left(\frac{az+d}{b}\right) \tag{A1}$$

where the normalizing factor $n^{-1/2}$ is not consistent with the formula (3.88) in [34], as the Maass waveforms have 0 weight , but it has become standard in the literature. □
A slightly different normalization, however, can be used for Hecke operators for the use of the Siegel modular group $SP(n,\mathbb{Z})$, i.e. by avoiding multiplication by the factor $n^{-1/2}$.
From the Bochner-Hecke formula pp. 108-109 in [34], one has the result about the hyergeometric functions

$$\int_{-1}^{+1} exp^{-2\pi i r t s}P_n(t)dt=(-i)^n(rs)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(2\pi rs) \tag{A2}$$

with the definition $P_n(t)=(2^n n!)^{-1}\frac{d^n}{dx^n}(x^2-1)^n$.
Similar results hold for $GL(n)$.
Whittacker functions are confluent hypergeometric functions; the Fourier expansion of automorphic forms can also be performed as sums of Whittacker functions (of similar argument) ([34] p.60).
The Helgason transform for P_n is listed in [35] p. 90 and commented on p.248.
The application of the Helgason transform will be useful in the study of the Fourier expansions of Eisenstein series for the congruence subgroups analysed in the below.
The Fourier expansion of the Eisenstein series has many applications in number theory. As an example, the Fourier expansion of the Epstein's zeta function contains still several aspects under investigation.
The Fourier expansion of automorphic forms for general discrete groups acting on symmetric spaces allows one to describe fundamental domains with cusps. The Siegel modular form acting on a symmetric space is obtained after the Fourier expansion of holomorphic Eisenstein series.
Further applications are found in the study of the Gauss sums and in that of elliptic curves.
In the non-holomorphic case, the non arithmetical part of the Fourier decomposition of the Eisenstein series consists of a matrix analogue of a type of confluent hypergeometric functions.
The Fourier exponents of the Eisenstein series is defined as ([34] p. 208 and following) after

$$\Lambda(s)=\pi^{-s}\Gamma(s)\zeta(2s) \tag{A3}$$

as

$$E_s^*(z)=\Lambda_s E_s(z). \tag{A4}$$

[2] It is convenient, for the sake of easy notation, to distinguish between θ-functions and Θ operators.

$E^*_s(z)$ admits the following Fourier expansion

$$E^*_s(z) = y^s\Lambda(s) + y^{1-s}\Lambda(1-s) + 2\sum_{n\neq 0} \mid n \mid^{s-1/2} \sigma_{1-2s}(n) y^{1/2} K_{s-1/2}(2\pi \mid n \mid y) exp^{2\pi i n x} \tag{A5}$$

with $\sigma(n)$ the divisor function.
The poles cancel at $s = 1/2$.
After the Mellin transform of $E^*_s(z)$,

$$E^*_s(z) = \Lambda(W_z, s) \tag{A6}$$

with $W_z(a, b) = y^{-1}(a - xb)^2 + yb^2$. The term containing $b = 0$ is integrated as for the obtaintion of $\Lambda(s)y^s$.
Let S be is a subset of M_m the set of all $n \times n$ integral matrices of determinant m, and let $T(S)$ be the operator s.t.

$$T(S)f(Y) = \sum_{A\in s} f(Y[A]^0). \tag{A7}$$

The corresponding formal power series provides important identities:

$$T_{p^r}T_p \equiv T_{p^{r+1}} + T(S^r_1) + T(S^R_2) \tag{A8}$$

$$T_{p^r}[(T_p)^2 - T_{p^2}] = p^3 T_{p^{r-1}} + T_{p^{r+1}}T_p + T_{p^{r+2}} \tag{A9}$$

The generalized Hecke operators

$$T_n f(z) = n^{-1/2} \sum_{ad=n,\quad d>0,\quad b\ mod\ d} f\frac{az + b}{d} \tag{A10}$$

for $S_p(n, \mathbb{Z})$ are defined.
For $A \in \mathbb{Z}^{n\times n}$ of rank n, there exists a coset decomposition $\Gamma A \Gamma = \bigcup_{B\in S_A} B\Gamma$ (disjoint union).
According to Tamagawa’s formulation, for prime p
for $\Gamma \equiv GL(3, \mathbb{Z})$,

$$(T_p)^2 - T_{p^2} = pT(SA_2). \tag{A11}$$

The Hecke operator T_A admits a matrix representation as a matrix $A \in Q^{n\times n}$, $\mid A \mid > 0$.

After using the Mellin transform Eq. (A6) to the Poisson summation formula, to obtain the singular series form of the Fourier coefficients of $E_s(z)$, for $b = 0$ in Eq. (A6), one obtains

$$\frac{y}{t} = \sum_{a\in Z} e^{-\frac{\pi y a^2}{t}}. \tag{A12}$$

After applying the mapping $z \to g(z)$, one obtains the conformal or angle-preserving mapping $H \leftrightarrow H$.

The singular series form of the Fourier coefficients of $E_s(z)$ is proven to admit the Ramanujan sum formula.

The arithmetic parts of the holomorphic Eisenstein series and of the non-holomorphic Eisenstein series are singular series (in the sense defined after divisor-like functions).
The non-arithmetic part or analytic part in the holomorphic case is given by e^{-trNY}, $Y \in \mathcal{P}_n$, N a non-negative symmetric half-integral $n \times n$ matrix; an half-integral matrix is one such that, given $2n_{ij} \in Z\ \forall i \neq j$, n_{ii}, N is non-negative iff $N[x] \geq 0 \forall \in R^n$.
In the non-holomorphic case, the non-arithmetic part is a matrix analogue of a confluent hypergeometric function.

It is relevant to study the Fourier expansions of Eisenstein series for congruence subgroups of $SL(2, \mathbb{O}_K)$, where $\mathbb{O}_K$ is the ring of integers of a number field.
It is therefore possible to define the Hecke operators associated to a matrix $N \in Z^{m\times(n-m)}$ defined for f s.t. $f : P_m \to C$ as

$$T_N f(Y) \equiv \sum_{B\in Z^{m\times m},\quad N=b^tc,\quad for\quad some\quad c\in Z^{(n-m)\times m}} f([B]) \tag{A13}$$

The theta function associated with a positive-definite symmetric $n \times n$ real matrix P and a point z on the UPHP writes

$$\theta(P, z) = \sum_{a \in Z^n} e^{i\pi Pa(z)} \equiv \mid P \mid \frac{z}{i}^{-1/2} (\frac{z}{i})^{(-n/2)} (\theta p^{-1}, -\frac{1}{z}). \tag{A14}$$

As far as the theta function is concerned, one defined as formula (3.60) in [34], i.e.

$$\theta(P, z) = \sum_{a \in \mathbb{Z}^n} exp^{i\pi P[a]z} \tag{A15}$$

is found to be a modular form as a function of z (endowed with trivial multiplier system for the group $SL(2, \mathbb{Z})$) with the following restrictions place two restrictions on P.
Theorem A1: Be P an even integral positive symmetric $n \times n$ matrix of determinant 1, then 8 divides n, and $\theta(P, z) \in \mathcal{M}(SL(2, \mathbb{Z}), n/2)$, with P an even integral if $P[a] \in 2\mathbb{Z} \forall a \in \mathbb{Z}^n$ from Eq. (A15). □
Accordingly ([34] p.193), after allowing P to have an arbitrary determinant, the modular form attached to a congruence subgroup $\Gamma_0(N)$: the level N of the congruence subgroup $\Gamma_0(N)$ is named the level of the form P and is defined to be the least positive integer such that NP^{-l} is an even integer.
The operator Θ is defined as

$$\Theta \equiv \sum exp^{2i\pi nt}, \tag{A16}$$

with $f \in H$.
The modular automorphic forms

$$\Theta(z + 2) = \Theta(z), \tag{A17a}$$

$$\Theta(-\frac{1}{z}) = (z)^{1/2}\Theta(z), \tag{A17b}$$

define the Θ operator; as a result, it is half-integral: $2n_{ij} \in Z$, $\forall i \neq j$, with $n_{ii} \in Z$, N non-negative in Z, $N[x] \geq 0 \forall x \in R^n$.
Further insight about the Fourier expansion of the Eisenstein series for $GL(3, \mathbb{Z})$ has been gained in [25].

1. Integration over fundamental domains

The integration over fundamental domains $\mathcal{P}_n/GL(n, \mathbb{Z})$ is performed by the Jacobi formula

$$\frac{dY}{d(t, W)} = (nt)^(-1)t^{(n+1)/2} \tag{A18}$$

with the $SL(n, \mathbb{R})$ invariant volume.

Theorem A2: The volume of the fundamental domain is found after normalization of measure. □
Corollary A1: As far as the fundamental domain $\Gamma^0 = SL(n, \mathbb{Z})$ is concerned, the volume of the fundamental domain admits the normalization of measure $\Lambda(s)$. □.
Definition A2: Functions Γ/H are automorphic functions s.t. $f : R^+ \to \mathbb{C}$ in the fundamental domain of the grouppal structure $\Gamma^0 = SL(n, \mathbb{Z})$ as

$$V\hat{f}(0) + c^+ f(0) = V f(0) + c^+ \hat{f}(0) \tag{A19}$$

from [34], [35]. □
Definition A3: θ functions are defined as

$$\theta(P, z) = \sum_{a \in Z^n} e^{i\pi P[a]z} \equiv \mid P \mid^{-1/2} (\frac{z}{i})^{-n/2} \left(\theta(P^{-1}, -\frac{1}{z})\right), \tag{A20}$$

with

$$P[a] = {}^t aPa. \tag{A21}$$

□

a. Hecke operators for cusp forms From [34] and [35], as Y approaches the boundaries, the following equalities hold

$$\frac{a_1}{a_3} = \frac{a_1/a_4}{a_3/a_4} \tag{A22}$$

$$\frac{a_2}{a_3} = \frac{a_2/a_4}{a_3/a_4} \ \ approach \ \ to \ \ zero; \tag{A23}$$

in particular, a cusp form v_2 is defined as

$$\lim \frac{a_2}{a_3} \to 0 \tag{A24}$$

as explained from the Fourier expansion

$$f(Y) = \sum_{N \in Z^{n\times(n-m)}} C_N(V,W) e^{2\pi i tr[^t NX]} : \tag{A25}$$

$C_N(V,W)$ can be divided into two parts, in the case the subgrouppal structure associated is arithmetic: one singular, and one division-type function.
The following demonstration tools for the integral can be easily found from the ordinary K-Bessel function.
The non-arithmetic part $C_{N \ \ n.a.}$ in the holomorphic case id found as

$$C_{N \ \ n.a.} = e^{-tr[NY]}, \ \ Y \in P_n. \tag{A26}$$

For $i \neq j$, N is non-negative, i.e. means that

$$N[x] \geq 0 \ \ \forall i \neq j. \tag{A27}$$

The highest rank terms in the Fourier expansion of an Eisenstein series are from p.224 of [35].
If Y admits a decomposition $v > 0$, $W \in P_{n-1}$, $X \in R^{1\times(n-1)}$, for $m = 1$ and in the case $Re(s) > n/2$, then

$$E_{1, \ \ n-1}(1, s \mid Y) \sim v^{-s} \tag{A28}$$

W approaches infinity in the sense that the diagonal elements in the the suitable diagonalizability/triangular properties of the decomposition of W all approach infinity; the functional equation of Epstein's ζ function is easily found; an analogue of the Kronecker limit formula can be found (ex. 6 p.210 in [34]): in particular, the Kronecker-limit formula is defined through the definition of Dedekind η function as an infinite product. The Kronecker limit formula provides one with a solution of the Pell's equation in terms of some elliptical modular functions.

Given $E_{m,n-m}(\phi, s \mid Y)$, $\phi(W) = E_{(m)}(r \ | \in W)$, $r \in C^{n-1}$, $m \leq n/2$, $N \in Z^m$ non singular, the $N-th$ Fourier coefficient ca be normalized inn the Eisenstein series.
Given $A, B \in P_m, r \in C^m, Re[r_i] > 1$, the Eisenstein series coefficients exists $\forall E_m(r \mid Y)$, i.e. admits a series expansion i the pertinent integrals in terms of KS Bessel functions. Given B nihilpotent of the subgroup $GL(n,R)$ there exists an upper triangular matrix for the subgroup.

Appendix B: ABOUT HYPERBOLIC TRIANGLE GROUPS

For hyperbolic triangle groups, the following procedures hold. As in [4], a description of hyperbolic triangles can be found as the Gauss hypergeometric function(s) (i.e. as eigenfunctions for some operator). For this purpose, a Halphen system can be defined, which encodes the hyperbolicity properties properly. In particular, the Halphen system

$$\dot{t_1} = (a-1)(t_1t_2 + t_1t_3 - t_2t_3)(b+c-1)t_1^2, \tag{B1a}$$

$$\dot{t_2} = (b-1)(t_2t_1 + t_2t_3 - t_1t_3)(a+c-1)t_2^2, \tag{B1b}$$

$$\dot{t_3} = (c-1)(t_3t_1 + t_3t_1 - t_1t_2)(a+b-1)t_1^2, \tag{B1c}$$

with the conditions

$$1 - a - b = \frac{1}{m_1}, \tag{B2a}$$

$$1 - b - c = \frac{1}{m_2}, \tag{B2b}$$

$$1 - a - c \equiv 1m_3 \equiv 0 \tag{B2c}$$

with $m_1 \le m_2 \in \mathbb{N} \cup \{\ \infty\}$, after which the hyperbolicity condition is defined

$$\frac{1}{m_1} + \frac{1}{m_2} < 1 \tag{B3}$$

is analysed.
The initial conditions are chosen as

$$t_1(0) = 0, \tag{B4a}$$
$$t_3(0) = 0. \tag{B4b}$$

with $\frac{1}{m_1} + \frac{1}{m_2} < 1$.
By recursion (of the Halphen system), the t's are uniquely determined if $q = e^{2i\pi\tau/h}$:
with $h = \cos\frac{\pi}{m_1} + \cos\frac{\pi}{m_2}$, by rescaling q by a constant, the t's are meromorphic on $Im(\tau) > \tau_0$ with $\tau_0 \in \mathbb{R}_+$, and exhibit properties useful for the study of triangle groups. Indeed, the t_i become meromorphic functions on the whole UPHP under the suitable hypotheses.

The following triangle group is chosen

$$\Gamma_t :=< \gamma_1, \gamma_2, \gamma_3 >\subset SL(2, \mathbb{R}) \tag{B5}$$

of the type

$$\mathbf{t} = m_1, m_2, \infty \tag{B6}$$

, with

$$\gamma_1\gamma_2\gamma_3 = {\gamma_1}^{m_1} = {\gamma_2}^{m_2} = -\hat{1}_{2\times 2} \tag{B7}$$

where

$$\gamma_1 = \begin{pmatrix} 2\cos\frac{\pi}{m_1} & 1 \\ -1 & 0 \end{pmatrix}, \tag{B8a}$$
$$\gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 2\cos\frac{\pi}{m2} \end{pmatrix}, \tag{B8b}$$
$$\gamma_3 = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}. \tag{B8c}$$

The Hauptmodul for the chosen triangle group J equals

$$J = \frac{t_3 - t_1}{t_3 - t_2}. \tag{B9}$$

Appendix C: ABOUT TRACES

Be $\mathcal{A}$ a commutative ring endowed with 1, and q and q in $\mathcal{A}$ fixed invertible elements.
The elements

$$\{\ r'_1 ... r'_n \mid r'_i \in \mathcal{R}'_i\} \tag{C1}$$

formed an $\mathcal{A}$-basis of H_n. The matrix whose elements are

$$\{\ g_{r_1} ... g_{r_n} \mid r_i \in \mathcal{R}_i \tag{C2}$$

the the basis-change matrix; it is triangular with powers of q on the diagonal. More in detail, the following propositions, which follow Proposition (3.2) in [22] p. 11 and p. 12, respectively, will not hold any more.
To state the propositions, it is necessary to introduce h_j an $\mathcal{A}$-linear combination of the basis elements g_w, $w \in W$; more in particular, $w = r_1 ... r_n$, which implies g_w be obtained as $g_w = g_{r_1} ... g_{r_n}$. The element r'_i is the corresponding element in H_n. Given $n_w \ge 0$ the total number of of inverses in the terms $r'_1, ..., r'_n$, the inversion formula allows one

to obtain Eq. (14).

Proposition C1: There exists a finite non-empty subset of linear combinations of basis such that

$$h_j = \sum_{n_{j1}, r'_j} h_{j-1} r'_j. \tag{C3}$$

□.

Proposition C2:

$$h_j d'_{j+1} ... d'_n = \sum_{(h_{j-1}, r'_j) \in R(h_j)} h_{j-1} d'_j d''_{j+1} ... d''_n r''_{j-1} \tag{C4}$$

□

where the notation of [22] is followed.

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