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The General Stability – Like Concepts for Differential and Control System

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ABSTRACT

Theorems corresponding to the concepts of stability discussed in the works under reference should follow the general pattern of the direct method of Lyapunov.
Theorems proved for such stability contain, as special cases results on - (i) stability for the origin, (ii) stability with respect to some components, (iii) stability of a set A, (iv) stability of a set A (t), (v) stability of conditional invariant set B relative to A, (vi) Stability of asymptotic invariant set A and (vii) stability of conditional asymptotic invariant set B relative to A.
It is shown that our results on the stability properties can be extended to cover control systems with the set of controls compact in R^m.

KEYWORDS: Stability, Lyapunov Functions, control system.

INTRODUCTION

It was observed, while studying stability behaviour of an invariant set A for a given differential system, that Lyapunov's direct method for stability of equilibrium point is carried over in almost similar way. The conditions on differential inequalities remain the same and those of positive definiteness and decrescence on Lyapunov functions can be interpreted relative to the set A, in a natural way (15, 31, 64, 65, 66). This is evident in the works of Oziraner (45, 46, 47), Oziraner and Rumiantsev (48), Rumiantsev (52, 53, 54), Peiffer and Rouche (50) and Corduneanu (10, 11), on partial stability or stability with respect to some components. Similarly, one can observe this phenomena in the work, on conditional invariant sets, of Kayande and Lakshmikanthan (26), Leela (31, 38). Lakshmikantham, Lella and Ladde (33) have specifically observed that similar extension is possible, while discussing the stability properties of conditional asymptotic invariant sets. The results on asymptotic invariant sets in (31) can also be cited as an example of the observed phenomena. Deo (13) has observed the similarity in conditional and partial stability, while discussing the strict partial stability results.

It is shown that it is natural that the theorems corresponding to the concepts of stability discussed in the works under reference should follow the general pattern of the direct method of Lyapunov. It is felt that if a general concept of stability in terms of continuous functions, instead of norms, is introduced for a differential system, then the theorems proved for such stability contain, as special cases results on - (i) stability for the origin, (ii) stability with respect to some components, (iii) stability of a set A, (iv) stability of a set A (t), (v) stability of conditional invariant set B relative to A, (vi) Stability of asymptotic invariant set A and (vii) stability of conditional asymptotic invariant set B relative to A. Thus the work is of a very general nature. (For unification in a different direction see (19) It is shown that our results on the stability properties can be extended to cover control systems with the set of controls compact in R^m.

GENERAL SUFFICIENCY CONDITIONS FOR STABILITY

Preliminaries:

1) Consider the differential system –

$$X' = f(t, x) \quad (' = d/dt) \qquad .. \ (2.1)$$

Where $f \in C(I \times D, R^n)$, $I = [0, \infty)$ and D is a region of the real n-space R^n, invariant for the system (2.1) so that the solutions starting in D remain in D for all $t \in I$. Further we assume the Lipschitz condition of f that.

$$|f(t, x) - f(t, y)| \leq k(t)\,|x-y| \qquad .. \ (2.2)$$

Denoting the norm in R^n, $k \in C(I, R_+)$, where $R_+ = [0, \infty)$, (2.2) being satisfied for each $t \in I$, $x, y \in D$. The Lipschitz condition ensures the uniqueness of solution of (2.1) and the continuous dependence of solutions on initial conditions.

Let $x(t, t_o, x_o)$ denote the solution of (2.1), through $(t_o, x_o) \in I \times D$, with $x(t_o, t_o, x_o) = x_o$.

(2) In addition to the class of monotone functions. K^*, K, L^*, L that we recalled. We mention one more class of functions defined in (15).

Definition – (2.1) : A function $b \in C(I \times R_+, H_+)$ is said to belong to the class D (i.e. $b \in D$) if b (t, r) is decreasing in t for each $r \in R_+$ and increasing in r for each $t \in I$ and lim b (t, r) = 0 as $t \to \infty$ and $r \to 0^+$.

(3) Let V = V (t, x) $\in$ r, if $V \in C_{lip}(I \times D, R_+)$, the class of continuous functions satisfying Lipschitz condition in x. We recall –

$$D^+V(t,x) = \limsup_{h \to 0^+} \frac{1}{h}\left[(V(t+h, x+hf(t,x)) - V(t,x)\right] \quad \ldots (2.3)$$

And

$$D^-V(t,x) = \liminf_{h \to 0^+} \frac{1}{h}\left[(V(t+h, x+hf(t,x)) - V(t,x)\right] \quad \ldots (2.4)$$

The following comparison theorems are well known (2).

Theorem – 2.1 : Let there exist $V \in \gamma$ such that

$D^+(V(t,x)) \le g(t, V(t,x)), (t, x) \in I \times D$ ... (2.5)

Where $g \in C(I \times R_+, R)$. Let r (t, t_o, r_o) denote the maximal solution of the differential equation –

$x' = g(t,r), r(t_o) = r_o$ (' = d/dt) ... (2.6)

Then $V(t_o, x_o) \le r_o$

Implies $V(t, x(t, t_o, r_o)) \le r(t, t_o, r_o)$ ... (2.7)

For all $t \ge t_o$ for which $r(t, t_o, r_o)$ exists.

Theorem – 2.2 : Let there exist $V \in \gamma$ such that $D^-(V(t,x)) \ge h(t, V(t,x)), (t,x) \in I \times D$

... (2.8)

Where $h \in C(I \times R_+, R)$. Let u (t, t_o, u_o) denote the minimal solution of

$u' = h(t,u), u(t_o) = u_o$ (' = d/dt) ... (2.9)

Then $V(t_o, x_o) \ge u_o$

Implies $V(t, x(t, t_o, x_o)) \ge u(t, t_o, u_o)$ ...(2.10)

For all $t \ge t_o$ for which $u(t, t_o, u_o)$ exists.

Note: (1) Functions g and h are assumed to be smooth enough to ensure the existence of maximal / minimal solution of the equations (2.6) / (2.9) respectively, for all $t \ge t_o, r_o, u_o \le p, p > 0$.

(2) If $g \equiv 0$ in (2.5) then (2.7) reduces to

$V(t, x(t, t_o, x_o)) \le V(t_o, x_o)$ ...(2.11)

(3) If $h \equiv 0$ in (2.8) then (2.10) reduces to

$V(t, x(t, t_o, x_o)) \ge V(t_o, x_o)$ ...(2.12)

Now we state and prove some general theorems giving the sufficiency criteria for stability –

Theorem -2.3 : Let $\delta_1 \in C(I \times D, R_+)$ and $\delta_2 \in C(D, R_+)$.

If there exists a function $V = V(t,x) \in \gamma$ such that for all $(t, x) \in I \times D$,

(i) $\delta_2(x) \le V(t,x) \le \delta_1(t,x)$ ...(2.13)

And

(ii) $D^+V(t,x) \le 0$.

Then $\delta_2(x(t, t_o, x_o)) \le \delta_1(t_0, x_0)$.

Proof: Because of (ii) and (2.11) holds.

i.e. $V(t, x(t, t_o, x_o)) \le V(t_0, x_0)$.

Hence by (i)

$\delta_2(x \le V(t, x(t, t_o, x_o)) \le V(t_o, x_o) \le \delta_1(t_0, x_0)$.

i.e. $\delta_2(x(t, t_o, x_o)) \le \delta_1(t_0, x_0)$.

Theorem – 2.4: Let the hypothesis of theorem (2.3) hold, with the exception that δ_1 is independent of t.

i.e. $\delta_1(t, x) = \delta_3(x)$ for all $t \in I$.

Then $\delta_2(x(t, t_o, x_o)) \le \delta_3(x_0)$.

Proof: Obvious.

Remark – 1 : Consider the following choices for the functions : viz., δ_1 and δ_2 in theorem (2.2) :

i) $\delta_1(t, x) = a(t, |x|) \in K^*$ and $\delta_2(x) = b(|x|) \in K$ gives equistability of the origin (15, 31).

ii) $\delta_1(t, x) = a(t,x) \in K^*$ and $\delta_2(x) = b\left[\sum_{i=1}^{k} x_i^2\right]^{y2}_{k \le n,} \in K$

Where $x = (x_1, x_2, x_3, \ldots x_k, x_{k+1}, \ldots x_n)$,

Gives equi-stability of the origin with respect to some components of x (15, 31, 45, 64).

iii) $\delta_1(t, x) = a(t, d(x, A)) \in K^*$ and $\delta_2 = b(d(x,A) \in K$, A being a compact closed subset of D, gives equi-stability of the set A (15, 31, 66).

iv) $\delta_1(t, x) = a(t, d(x, A)) \in K^*$ and $\delta_2 = b(d(x, B)) \in K$, where A, B $\subset$ D, A, B being compact closed sets in D, gives equistability of the conditionally invariant set B, relative to the set A (31, 26).

v) Let $\lambda \in D$. Choose $\delta_1(t, x) = M(\alpha)\, d(x, A) + \lambda(t)$ and $\delta_2(x) = d(x,B)$. We obtain the uniform stability of the conditional asymptotically invariant set B relative to the set A (33).

vi) If in (V), we choose $\delta_2(x) = d(x, A)$, then we get uniform stability of asymptotically invariant set A (31, 33).

vii) If $\delta_1(t,x) = a(t, d(x,A(t)))$ and

$\delta_2(x) = b(d(x, A(t)))$, where $a \in K^*$ and $b \in K$, then the theorem implies the equistability of the set A(t) (68).

Remark – 2 : By choosing δ_1 independent of t, in (i), (ii) (iii), (iv) and (vii) above, we obtain the corresponding uniform stability of the sets mentioned therein.

Theorem – 2.5 : Let there exist a function $V \in \gamma$ such that the hypothesis of theorem (2.3) holds and (2.5) is satisfied. Let the maximal solution r (t, t_o, r_o) of the equation (2.6) satisfy the estimate below –

$r(t, t_o, r_o) \le a(t_o, r_o), t \ge t_o, a \in K^*$

... (2.14)

for some $p > o, r_o \le p$.

Then for $t \ge t_o$,

$\delta_2(x(t, t_o, x_o)) \le a(t_o, \delta_1(t_o, x_o))$

... (2.15)

for $x_o \in \Omega\, t_o(P)$ where $x_o \in \Omega_{to}(P)$ means that

$\delta_1 (t_o, x_o)) \leq p$.

Proof: Let $x_o \in \Omega_{to}$ (P) so that $\delta_1 (t_o, x_o) \leq (P)$.
Choose $r_o = V (t_o, x_o)$. By (i) of theorem (2.3).
$r_o = V (t_o,x_o) \leq \delta_1 (t_o, x_o) \leq (P)$
Thus, $x_o \in \Omega_{to}$ (P) implies $r_o \leq p$
Hence (2.14) is satisfied.
By the choice of r_o, we have

$$\begin{aligned}\delta_2 (x (t, t_o, x_o)) &\leq V(t,x (t, t_o, x_o)) \\ &\leq r (t, t_o, r_o) \\ &\leq a (t_o,r_o) = a (t_o,V(t_o, x_o)) \\ &\leq a (t_o, \delta_1 (t_o, x_o)),\end{aligned}$$

Which is what (2.15) asserts.
Corollary : Theorem (2.3).
Proof: Choose $g \equiv 0$, in the above theorem, so that

$$\begin{aligned}\delta_2 (x (t, t_o, x_o)) &\leq V(t,x (t, t_o, x_o)) \\ &\leq V (t_o, x_o) \\ &\leq \delta_1 (t_o, x_o),\end{aligned}$$

which is what the theorem (2.3) claims.

Theorem – 2.6 : Let the hypothesis of theorem (2.5) hold except that, instead of the condition (2.14), we have

$r(t,t_o,r_o) \leq a (r_o), t \geq t_o, a \in K$ … (2.16)

for some p>o, $r_o \leq p$, and $\delta_1 (t, x)= \delta_3 (x)$ for all $(t,x)\in I \times D$.

Then $\delta_2 (x (t, t_o, x_o)) \leq a(\delta_3(x_0))$ . . (2.17)

for all $t \geq t_o, x_o\in\Omega_1(p)$ meaning that $\delta_3 (x_o) \leq p$.

Proof: Let $x_o \in \Omega_1(P)$ so that $\delta_3 (x_o) \leq p$.
Choose $r_o = V (t_o, x_o)$.
$r_o = V (t_o,x_o) \leq \delta_3 (x_o) \leq P$.
Now $x_o\in\Omega_1$ (p) implies $r_o \leq p$. Hence (2.16) holds.
Therefore,

$$\begin{aligned}\delta_2 (x (t, t_o, x_o)) &\leq V(t,x (t, t_o, x_o)) \\ &\leq r (t, t_o, r_o) \\ &\leq a (r_o) = a (V(t_o, x_o)) \\ &\leq a (\delta_3 (x_o))\end{aligned}$$

for $t \geq t_o, x_o\in\Omega_1$ (p). Hence the proof.

Theorem – 2.7: Let the hypothesis of theorem (2.5) hold except that the condition (2.14) is replaced by

$r(t, t_o, r_o) \leq a (t_o,r_o) b (t_o, t-t_o)$
… (2.18)

When $a \in K^*$, $b \in L^*$, for some p > O, $r_o \leq p$, $t\geq t_o$.
Then for $x_o \in\Omega\, t_o$ (p), $t \geq t_o$

$\delta_2 (x (t, t_o, x_o)) \leq a (t_o, \delta_2 (t_o, x_o)) b (t_o, t-t_o)$
… (2.19)

Proof: The argument here is parallel to that in the proof of theorem (2.3) but for the deviation -

$$\begin{aligned}\delta_2 (x (t, t_o, x_o)) &\leq V (t, x(t, t_o, x_o)) \\ &\leq t(t, t_o, r_o) \\ &\leq a(t_o, r_o) b (t_o, t-t_o) = a (t_o, V(t_o, x_o)) b (t_o, t-t_o) \\ &\leq a(t_o, \delta_1 (t_o, x_o)) b (t_o, t-t_o),\end{aligned}$$

for $t \geq t_o$, and $x_o \in \Omega_{to}$ (p).

Theorem – 2.8 : In the hypothesis of theorem (2.5), let (2.14) be replaced by $r (t, t_o, r_o) \leq a (r_o) b(t-t_o) \geq$
… (2.20)

Where $a \in K$, $b\in L$ and $\delta_1(t, x) = \delta_3 (x)$, in hypothesis (i) of theorem (2.5). Then for $x_o\in \Omega_1(p)$, $t \geq t_o$,

$\delta_2 (x(t, t_o, x_o)) \leq a (\delta_3(x_o)) b (t-t_o)$
… (2.21)

Proof : Similar to that of theorem (2.6).

Remarks :

1) The theorems (2.5) to (2.8) are comparison theorems and the conditions (2.14), (2.16), (2.18) and (2.20) assure equi, uniform, equi-asymptotic and uniform asymptotic stability respectively for trivial solution of the equation (2.6).
   By proper choice of the δ - functions in the above theorems we can derive stability properties in each of the cases mentioned under remarks 1 and 2 following theorem (2.4).
2) The theorems (2.3) to (2.8) give upper estimates on the values of δ_2 – function along the trajectories of the solutions of the equation (2.1).
   We now state some theorems in which lower estimates are got for the δ_2 – function along the trajectories of the solutions of the equation (2.1).

Theorem – 2.9 : Let $\delta_4 \in C(I \times D, R_+)$ and $\delta_5 \in C(D, R_+)$.
If there exists a function $V_1 \in \gamma$ such that for $(t, x) \in I \times D$,

(i) $\delta_4 (t, x) \leq V_1 (t, x)\ \delta_5 (x)$ … (2.22)

and

(ii) $D^- V_1 (t,x) \geq O$.

Then, for $t \geq t_o$, $\delta_5 (x (t, t_o, x_o)) \geq \delta_4 (t_o, x_o)$ … (2.23)

Proof: From (ii) and (2.12)

$V1 (t, x (t, t_o, x_o)) \geq V_1 (t_o, x_o)$.

From (2.22) it follows that

$\delta_5 (x (t, t_o, x_o)) \geq V_1 (t, x (t, t_o, x_o)) \geq V_1 (t_o, x_o)\ \delta_4 (t_o, x_o)$.

Hence the result

Theorem 2.10 : Let there exist a function $V1 \in r$ satisfying the hypothesis of theorem (2.9) except that δ_4 is independent of t i.e. $\delta_4 (t,x) = \delta_6 (x)$, for $(t,x) \in I \times D$. Then $\delta_5 (x (t, t_o, x_o)) \geq \delta_6 (x_o)$, for$\in t_o$.… (2.24)

Proof: On the same lines as in the theorem (2.9).

Combing the results of theorem (2.3) and (2.9) we obtain the following theorem, which corresponds to strict-stability properties –

Theorem – 2.11 : Let there exist functions V and V_1 both belonging to r satisfying the hypothesis of theorems (2.3) and (2.9) respectively with $\delta_5 = \delta_2$ in hypothesis (i) of theorem (2.3). Then for $t \geq t_o$,

$\delta_4 (t_o, x_o) \leq \delta_2 (x (t, t_o, x_o)) \leq \delta_1 (x (t_o, x_o)$ (2.25)

A similar combination of theorems (2.4) and (2.10), corresponding to uniform strict stability properties yields.

Theorem – 2.12: Let there exist functions V and V_1 both belonging to r satisfying the hypothesis of theorems (2.4) and (2.10) respectively with $\delta_5 = \delta_2$ in hypothesis (i) of theorem (2.4).

Then for $t \geq to$

$\delta_6 (x_o) \leq \delta_2 (x (t, t_o, x_o)) \leq \delta_3 (x (x_o)$ (2.26)

Note – 1 : In theorems (2.4) and (2.10), consider the following choice of δ- functions –

(a) Let $a_1, b_i \in K$, I = 1,2. Choose $\delta_1 (t,x) = a_1 (|x|)$

$\delta_2\ b_1 (|x|)$, $\delta_5 (x) = a_2 (|x|)$ and $\delta_6 (= b_2 (|x|)$

Then

$$a_2^{-1}\ b_2 (|x_o|) \leq |x (t, t_o, x_o) | \leq b_2^{-1}\ a_1 (x (|x_o|)$$

Thus the origin is uniformly strictly stable.

(b) In the same theorems, choose $\delta_3 (x) = a_1 (|x|)\ a_1 \in K$

$$\delta_2 (x) = b_1 \left(\sum_{i=1}^{k} x_1^2\right)^{1/2} \quad \delta_5 (x) = b_2 \left(\sum_{i=1}^{k} x_1^2\right)^{1/2}$$

$\delta_6 (x) = a_2 (|x|)$, $b_1, b_2, a_2 \in K$

We obtain strict-partial stability of the origin (38).

(c) In theorem (2.12), let $\delta_2 (x) = \phi (d(x,B))$,

$\delta_5 (x) = a (d(x,A))$ and $\delta_6 x) = b (d (x,B))$, where

ϕ, a, b $\in$ K Then we obtain the uniform strict stability of the conditionally invariant set B relative to the set A.

(d) In theorem (2.12), let $\delta_2 (x) = \phi (d(x, A))$,

$\delta_3 (x) = a (d(x,A))$ and $\delta_6 (x) = b (d (x,A)$, ϕ, a, b $\in$ K.

This yields uniform strict stability of the set A.

(e) In theorem (2.11), let $\delta_2 (x) = \phi (d(x, A))$, $\phi \in K$.

$\delta_4 (t,x) = a (t,d(x,A))$ and $\delta_6 (t,x) = b (t,d (x,A))$, a, b $\in K^*$.

Then we obtain equistrict stability of the set A.

Other results can be obtained similarly.

Some comparison theorems giving the lower estimates are stated below:

Theorem 2.13: Let there exist a function V1 $\in \gamma$ such that (2.22) and (2.8) are satisfied. Let the minimal solution u (t, t_o, u_o) of (2.9) satisfy the estimate :

$u (t, t_o, u_o) \geq a (t_o,u_o)$ …(2.27)

for all $t \geq t_o$, a $\in K^*$, for some p ≥ o, u_o, ≥ p.

Then, for $t \geq t_o$, and $x_o \in \Omega_1$ to (p) which means $\delta_4 (t_o,x_o) \geq P$

$\delta_5 (x (t, t_o, x_o)) \leq a (t_o, \delta_4 (t_o, x_o))$ …(2.28)

Proof: Let xo $\in \Omega_1\ t_o$ (p) so that (t_o, x_o) p.

Choose $u_o = V_1 (t_o,x_o)$. By (2.22) $u_o = V_1 (t_o,x_o) \geq \delta_4 (t_o, x_o) \geq p$.

Thus $x_o \in \Omega_1\ t_o$ (p) implies $u_o \geq p$.

Therefore, $\delta_5 (x (t, t_o, x_o)) \geq V_1 (t, x(t, t_o, x_o))$

$\geq u(t, t_o, u_o)$

$\geq a(t_o, u_o) = a (t_o, V_1(t_o, x_o))$

Therefore, $\delta_5 (x (t, t_o, x_o)) \geq a (t_o, \delta_4 (t_o, x_o))$.

Theorem – 2.14 : Let the assumptions of theorem (2.15) hold with $\delta_4 (t, x) = \delta_6 (x)$ (i.e. independent of t), for all (t,x) $\in$ I X D and (2.26) be replaced by

$u(t, t_o, u_o) \geq a (u_o)$, a $\in$ K …(2.29)

Then for $t \geq t_o$, $\delta_5 (x (t, t_o, x_o)) \geq a\ \delta_6 (x_o))$ …(2.30)

for all $x_o \in \Omega_1\ t_o$ (p) which means $\delta_6 (x_o) \geq p$.

Proof: Runs on the same lines as the proof of the theorem (2.13)

Theorem – 2.15 : Let the assumptions of theorem (2.13) hold with the condition (2.27) replaced by

$u(t, t_o, u_o) \geq a (t_o, u_o)\ b(t_o, t-t_o)$ … (2.31)

where a $\in K^*$ and b $\in L^*$:

Then for $t \geq t_o$, $x_o \in \Omega_1\ t_o$ (p) which means $\delta_4 (t_o, x_o) \geq p$

$\delta_5 (x (t, t_o, x_o)) \geq a (to, \delta_4 (t_o,x_o))\ b (to, t-to)$ … (2.32)

The proof is similar to that for theorem (2.13) but for the deviation –

$(x (t, t_o, x_o)) \geq V_1 (t, x (t, t_o, x_o))$

$\geq u (t, t_o, x_o)$

$\geq a (t_o, u_o)\ b (t_o, t-t_o)$

$\geq a (t_o, V_1(t_o, x_o))\ b (t_o, t-t_o)$

$\geq a (t_o, (t_o, x_o))\ b (t_o, t-t_o)$

For all $t \geq t_o$, $x_o \in \Omega_1 t_o$ (p).

Theorem – 2.16 : Let the assumptions of theorem (2.13) be satisfied, with $\delta_4 (t,x) = \delta_6 (x)$ for all (t, x) $\in$ I X D and the condition (2.27) replaced by

$U(t, t_o, u_o) \geq a (u_o)b(t-t_o)$, a $\in$ K, b $\in$ L … (2.33)

Then for $t \geq t_o$, $x_o \in \Omega_1$ (p) which means $\delta_6 (x) \geq p > O$,

$\delta_5 (x(t,t_o,x_o)) \geq a (\delta_6 (x_o))\ b (t-t_o)$ … (2.34)

Proof: Just on the same lines as for theorem (2.15).

Combing the theorems (2.5) and (2.13), we obtain –

Theorem – 2.17 : Let there exist functions V and $V_1 \in$ r satisfying the hypothesis of theorems (2.5) and (2.13) respectively, with the function δ_2 (of theorem (2.5)) identical with the function δ_5 (of theorem (2.13)).

Then for $t \geq t_o$

$\delta_4 (t_o,x_o)) \geq \delta_2 (x (t, t_o, x_o)) \leq \delta_1 (t_o,x_o)$ … (2.35)

By proper choice of the functions, we get strict stability properties of sets.

Combining the theorems (2.6) and (2.14), we obtain –

Theorem 2.18 : Let there exist functions V and $V_1 \in$ r satisfying the conditions of theorems (2.6) and (2.14) respectively, with the function δ_2 (of theorem (2.6) identical with the function δ_5 (of theorem (2.14)).

Then for $t \geq t_o$,

$\delta_4 (x_o)) \geq \delta_2 (x (t, t_o, x_o)) \leq \delta_1(x_o)$ … (2.36)

Proper choice of δ - functions lead to strict uniform stability of sets.

Combining the theorems (2.8) and (2.16), we obtain

Theorem – 2.19 : Let there exist functions V and $V_1 \in$ r satisfying the conditions of theorems (2.8) and (2.14) respectively, with $\delta_2=\delta_5$. Then for $t \geq t_o$,

$\delta (x_o)\ \sigma (t-t_o) \leq \delta_2 (x (t, t_o, x_o)) \leq \bar{\delta} (x_o)\ \bar{\sigma} (t-t_o)$ … (2.37)

Proper choice of functions δ and σ lead to strict uniform asymptotic stability properties of sets.

3. THEOREMS ON THE EXISTENCE OF LYAPUNOV FUNCTIONS

In this section, the existence of Lyapunov functions is proved on the assumptions that the system (2.1) satisfies the condition (2.2) so that its solution is unique and continuously depends on initial conditions.

Theorem – 3.1 : Let the trajectories x (t, t_o, x_o) of (2.1) with (2.2) satisfy the estimates –

δ_2 (x (t, t_o, x_o)) ≤ $\delta_3(x_o)$, t ≥ t_o, t_o ∈ I … (3.1)

Where δ_2 ∈ C(D,R_+) and δ_2∈C(D,R_+) as well, with

|δ_3 (x) - δ_3(y) | ≤ k |x-y|, for x, y ∈ D … (3.2)

Then there exists a V-function satisfying the hypotehses of theorem (2.4).

Proof : Define V (t,x) =

$$\inf_{O \le T \le t} \delta 3(x(T,t,x_o)), \ for\,(t,x)\ I\ X\ D.$$

Because of (2.2), |f(t,x) – f (t,y) |≤ λ (t) |x-y|.

The continuity of and continuous dependence of x on initial conditions imply the continuity of V (t,x).

Also V (t,x) = δ_3 (x (T_1, t, x)) for some T_1∈[o,t]

and V (t,y) = δ_3 (x (T_2, t, y)) for some T_2∈[o,t].

Let T_1 ≤ T_2.

Then |V (t,x) – V(t,y) |= | δ_3 (x (T_1, t, x)) - δ_3 (x (T_2, t, y))|

≤ k |x (T_2, t,x) – x (T_2, t, y) |

≤ k |x-y| exp $\int_o^t \lambda(s)\,ds.$

This shows that V satisfies Lipachitz condition in x. for each t ∈ I.

Thus V ∈ r.

Along the trajectory x (t, t_o, x_o) of (2.1)

$$V\,(t,x\,(t,t_o,x_o)\,) = \inf_{O \le T \le t} \delta_3(x(T,t,x(t,t_o,x_o))$$

$$\inf_{O \le T \le t} \delta_3(x(T,t_o,x_o)$$

.. (3.3)

Since the solution of (2.1) is unique.

Similarly,

$$V\,(t+h,x(t+h,\ t_o,\ x_o)\ ) = \inf_{O \le T \le t+h} \delta 3(x(T,t_o,x_o)\ )$$

.. (3.4)

And V (t+h,x(t+h, t_o, x_o)) – V (t,x (t, t_o, x_o)) ≤ O.

Hypothesis (ii) of theorem (2.3) follows easily.

From the definition of V, V (t,x) ≤ δ_3 (x) …(3.5)

From (3.1) for each T ∈ [o,t], δ_3 (x(T)) ≥ δ_2 (x)

Thus V (t,x) ≥ δ_2 (x) …(3.6)

(3.5) and (3.6) verify (i) of theorem (2.3)

With δ_1 (t,x) = δ_3 (x)

This completes the proof.

Theorem – 3.2: Let the trajectories x (t, t_o, x_o) of (2.1) with (2.2) satisfy the estimates –

δ_5 (x (t,t_o,x_o)) ≥ δ_6 (x_o), t ≥ t_o, t_o ∈ I … (3.7)

Where δ_3 ∈ C (D, R_+), δ_6∈ C (D, R_+) with

| δ_6 (x) - δ_6 (y) k |x-y| for x, y ∈ D. … (3.8)

Then there exists a function V satisfying the hypothesis of theorem (2.10).

Proof : For (t, x) ∈ I X D, define

$$V\,(t,x) = \sup_{O \le T \le t} \delta 6(x(T,t,x)\ )$$

Clearly V (t,x) ≥ δ_6(x) … (3.9)

And from (3.7), V (t,x) ≤ δ_5(x) … (3.10)

Since for each T ∈ [o,t], δ_6 (x(T)) ≥ δ_5 (x).

(3.9) and (3.10) together verify (i) of theorem (2.10) with δ_4(t,x) = δ_6(x).

V∈r follows, because V is continuous as δ_6 is so and x continuously depends on the initial conditions. Also V satisfies Lipschitz condition because of (3.8) and (2.2). Which can be proved on the same lines as in previous theorem.

$$V\,(t,x\,(t,t_o,x_o)\,) = \sup_{O \le T \le t} \delta_6(x(T,t,x(t,t_o,x_o))$$

$$\sup_{O \le T \le t} \delta_6(x(T,t_o,x_o)$$

and V (t+h,x(t+h,t_o,x_o)) =

$$\sup_{O \le T \le t+h} \delta_2(x(T,t,x_o)$$

$$\sup_{O \le T \le t} \delta_6(x(T,t,x_o)) = V(t,x)$$

Therefore D^-V (t,x) ≥ 0,which is the hypothesis (ii) of theorem (2.9), and this completes the proof.

Theorem – 3.3: Let the trajectories x (t, t_o, x_o) of (2.1) with (2.2) satisfy the estimate :

δ_1 (x (t, t_o,x_o)) ≤ δ_2 (x_o) σ (t-t_o), t ≥ t_o, t_o ∈ I … (3.11)

Where δ_1 , δ_2 ∈ C (D,R_+) and σ ∈ L, σ being differentiable on I and σ′ (t) = - λ_1σ(t), t ≥ 0, λ> 0 …(3.12)

For each t ∈I. Let δ_2 satisfy the Lipechite condition

|δ_2 (x) – δ_2 (y)| ≤ k |x-y|, x, y ∈ D …(3.13)

Then there exists a V-function, satisfying –

1) V ∈ γ

2) $D^+ V(t,x) \leq -\lambda_1 V(t,x)$, and
3) $\delta_1(x) \leq V(t,x) = \delta_2(x)\,\sigma(0)$, for $(t,x) \in I \times D$.

Proof: Define $V(t,x) = \inf_{o \leq T \leq t} \delta_2(x(T,t,x))\,(t-T)$

For $(t,x) \in I \times D$

The continuity of δ_2 and σ and the continuous dependence of $x(t,t_o,x_o)$ on initial conditions ensure (1). The continuity of V.

Also as in theorem (3.1), it can be easily shown that

$$|V(t,x) - V(t,y)| \leq k\,|x-y|\,\sigma(o)\int_0^t \lambda(s)\,ds.$$

Hence (1) is satisfied.

Next, $V(t, x(t, t_o, x_o)) =$

$$\inf_{O \leq T \leq t} \delta_2(x(T,t,x(t,t_o,x_o)))\sigma(t-T)$$

$$= \inf_{O \leq T \leq t} \delta_2(x(T,t,x))\sigma(t-T)$$

Thus $V(t,x(t,t_o,x_o)) = \delta_2(x(T_1,t,x_o))\,\sigma(t,-T_1)$.

for some $T_1 \in [O,t]$ due to the compactness of $[O, t]$.

Thus $V(t+h,x(t+h,t_o,x_o)) =$

$$\inf_{QT \leq t+h} \delta_2(x(T,t_o,x_o))\sigma(t+h-T)$$

$$= \inf_{QT \leq t+h} \delta_2(x(T,t_o,x_o))\sigma(t+h-T)$$

$$= \delta_2(x(T_1,t_o,x_o))\,\sigma(t+h\,T_1).$$

Hence,

$$D^+ V(t,x) = \lim_{h \to o^+} \sup \delta_2(x(T_1,t_o,x_o))\left[\frac{\sigma(t+h-T_1)-\sigma(t-T_1)}{h}\right]$$

$$= \delta_2(x(T_1,t_o,x_o))\,\sigma'(t-T_1).$$

$$= \delta_2(x(T_1,t_o,x_o))\,\sigma(t-T_1)\frac{\sigma'(t-T_1)}{\sigma(t-T_1)}.$$

$\leq -\lambda_1 V(t,x(t,t_o,x_o))$ due to (3.12). Hence (2).

From the definition of V and (3.11). it follows that $V(t,x) \leq \delta_2(x)\,\sigma(0)$ and $V(t,x) \geq \delta_1(x)$ due to the fact that for each $T \in [0,t]$, $\delta_2(x(T, t, x))\,\sigma(t-T) \geq (x)$. This verifies (3) and hence the proof.

Remark:

The condition (3.11) verifies a result corresponding to uniform asymptotic stability. The condition (3.12) shows that this stability is exponential. In the above theorem, if $\delta_1(x) = d(x, B)$. $\delta_2(x) = d(x, A)$, $\delta_3(x) = d(x, B)$, then we have the converse theorem on exponential uniform asymptotic stability of the C.I. Set B with respect to A. Other deductions can similarly be made obtaining converse theorems in the case of exponential asymptotic partial stability (47) and exponential asymptotic stability of the set A.

4. CONTROL SYSTEMS

In this it is shown that our results on the stability properties can be extended to control systems with the set of controls compact in R^m.

Consider the control system:

$$x' = f(t,x,u) \quad (' = d/dt) \qquad \ldots (4.1)$$

where $f \in C(I \times D \times E, R^n)$, D being a region in R^n and E is a compact set in R^m.

For converse results, it is assumed that for each fixed $u \in E$,

$$|f(t,x,u) - f(t,y,u)| \leq \lambda(t)\,|x-y| \qquad \ldots (4.2)$$

Where $\lambda \in C(I, R_+)$. It is further assumed that the region D is invariant for the system (4.1).

<u>An extension of comparison theorems (2.1) and (2.2) to the control system (4.1)</u>

Let $x_u(t, t_o, x_o)$ denote a trajectory of (4.1) corresponding to a fixed $u \in E$.

Theorem – 4.1 : Let there exist a function $V \in \gamma$, such that

$$D^+ V(t,x) = \lim_{h \to O^+} \sup \frac{1}{h}\left[(V(t+h, x+hf(t,x,u)) - V(t,x)\right]$$

$$\leq g(t, V(t,x)) \qquad \ldots (4.3)$$

for all $(t,x) \in I \times D$, $u \in E$ and where $g \in C(I \times R_+, R)$.

Let $r(t, t_o, r_o)$ denote the maximal solution of the differential equation:

$$\left.\begin{array}{l} r' = g(t, r) \\ r(t_o) = r_o \end{array}\right\} \qquad \ldots (4.4)$$

$$\left.\begin{array}{l} \text{Then } V(t_o, x_o) \leq r_o \text{ implies} \\ V(t,x_u(t, t_o, x_o)) \leq r(t, t_o, r_o) \end{array}\right\} \qquad \ldots (4.5)$$

for all $t \geq t_o$ for which $r(t, t_o, r_o)$ exists.

Theorem – 4.2 : Let there exist a function V such that for all $(t,x) \in I \times D$, $u \in E$,

$$D^- V(t,x) = \lim_{h \to O^+} \inf \frac{1}{h}\left[(V(t+h, x+hf(t,x,u)) - V(t,x)\right]$$

$$\geq \overline{h} \quad g(t, V(t,x)) \qquad \ldots (4.6)$$

Where $\overline{h} \in C(I \times R_+, R)$. Let $(t, t_o, \int)$ be the minimal solution of

$$\begin{rcases} \int' = (t, \overline{h}\ \int) \\ \int(t_o) = \int_o \end{rcases} \quad \ldots (4.7)$$

Then $V(t_o, x_o) \geq \int_o$.

Implies $V(t, x_u(t, t_o, x_o)) \geq \int(t, t_o, \int_o)$ ... (4.8)

for all $t \geq t_o$ for which $(t, t_o, \int_o)$ exists.

As in section 2, g and $\overline{h}$ are smooth enough to ensure the existence of solutions of the equations (4.4) and (4.7) respectively, for all $t \geq t_o$, r_o, $\int_o \leq p$, $p > 0$.

In case of $g \equiv o$, $\overline{h} \equiv 0$, (4.5) and (4.8) reduce to

$$V(t, x_u(t, t_o, x_o)) \leq V(t_o, x_o) \quad .. (4.9)$$

And

$$V(t, x_u(t, t_o, x_o)) \geq V(t_o, x_o) \quad .. (4.10)$$

respectively.

Now we prove an extension of theorem (2.3) to control systems and its converse as well.

Theorem 4.3 : (Corresponding to theorem (2.3))

Let $\delta_1 \in C(I \times D, R_+)$ and $\delta_2 \in C(D, R_+)$. If there exists a function $V \in r$ such that, for all $(t, x) \in I \times D$

(i) $\delta_2(x) \leq V(t,x) \leq \delta_1(t,x)$ .. (4.11)

And (ii) (4.3) holds with $g \equiv 0$.

Then $\delta_2(x_u(t,t_o,x_o)) \leq \delta_1(t_o, x_o)$,

Proof : Because of (ii), (4.9) follows at once.

i.e. $V(t, x_u(t, t_o, x_o)) \leq V(t_o, x_o)$

Now $\delta_2(x_u(t,t_o,x_o)) \leq V(t, x_u(t, t_o, x_o))$

$\leq V(t, x_o) \leq \delta_1(t_o, x_o)$

Theorem 4.4 : (Corresponding to theorem (2.4))

Let $\delta_i \in C(D,R_+)$, $(i = 2, 3)$. If there exists $V \in r$ such that, for all $(t, x) \in I \times D$,

(i) $\delta_2(x) \leq V(t,x) \leq \delta_3(x)$

And

(ii) (4.3) holds with $g \equiv 0$, them $\delta_2(x_u(t,t_o,x_o)) \leq \delta_3(x_o)$,

Proof : On the same lines as in theorem (4.3) with $\delta_1(t,x) \equiv \delta_3(x)$.

Note : The remarks following theorem (2.4) are quite relevant, in the context of control systems aswell.

Theorem – 4.5 : (Converse of theorem (4.4))

Let the system (4.1) satisfy the condition (4.2). Let the trajectories $x_u(t, t_o, x_o)$ of (4.1) satisfy the estimate -

$\delta_2(x_u(t,t_o,x_o)) \leq \delta_3(x_o)$, $t \geq t_o$, $t_o \in I$ .. (4.13)

Where $\delta_2, \delta_3 \in C(D,R_+)$

And $|\delta_3(x) - \delta_3(y)| \leq k|x-y|$ .. (4.14)

for $x,y \in D$.

Then there exists a V-function satisfying the hypotheses of theorem (4.4).

Proof : For $(t, x) \in I \times D$; define

$$V(t,x) = \left[\inf_{u \in E} \inf_{O \leq T \leq t} (x_u(T,t,x)) \right]$$

The continuity and Lipschitz'u condition $\delta_3$on, besides the continuous dependence on initial conditions of x_u and the compactness of E, show that $V \in r$.

The hypothesis (ii) of theorem (4.4) can be shown to be satisfied by arguments similar to these in theorem (3.1).

Trivially, $V(t,x) \leq \delta_3(x)$.

Also for each $T \in [0, t]$, $\delta_3(x_u(T,t,x)) \geq \delta_2(x)$ due to the uniqueness of the solutions of (4.1) and (4.14).

Thus

$$\inf_{O \leq T \leq t} (x_u(T,t,x)) \geq \delta_2(x)$$

Noting that δ_2 is now independent of u, it follows that $V(t,x) \geq \delta_2(x)$. Hence the proof.

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