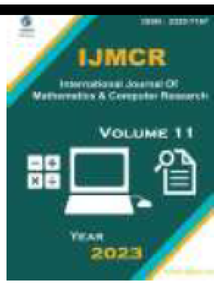
International Journal of Mathematics and Computer Research
ISSN: 2320-7167
Volume 11 Issue 01 January 2023, Page no. – 3113-3131
Index Copernicus ICV: 57.55, Impact Factor: 7.362
DOI: 10.47191/ijmcr/v11i1.03



GROUP ANALYSIS OF ANONLINEAR HEAT-LIKE EQUATION

Joseph Owuor Owino
*
Department of Pure and Applied Mathematics, Faculty of Applied Science and Technology,
Technical University of Kenya, Kenya,

Abstract- *We study a nonlinear heat like equation from a lie symmetry stand point. Heat equation have been employed to study flow of current, information and propagation of heat. The Lie group approach is used on the system to obtain symmetry reductions and the reduced systems studied for exact solutions. Solitary waves have been constructed by use of a linear span of time and space translation symmetries. We also compute conservation laws using multiplier approach and by a conservation theorem due to Ibragimov.*
Keywords-*Invariant, Lie group analysis, Multiplier, Nonlinear heat-like, Stationary solution, Soliton, Symmetry reduction*

1. Introduction

The heat equation [19] has been studied by many scientists. A coupled system of such equations can be used to study polydispersive sedimentation. This coupled system can be used in studying movement of particles in fluids and how gravity affects them. As the particles spread and mix with the fluid, suspensions or colloids form depending on size of particles. We have derived the nonlinear heat-like equation from the classical heat equation

$$\Delta \equiv u_t + \beta u_{xx} = 0, \tag{1}$$

and if we let the thermal conductivity to depend on u and α, that is, $\beta = \alpha u$, Equation (1) becomes

$$\Delta \equiv u_t + \alpha u u_{xx} = 0. \tag{2}$$

The role of this paper is study the nonlinear system 2 by Lie group analysis. We in the first section give preliminaries before studying the system.

2. Preliminaries

This section presents a prelude that is used in what comes after.

Local Lie groups

[6]

We will consider the transformations

$$T_\epsilon : \quad \bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \tag{3}$$

in the Euclidean space $\mathbb{R}^n$ of $x = x^i$ independent variables and $\mathbb{R}^m$ of $u = u^\alpha$ dependent variables. The continuous parameter ϵ ranges from a neighbourhood $\mathcal{N}' \subset \mathcal{N} \subset \mathbb{R}$ of $\epsilon = 0$ for φ^i and ψ^α differentiable and analytic in the parameter ϵ.

Definition 1 *Let $\mathcal{G}$ be a set of transformations in (3) . Then $\mathcal{G}$ is a local Lie group if:*

(i). Given $T_{\epsilon_1}, T_{\epsilon_2} \in \mathcal{G}$, for $\epsilon_1, \epsilon_2 \in \mathcal{N}' \subset \mathcal{N}$, then $T_{\epsilon_1} T_{\epsilon_2} = T_{\epsilon_3} \in \mathcal{G}$, $\epsilon_3 = \phi(\epsilon_1, \epsilon_2) \in \mathcal{N}$ (Closure).

(ii). There exists a unique $T_0 \in \mathcal{G}$ if and only if $\epsilon = 0$ such that $T_\epsilon T_0 = T_0 T_\epsilon = T_\epsilon$(Identity).

(iii). There exists a unique $T_{\epsilon^{-1}} \in \mathcal{G}$ for every transformation $T_\epsilon \in \mathcal{G}$, where $\epsilon \in \mathcal{N}' \subset \mathcal{N}$ and $\epsilon^{-1} \in \mathcal{N}$ such that $T_\epsilon T_{\epsilon^{-1}} = T_{\epsilon^{-1}} T_\epsilon = T_0$ (Inverse).

Remark 1 *The condition (i) is sufficient for associativity of $\mathcal{G}$.*

Prolongations

Consider the system,

$$\Delta_\alpha \left(x^i, u^\alpha, u_{(1)}, \dots, u_{(\pi)}\right) = \Delta_\alpha = 0, \tag{4}$$

where u^α are dependent variables with partial derivatives $u_{(1)} = \{u_i^\alpha\}$, $u_{(2)} = \{u_{ij}^\alpha\}, \dots, u_{(\pi)} = \{u_{i_1 \dots i_\pi}^\alpha\}$, of the first, second, ..., up to the πth-orders. We shall denote by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \tag{5}$$

the total differentiation operator with respect to the variables x^i and δ_i^j, the Kronecker delta. Then

$$D_i(x^j) = \delta_i^j, ', \; u_i^\alpha = D_i(u^\alpha), \; u_{ij}^\alpha = D_j(D_i(u^\alpha)), \; \dots, \tag{6}$$

where u_i^α defined in (6) are differential variables [6].

1. **Prolonged groups** Let $\mathcal{G}$ given by

$$\bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \varphi^i\Big|_{\epsilon=0} = x^i, \; \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \tag{7}$$

where $\Big|_{\epsilon=0}$ means evaluated on $\epsilon = 0$.

Definition 2 *The construction of $\mathcal{G}$ in (7) is equivalent to the computation of infinitesimal transformations*

$$\begin{aligned} \bar{x}^i &\approx x^i + \xi^i(x^i, u^\alpha)\epsilon, \quad \varphi^i\Big|_{\epsilon=0} = x^i, \\ \bar{u}^\alpha &\approx u^\alpha + \eta^\alpha(x^i, u^\alpha)\epsilon, \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \end{aligned} \tag{8}$$

obtained from (3) by a Taylor series expansion of $\varphi^i(x^i, u^\alpha, \epsilon)$ and $\psi^i(x^i, u^\alpha, \epsilon)$ in ϵ about $\epsilon = 0$ and keeping only the terms linear in ϵ, where

$$\xi^i(x^i, u^\alpha) = \frac{\partial \varphi^i(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}, \quad \eta^\alpha(x^i, u^\alpha) = \frac{\partial \psi^\alpha(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}. \tag{9}$$

Remark 2 *By using the symbol of infinitesimal transformations, X, (8) becomes*

$$\bar{x}^i \approx (1 + X)x^i, \quad \bar{u}^\alpha \approx (1 + X)u^\alpha, \tag{10}$$

where

$$X = \xi^i(x^i, u^\alpha)\frac{\partial}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial}{\partial u^\alpha}, \tag{11}$$

is the generator $\mathcal{G}$ in (7).

Remark 3 *The change of variables formula*

$$D_i = D_i(\varphi^j)\bar{D}_j, \tag{12}$$

is employed to construct transformed derivatives from (3). The $\bar{D}_j$ is total differentiation $\bar{x}^i$. As a result

$$\bar{u}_i^\alpha = \bar{D}_i(\bar{u}^\alpha), \ \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha). \tag{13}$$

If we apply the change of variable formula given in (12) on $\mathcal{G}$ given by (7), we get

$$D_i(\psi^\alpha) = D_i(\varphi^j), \ \bar{D}_j(\bar{u}^\alpha) = \bar{u}_j^\alpha D_i(\varphi^j). \tag{14}$$

If we expand (14), we obtain

$$\left(\frac{\partial \varphi^j}{\partial x^i} + u_i^\beta \frac{\partial \varphi^j}{\partial u^\beta}\right)\bar{u}_j^\beta = \frac{\partial \psi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \psi^\alpha}{\partial u^\beta}. \tag{15}$$

The $\bar{u}_i^\alpha$ can be written as functions of $x^i, u^\alpha, u_{(1)}$, meaning that,

$$\bar{u}_i^\alpha = \Phi^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon), \quad \Phi^\alpha\Big|_{\epsilon=0} = u_i^\alpha. \tag{16}$$

Definition 3 *The transformations in (7) and (16) give the first prolongation group $\mathcal{G}^{[1]}$.*

Definition 4 *Infinitesimal transformation of the first derivatives is*

$$\bar{u}_i^\alpha \approx u_i^\alpha + \zeta_i^\alpha \epsilon, \quad \text{where} \quad \zeta_i^\alpha = \zeta_i^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon). \tag{17}$$

Remark 4 *In terms of infinitesimal transformations, $\mathcal{G}^{[1]}$ is given by (8) and (17).*

2. **Prolonged generators**

Definition 5 *By the relation (14) on $\mathcal{G}^{[1]}$ from 3, we obtain [6]*

$$D_i(x^j + \xi^j\epsilon)(u_j^\alpha + \zeta_j^\alpha\epsilon) = D_i(u^\alpha + \eta^\alpha\epsilon), \quad \text{which gives} \tag{18}$$

$$u_i^\alpha + \zeta_j^\alpha\epsilon + u_j^\alpha\epsilon D_i\xi^j = u_i^\alpha + D_i\eta^\alpha\epsilon, \tag{19}$$

and thus

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \tag{20}$$

is the first prolongation formula.

Remark 5 *Analogously, one constructs higher order prolongations [6],*

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u_{i\kappa}^\alpha D_j(\xi^\kappa), \quad \ldots, \quad \zeta_{i_1,\ldots,i_\kappa}^\alpha = D_{i_\kappa}(\zeta_{i_1,\ldots,i_{\kappa-1}}^\alpha) - u_{i_1,i_2,\ldots,i_{\kappa-1}j}^\alpha D_{i_\kappa}(\xi^j). \tag{21}$$

Remark 6 *The prolonged generators of the prolongations $\mathcal{G}^{[1]}, \ldots, \mathcal{G}^{[\kappa]}$ of the group $\mathcal{G}$ are*

$$X^{[1]} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \quad \ldots, X^{[\kappa]} = X^{[\kappa-1]} + \zeta_{i_1,\ldots,i_\kappa}^\alpha \frac{\partial}{\partial \zeta_{i_1,\ldots,i_\kappa}^\alpha}, \quad \kappa \geq 1, \tag{22}$$

for the group generator X in (11).

Group invariants

Definition 6 *A function $\Gamma(x^i, u^\alpha)$ is said to be an invariant of $\mathcal{G}$ of in (3) if*

$$\Gamma(\bar{x}^i, \bar{u}^\alpha) = \Gamma(x^i, u^\alpha). \tag{23}$$

Theorem 1 *A function $\Gamma(x^i, u^\alpha)$ is an invariant of the group $\mathcal{G}$ given by (3) if and only if it solves the following first-order linear PDE: [6]*

$$X\Gamma = \xi^i(x^i, u^\alpha)\frac{\partial\Gamma}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial\Gamma}{\partial u^\alpha} = 0. \tag{24}$$

From Theorem (1), we have the following result.

Theorem 2 *The Lie group $\mathcal{G}$ in (3) [6] has precisely $n-1$ functionally independent invariants and one can take as the basic invariants, the left-hand sides of the first integrals*

$$\psi_1(x^i, u^\alpha) = c_1, \ldots, \psi_{n-1}(x^i, u^\alpha) = c_{n-1}, \tag{25}$$

of the characteristic equations for (24):

$$\frac{\mathrm{d}x^i}{\xi^i(x^i, u^\alpha)} = \frac{\mathrm{d}u^\alpha}{\eta^\alpha(x^i, u^\alpha)}. \tag{26}$$

Symmetry groups

Definition 7 *We define the vector field X (11) as a Lie point symmetry of (4) if the determining equations*

$$X^{[\pi]}\Delta_\alpha\Big|_{\Delta_\alpha=0} = 0, \quad \alpha = 1,\dots,m, \quad \pi \geq 1, \tag{27}$$

are satisfied for the π-th prolongation of X, namely $X^{[\pi]}$.

Definition 8 *The Lie group $\mathcal{G}$ is a symmetry group of (4) if (4) is form-invariant, that is*

$$\Delta_\alpha\left(\bar{x}^i, \bar{u}^\alpha, \bar{u}_{(1)}, \dots, \bar{u}_{(\pi)}\right) = 0. \tag{28}$$

Theorem 3 *The Lie group $\mathcal{G}$ (3) can be constructed from the infinitesimal transformations in (7) by integrating the Lie equations*

$$\frac{\mathrm{d}\bar{x}^i}{\mathrm{d}\epsilon} = \xi^i(\bar{x}^i, \bar{u}^\alpha), \quad \bar{x}^i\Big|_{\epsilon=0} = x^i, \quad \frac{\mathrm{d}\bar{u}^\alpha}{\mathrm{d}\epsilon} = \eta^\alpha(\bar{x}^i, \bar{u}^\alpha), \quad \bar{u}^\alpha\Big|_{\epsilon=0} = u^\alpha. \tag{29}$$

Lie algebras

Definition 9 *A vector space $\mathcal{V}_r$ of operators [6] X (11) is a Lie algebra if for any $X_i, X_j \in \mathcal{V}_r$,*

$$[X_i, X_j] = X_iX_j - X_jX_i, \tag{30}$$

is in $\mathcal{V}_r$ for all $i, j = 1, \dots, r$.

Remark 7 *The commutator is bilinear, skew symmetric and admits to the Jacobi identity [6].*

Theorem 4 *The set of solutions of (27) forms a Lie algebra[6].*

Exact solutions

The methods of (G'/G)-expansion method [22], Extended Jacobi elliptic function expansion [23] and Kudryashov [20] are usually applied after symmetry reductions.

Conservation laws

[6]

Fundamental operators

Definition 10 *The Euler-Lagrange operator* $\frac{\delta}{\delta u^\alpha}$ *is*

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{\kappa \geq 1} (-1)^\kappa D_{i_1}, \dots, D_{i_\kappa} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_\kappa}}, \tag{31}$$

and the Lie- Bäcklund operator in abbreviated form [6] is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \dots. \tag{32}$$

Remark 8 *The Lie- Bäcklund operator (32) in its prolonged form is*

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{\kappa \geq 1} \zeta_{i_1 \dots i_\kappa} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_\kappa}}, \tag{33}$$

for

$$\zeta^\alpha_i = D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \qquad \dots, \zeta^\alpha_{i_1 \dots i_\kappa} = D_{i_1 \dots i_\kappa}(W^\alpha) + \xi^j u^\alpha_{j i_1 \dots i_\kappa}, \quad j = 1, \dots, n. \tag{34}$$

and the Lie characteristic function

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{35}$$

Remark 9 *The characteristic form of Lie- Bäcklund operator (33) is*

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + D_{i_1 \dots i_\kappa}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_\kappa}}. \tag{36}$$

The method of multipliers

Definition 11 *A function* $\Lambda^\alpha \left(x^i, u^\alpha, u_{(1)}, \dots\right) = \Lambda^\alpha$, *is a multiplier of (4) if [22]*

$$\Lambda^\alpha \Delta_\alpha = D_i T^i, \tag{37}$$

where $D_i T^i$ *is a divergence expression.*

Definition 12 *To find the multipliers* Λ^α, *one solves the determining equations (38) [21],*

$$\frac{\delta}{\delta u^\alpha}(\Lambda^\alpha \Delta_\alpha) = 0. \tag{38}$$

Ibragimov's conservation theorem

The technique [6] enables one to construct conserved vectors associated with each Lie point symmetry of (4).

Definition 13 *The adjoint equations of (4) are*

$$\Delta_{\alpha}^{*}\left(x^{i},u^{\alpha},v^{\alpha},\ldots,u_{(\pi)},v_{(\pi)}\right)\equiv\frac{\delta}{\delta u^{\alpha}}(v^{\beta}\Delta_{\beta})=0, \tag{39}$$

for a new dependent variable v^{α}*.*

Definition 14 *The Formal Lagrangian* $\mathcal{L}$ *of (4) and its adjoint equations (39) is [6]*

$$\mathcal{L}=v^{\alpha}\Delta_{\alpha}(x^{i},u^{\alpha},u_{(1)},\ldots,u_{(\pi)}). \tag{40}$$

Theorem 5 *Every infinitesimal symmetry* X *of (4) leads to conservation laws [6]*

$$D_{i}T^{i}\Big|_{\Delta_{\alpha}=0}=0, \tag{41}$$

where the conserved vector

$$\begin{aligned}T^{i}=\xi^{i}\mathcal{L}+W^{\alpha}\left[\frac{\partial\mathcal{L}}{\partial u_{i}^{\alpha}}-D_{j}\left(\frac{\partial\mathcal{L}}{\partial u_{ij}^{\alpha}}\right)+D_{j}D_{k}\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^{\alpha}}\right)-\ldots\right]+\\ D_{j}(W^{\alpha})\left[\frac{\partial\mathcal{L}}{\partial u_{ij}^{\alpha}}-D_{k}\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^{\alpha}}\right)+\ldots\right]+D_{j}D_{k}(W^{\alpha})\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}^{\alpha}}-\ldots\right].\end{aligned} \tag{42}$$

3. Main results

3.1. Lie point symmetries of nonlinear heat-like Equation(2)

We start first by computing Lie point symmetries of the system (2), which admits the one-parameter Lie group of transformations with infinitesimal generator

$$X=\tau(t,x,u)\frac{\partial}{\partial t}+\xi(t,x,u)\frac{\partial}{\partial x}+\eta(t,x,u)\frac{\partial}{\partial u} \tag{43}$$

if and only if

$$X^{[2]}\Delta\Big|_{\Delta=0}=0, \tag{44}$$

By definition

$$X^{[2]}=\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+\zeta_{1}\frac{\partial}{\partial u_{t}}+\zeta_{22}\frac{\partial}{\partial u_{xx}} \tag{45}$$

where

$$
\begin{aligned}
\zeta_1 &= \eta_t + u_t(\eta_u - \tau_t) + u_x(-\xi_t) + u_t u_x(-\xi_u) + u_t^2(-\tau_u),\\
\zeta_2 &= \eta_x + u_x(\eta_u - \xi_x) + u_t(-\tau_x) + u_t u_x(-\tau_u) + u_x^2(-\xi_u),\\
\zeta_{22} &= \eta_{xx} + u_x(2\eta_{xu} - \xi_{xx}) + u_t(-\tau_{xx}) + u_t u_x(-2\tau_{xu}) + u_t u_{xx}(-\tau_u)\\
&\quad + u_{tx}(-2\tau_x) + u_{xx}(\eta_u - 2\xi_x) + u_x u_{tx}(-2\tau_u) + u_x u_{xx}(-3\xi_u) + u_x^2(\eta_{uu} - 2\xi_{xu})\\
&\quad + u_t u_x^2(-\tau_{uu}) + u_x^3(-\xi_{uu})
\end{aligned}
\tag{46}
$$

Using the prolonged generator (45) in Equation (44) gives

$$
\alpha\eta u_{xx} + \zeta_1 + \alpha u\zeta_{22} = 0. \tag{47}
$$

If we substitute for ζ_1 and ζ_{22} in the determining Equation (44), we obtain the following;

$$
\begin{aligned}
&u_{xx}(\alpha\eta) + \{\eta_t + u_t(\eta_u - \tau_t) + u_x(-\xi_t) + u_t u_x(-\xi_u) + u_t^2(-\tau_u)\}\\
&\alpha u\{\eta_{xx} + u_x(2\eta_{xu} - \xi_{xx}) + u_t(-\tau_{xx}) + u_t u_x(-2\tau_{xu}) + u_t u_{xx}(-\tau_u)\\
&+ u_{tx}(-2\tau_x) + u_{xx}(\eta_u - 2\xi_x) + u_x u_{tx}(-2\tau_u) + u_x u_{xx}(-3\xi_u) + u_x^2(\eta_{uu} - 2\xi_{xu})\\
&+ u_t u_x^2(-\tau_{uu}) + u_x^3(-\xi_{uu})\}\big|_{u_{xx}=-\frac{u_t}{\alpha u}} = 0
\end{aligned}
\tag{48}
$$

Now replacing u_{xx} by $-\frac{u_t}{\alpha u}$ in the Equation (48), we obtain,

$$
\begin{aligned}
&\left[-\frac{u_t}{\alpha u}\right](\alpha\eta) + \left\{\eta_t + u_t(\eta_u - \tau_t) + u_x(-\xi_t) + u_t u_x(-\xi_u) + u_t^2(-\tau_u)\right\}\\
&\alpha u\left\{\eta_{xx} + u_x(2\eta_{xu} - \xi_{xx}) + u_t(-\tau_{xx}) + u_t u_x(-2\tau_{xu}) + u_t\left[-\frac{u_t}{\alpha u}\right](-\tau_u)\right.\\
&+ u_{tx}(-2\tau_x) + \left[-\frac{u_t}{\alpha u}\right](\eta_u - 2\xi_x) + u_x u_{tx}(-2\tau_u) + u_x\left[-\frac{u_t}{\alpha u}\right](-3\xi_u) + u_x^2(\eta_{uu} - 2\xi_{xu})\\
&\left.+ u_t u_x^2(-\tau_{uu}) + u_x^3(-\xi_{uu})\right\} = 0
\end{aligned}
\tag{49}
$$

or

$$
\begin{aligned}
&\eta_t + \alpha u\eta_{xx} + u_t\big(2\xi_x - \alpha u\tau_{xx} - \frac{\eta}{u} - \tau_t\big) + u_x(-\xi_t + \alpha u\{2\eta_{xu} - \xi_{xx}\})\\
&+ u_t u_x(2\xi_u - 2\alpha u\tau_{xu}) + u_{tx}(-2\alpha u\tau_x) + u_x u_{tx}(-2\alpha u\tau_u)\\
&+ u_x^2(\alpha u\{\eta_{uu} - 2\xi_{xu}\}) + u_t u_x^2(-\alpha u\tau_{uu}) + u_x^3(-\alpha u\xi_{uu}) = 0
\end{aligned}
\tag{50}
$$

Since the functions τ, ξ and η depend only on t, x and u and are independent of the derivatives of u, we can then split the above equation on the derivatives of u and obtain

$$\tau_x = \tau_u = \xi_u = \eta_{uu} = 0, \tag{51}$$
$$u\{2\xi_x - \tau_t\} - \eta = 0, \tag{52}$$
$$\alpha u\{2\eta_{xu} - \xi_{xx}\} - \xi_t = 0 \tag{53}$$
$$\eta_t + \alpha u \eta_{xx} = 0. \tag{54}$$

From Equation (51–52), it is evident that

$$\tau = \tau(t), \tag{55}$$
$$\xi = \xi(t, x), \tag{56}$$
$$\eta = u\{2\xi_x - \tau_t\}. \tag{57}$$

Using these functions in Equation (53),

$$3\alpha\xi_{xx}u - \xi_t = 0 \tag{58}$$

and separating on powers of u gives the system

$$\begin{aligned} u :& \xi_{xx} = 0, \\ u^0 :& \xi_t = 0. \end{aligned} \tag{59}$$

Equation (59) is necessary and sufficient for

$$\xi(t, x) = c_1 x + c_2. \tag{60}$$

Now using Equation (54), we have

$$\tau_{tt} = 0, \tag{61}$$

which can be integrated twice with respect to t to give us

$$\tau(t) = c_3 t + c_4. \tag{62}$$

and finally;

$$\tau = c_3 t + c_4, \tag{63}$$
$$\xi = c_1 x + c_2, \tag{64}$$
$$\eta = (2c_1 - c_3)u. \tag{65}$$

We have obtained a four dimensional Lie algebra of symmetries spanned by

$$X_1 = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}, \tag{66}$$
$$X_2 = \frac{\partial}{\partial x}, \tag{67}$$
$$X_3 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \tag{68}$$
$$X_4 = \frac{\partial}{\partial t}. \tag{69}$$

3.2. Commutator Table for Symmetries

We evaluate the commutation relations for the symmetry generators. By definition of Lie bracket [23], for example, we have that

$$[X_2, X_4] = X_2X_4 - X_4X_2 = \left(\frac{\partial}{\partial x}\frac{\partial}{\partial t}\right) - \left(\frac{\partial}{\partial t}\frac{\partial}{\partial x}\right) = 0. \tag{70}$$

Remark 10 *The remaining commutation relations are obtained analogously. We present all commutation relations in Table (1) below.*

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	-X_2	0	0
X_2	X_2	0	0	0
X_3	0	0	0	-X_4
X_4	0	0	X_4	0

Table 1. A commutator table for Lie algebra of a nonlinear heat-like equation.

3.3. Group Transformations

The corresponding one-parameter group of transformations can be determined by solving the Lie equations [24]. Let T_{ϵ_i} be the group of transformations for each $X_i, i = 1, 2, 3, 4$. We display how to obtain T_{ϵ_i} from X_i by finding one-parameter group for the infinitesimal generator $X_2 = \frac{\partial}{\partial x}$. In particular, we have the Lie equations

$$\begin{aligned} \frac{d\bar{t}}{d\epsilon_2} &= 0, \quad \bar{t}\Big|_{\epsilon_2=0} = t, \\ \frac{d\bar{x}}{d\epsilon_2} &= 1, \quad \bar{x}\Big|_{\epsilon_2=0} = x, \\ \frac{d\bar{u}}{d\epsilon_2} &= 0, \quad \bar{u}\Big|_{\epsilon_2=0} = u. \end{aligned} \tag{71}$$

Solving the system (71) one obtains,

$$\bar{t} = t, \quad \bar{x} = x + \epsilon_2, \quad \bar{u} = u, \tag{72}$$

and hence the one-parameter group T_{ϵ_2} corresponding to the operator X_2 is

$$T_{\epsilon_2}: \quad (\bar{t}, \bar{x}, \bar{u}) = (t, x + \epsilon_2, u). \tag{73}$$

All the five one-parameter groups are presented below :

$$\begin{aligned} T_{\epsilon_1}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (t, xe^{\epsilon_1}, ue^{2\epsilon_1}), \\ T_{\epsilon_2}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (t, x + \epsilon_2, u), \\ T_{\epsilon_3}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (te^{\epsilon_3}, x, ue^{\epsilon_3}), \\ T_{\epsilon_4}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (t + \epsilon_4, x, u). \end{aligned} \tag{74}$$

3.4. Symmetry transformations

The symmetries we have obtained can be used to transform special exact solutions of the nonlinear heat-like equation into new solutions. The Lie group analysis vouches for fundamental ways of constructing exact solutions of PDEs, that is, group transformations of known solutions and construction of group-invariant solutions. We will illustrate these methods with examples. If $\bar{u} = g(\bar{t}, \bar{x})$ is a solution of equation (2)

$$\phi(t, x, u, \epsilon) = g(f_1(t, x, u, \epsilon), f_2(t, x, u, \epsilon)), \tag{75}$$

is also a solution. The one parameter groups dictate to the following generated solutions:

$$\begin{aligned} T_{\epsilon_1} : u =& g\big(t, xe^{\epsilon_1}\big)e^{-2\epsilon_1}, \\ T_{\epsilon_2} : u =& g(t, x + \epsilon_2), \\ T_{\epsilon_3} : u =& g(te^{\epsilon_3}, x)e^{-\epsilon_3}, \\ T_{\epsilon_4} : u =& g(t + \epsilon_4, x). \end{aligned} \tag{76}$$

3.5. Construction of Group-Invariant Solutions

Now we compute the group invariant solutions of a non-linear heat-like equation.

(i). $X_1 = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}$

The associated Lagrangian equations

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{x} = \frac{\mathrm{d}u}{2u}, \tag{77}$$

yield two invariants, $J_1 = t$ and $J_2 = \frac{2u}{x}$. Thus using $J_2 = \Phi(J_1)$, we have

$$u(t, x) = \frac{x\Phi(t)}{2}. \tag{78}$$

The derivatives are given by :

$$\begin{aligned} u_t =& \frac{x\Phi'(t)}{2}, \\ u_x =& \frac{\Phi(t)}{2}, \\ u_{xx} =& 0. \end{aligned}$$

If we substitute these derivatives into Equation (2) , we obtain the first order ordinary differential equation

$$\frac{x\Phi'(t)}{2} = 0. \tag{79}$$

For any $x \neq 0$, Equation (79) is equivalent to

$$\Phi'(t) = 0, \tag{80}$$

whose solution is $\Phi(t) = C_1$. Hence the solution for Equation (2) obtained from this symmetry is

$$u(t,x) = \frac{C_1 x}{2}, \quad C_1 \in \mathbb{R}. \tag{81}$$

(ii). $X_2 = \frac{\partial}{\partial x}$

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{0}. \tag{82}$$

This gives the constants $J_1 = t$ and $J_2 = u$, giving the solution

$$u(t,x) = f(t). \tag{83}$$

We obtain the derivatives as follows:

$$u_t = f_t(t), \tag{84}$$

$$u_x = 0 = u_{xx}. \tag{85}$$

If we substitute the above derivatives in Equation (2), we obtain the first order ordinary differential equation

$$f_t(t) = 0, \tag{86}$$

whose solution is

$$f(t) = C_2, \quad C_2 \in \mathbb{R}. \tag{87}$$

Hence the solution for Equation (2) obtained from this symmetry is

$$u(t,x) = C_2, \quad C_2 \in \mathbb{R}. \tag{88}$$

(iii). $X_3 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$

The Lagrangian system associated with the operator X_3 is

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}x}{0} = -\frac{\mathrm{d}u}{u}, \tag{89}$$

whose invariants are $J_1 = x$ and $J_2 = tu$. So, $u(t,x) = \frac{\psi(x)}{t}$ is the group-invariant solution. The derivatives are namely;

$$u_t = -\frac{\psi(x)}{t^2}, \tag{90}$$

$$u_x = \frac{\psi'(x)}{t} \tag{91}$$

$$u_{xx} = \frac{\psi''(x)}{t} \tag{92}$$

Substituting of $u = \psi(x)$ into (2) yields

$$\alpha\psi(x)\psi''(x) - \psi(x) = 0. \tag{93}$$

For any $x \in \mathbb{R}$ such that $\psi(x) = 0$, we have the trivial solution. Otherwise, for $x \in \mathbb{R}$ such that $\psi(x) \neq 0$, Equation (93) is equivalent to

$$\alpha\psi''(x) - 1 = 0, \tag{94}$$

where $\alpha \neq 0$ lest we have a contradiction.

The solution to Equation (94) is

$$\psi(x) = \frac{x^2}{2\alpha} + C_3 x + C_4, \quad (0,0) \neq (C_3, C_4) \in \mathbb{R}^2. \tag{95}$$

Hence the associated group-invariant solution is

$$u(t,x) = \frac{1}{t}\left\{\frac{x^2}{2\alpha} + C_3 x + C_4\right\}, \qquad (0,0,0,0) \neq (t, \alpha, C_3, C_4) \in \mathbb{R}^4. \tag{96}$$

(iv). $X_4 = \frac{\partial}{\partial t}$.
Characteristic equations associated to the operator X_4 are

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{0}, \tag{97}$$

yields$J_1 = x$ and $J_2 = u$. As a result, the group-invariant solution of (2) for this case is $J_2 = \delta(J_1)$, for δ an arbitrary function. That is,

$$u(t,x) = \delta(x). \tag{98}$$

Substitution of the value of u from Equation (98) into Equation (2) yields a second order ordinary differential equation

$$\alpha\delta(x)\delta''(x) = 0. \tag{99}$$

. Whenever, $(\alpha, \delta(x)) = (0,0)$ for any roots of $\delta(x)$, we have the trivial solution. Otherwise Equation (99) is satisfied by

$$\delta(x) = C_5 x + C_6, \quad (0,0) \neq (C_5, C_6) \in \mathbb{R}^2. \tag{100}$$

Consequently, the group-invariant solution under X_4 is

$$u(t,x) = C_5 x + C_6 \quad (0,0,0) \neq (\alpha, C_5, C_6) \in \mathbb{R}^3. \tag{101}$$

3.6. Soliton

We obtain a traveling wave solution of the non-linear heat-like Equation(2) by considering a linear span of the symmetries X_2 and X_4, namely, [22]

$$X = cX_2 + X_4 = c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad \text{for some constant } c. \tag{102}$$

The characteristic equations are

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{c} = \frac{\mathrm{d}u}{0}. \tag{103}$$

We get two invariants, $J_1 = x - ct$ and $J_2 = u$. So the group-invariant solution is

$$u(t, x) = \Omega(x - ct), \tag{104}$$

for some arbitrary function Ω and c the velocity of the wave. The resulting derivatives are

$$u_t = -c\Omega'(x - ct), \tag{105}$$
$$u_x = \Omega'(x - ct), \tag{106}$$
$$u_{xx} = \Omega''(x - ct). \tag{107}$$

Substitution of u into (2) yields a second order nonlinear ordinary differential equation

$$-c\Omega' + \Omega\Omega'' = 0, \quad \Omega = \Omega(x - ct). \tag{108}$$

Integration of Equation (108) with respect to Ω' yields

$$-c\frac{\Omega'^2}{2} + \Omega\Omega' = 0, \tag{109}$$

where we have taken the zero on the right hand side as the constant of integration. If the function Ω is non constant, that is, $\Omega' \neq 0$, then we have from Equation (109) that

$$-c\Omega' + 2\Omega = 0. \tag{110}$$

By letting $\xi = x - ct$, we have

$$\frac{\mathrm{d}\Omega}{\Omega} = \frac{2\mathrm{d}\xi}{c}, \tag{111}$$

whose integration yields

$$\Omega = C_8 e^{\frac{2\xi}{c}}, \quad \xi = x - ct, \ C_8 = e^{C_7}. \tag{112}$$

Clearly, we have the one-soliton solution as

$$u(t, x) = C_8 e^{\frac{2(x-ct)}{c}}, \quad (0, 0) \neq (c, C_8) \in \mathbb{R}^2. \tag{113}$$

4. Conservation laws of equation (2)

We will employ multipliers in the construction of conservation laws.

4.1. The multipliers

We make use of the Euler-Lagrange operator defined as defined in [24] to look for a zeroth order multiplier $\Lambda = \Lambda(t, x, u)$. The resulting determining equation for computing Λ is

$$\frac{\delta}{\delta u}\left[\Lambda\{u_t + \alpha u u_{xx}\}\right] = 0. \tag{114}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} + \dots \tag{115}$$

Expansion of Equation (114) yields

$$\Lambda_u(u_t + \alpha u u_{xx}) + \alpha\Lambda u_{xx} - D_t(\Lambda) + \alpha D_x^2(\Lambda u) = 0. \tag{116}$$

Invoking the total derivatives

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots, \tag{117}$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \dots. \tag{118}$$

on Equation (116) produces

$$2\alpha(\Lambda_x + u\Lambda_{xu})u_x + 2\alpha(\Lambda + u\Lambda_u)u_{xx} + \alpha(2\Lambda_u + u\Lambda_{uu})u_x^2 - \Lambda_t + \alpha u\Lambda_{xx} = 0 \tag{119}$$

Splitting Equation (119) on derivatives of u produces an overdetermined system of four partial differential
equations, namely,

$$u_x : \Lambda_x + u\Lambda_{xu} = 0, \tag{120}$$

$$u_{xx} : \Lambda + u\Lambda_u = 0, \tag{121}$$

$$u_x^2 : 2\Lambda_u + u\Lambda_{uu} = 0, \tag{122}$$

$$\text{rest} : -\Lambda_t + \alpha u\Lambda_{xx} = 0 \tag{123}$$

Note that Equation (121) is sufficient for Equations (122) and (120) . We can write Equation (121) as

$$\frac{\mathrm{d}\Lambda}{\Lambda} = -\frac{\mathrm{d}u}{u} \tag{124}$$

if and only if

$$\frac{\mathrm{d}\Lambda}{\Lambda} = \mathrm{d}u, \tag{125}$$

giving the solution

$$\Lambda = \frac{c_1}{u}, \quad c_1 \in \mathbb{R}. \tag{126}$$

Essentially, we extract the one multiplier

$$\Lambda_1 = \frac{1}{u}. \tag{127}$$

Remark 11 *Recall that a multiplier Λ for Equation(2) has the property that for the density $T^t = T^t(t, x, u)$ and flux $T^x = T^x(t, x, u, u_x)$,*

$$\Lambda\left(u_t + \alpha u u_{xx}\right) = D_t T^t + D_x T^x. \tag{128}$$

We derive a conservation law corresponding to the multiplier.

4.1.1. Conservation law for the multiplier $\Lambda = \frac{1}{u}$

Expansion of equation (128) gives

$$\frac{1}{u}\{u_t + \alpha u u_{xx}\} = T^t_t + u_t T^t_u + T^x_x + u_x T^x_u + u_{xx} T^x_{u_x}. \tag{129}$$

Splitting Equation (129) on the second derivative of u yields

$$u_{xx} \;:\; T^x_{u_x} = \alpha, \tag{130}$$

$$\text{Rest} \;:\; \frac{1}{u}\{u_t\} = T^t_t + T^t_u u_t + T^x_x + T^x_u u_x. \tag{131}$$

The integration of Equation (130) with respect to u_x gives

$$T^x = \alpha u_x + A(t, x, u). \tag{132}$$

Substituting the expression of T^x from (132) into Equation (131) we get

$$\frac{1}{u}\{u_t\} = T^t_t + T^t_u u_t + A_x + A_u u_x. \tag{133}$$

which splits on first derivatives of u, to give

$$u_x \;:\; A_u = -0, \tag{134}$$

$$u_t \;:\; T^t_u = \frac{1}{u}, \tag{135}$$

$$\text{Rest} \;:\; 0 = T^t_t + A_x. \tag{136}$$

Integrating equations (134) and (135) with respect to u manifests that

$$T^t = \ln u + C(t, x), \tag{137}$$

$$A = B(t, x) \tag{138}$$

By substituting the obtained functions into Equation (136), we have

$$C_t(t,x) + B_x(t,x) = 0. \tag{139}$$

Since $C(t,x)$ and $B(t,x)$ contribute to the trivial part of the conservation law, we take $C(t,x) = B(t,x) = 0$ and obtain the conserved quantities

$$T^t = \ln u, \tag{140}$$
$$T^x = \alpha u_{xx}, \tag{141}$$

from which the conservation law corresponding to the multiplier $\Lambda = \frac{1}{u}$ is given by

$$D_t(\ln u) + D_x(\alpha u_x\}) = 0. \tag{142}$$

Remark 12 *It can be shown that the two sets of conserved quantities are conservation laws. Furthermore, since $u \neq 0$ and $\frac{1}{u} \neq 0$, the nonlinear heat-like equation is itself a conserved quantity.*

5. Conclusion

In this manuscript, four dimensional Lie algebra of Lie point symmetries has been applied to study a nonlinear heat-like equation. A commutator table has been constructed for the obtained Lie algebra. We have also used symmetry reductions to compute exact group-invariant solutions, including a soliton. A conservation law has been derived by the method of zeroth order multipliers.

Acknowledgement

The author thanks referees and the editor for their careful reading and comments.

Author's contribution

The author wrote the article as a scholarly duty and passion to disseminate mathematical research and hereby declares that there is no conflict of interest.

References

[1] Arrigo, D. J. *Symmetry analysis of differential equations: an introduction*. John Wiley & Sons, 2015.

[2] Bluman, G. and Anco, S. *Symmetry and integration methods for differential equations*, volume 154. Springer Science & Business Media, 2008.

[3] Bluman, G. W. and Kumei, S. *Symmetries and differential equations*, volume 81. Springer Science & Business Media, 1989.

[4] Bluman, G. W., Cheviakov, A. F., and Anco, S. C. *Applications of symmetry methods to partial differential equations*, volume 168. Springer, 2010.

[5] Cannon, J. R. *The one-dimensional heat equation.* Number 23. Cambridge University Press, 1984.

[6] Ibragimov, N. *Elementary Lie group analysis and ordinary differential equations.* Wiley, 1999.

[7] Ibragimov, N. H. *CRC handbook of Lie group analysis of differential equations*, volume 1-3. CRC press, 1994.

[8] Ibragimov, N. H. *Selected works. Volume 1-4.* ALGA publications, Blekinge Institute of Technology, 2006-2009.

[9] Ibragimov, N. H. A new conservation theorem. *Journal of Mathematical Analysis and Applications*, 333(1):311–328, 2007.

[10] Ibragimov, N. H. *A Practical Course in Differential Equations and Mathematical Modelling: Classical and New Methods. Nonlinear Mathematical Models. Symmetry and Invariance Principles.* World Scientific Publishing Company, 2009.

[11] Khalique, C. M. and Abdallah, S. A. Coupled Burgers equations governing polydispersive sedimentation; a lie symmetry approach. *Results in Physics*, 16, 2020.

[12] LeVeque, R. J. *Numerical methods for conservation laws*, volume 3. Springer, 1992.

[13] Lie, S. *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen.* BG Teubner, 1891.

[14] Mhlanga, I. and Khalique, C. Travelling wave solutions and conservation laws of the Korteweg-de Vries-Burgers Equation with Power Law Nonlinearity. *Malaysian Journal of Mathematical Sciences*, 11:1–8, 2017.

[15] Noether, E. Invariant variations problem. Nachr. Konig. Gissel. Wissen, Gottingen. *Math. Phys. Kl*, pages 235–257, 1918.

[16] Olver, P. J. *Applications of Lie groups to differential equations*, volume 107. Springer Science & Business Media, 1993.

[17] Ovsyannikov, L. *Lectures on the theory of group properties of differential equations.* World Scientific Publishing Company, 2013.

[18] Owino, J. O. A group approach to exact solutions and conservation laws of classical burger's equation. *International Journal of Mathematics And Computer Research*, 10(9):2894–2909, 2022.

[19] Owino, J. O. Group analysis on one-dimensional heat equation. *International Journal of Advanced Multidisciplinary Research and Studies*, 2(5):525–540, 2022.

[20] Owino, J. O. Group invariant solutions and conserved vectors for a special kdv type equation. *International Journal of Advanced Multidisciplinary Research and Studies*, 2(5):9–26, 2022.

[21] Owino, J. O. An application of lie point symmetries in the study of potential burger's equation. *International Journal of Advanced Multidisciplinary Research and Studies*, 2(5):191–207, 2022.

[22] Owino, J. O. and Okelo, B. Lie group analysis of a nonlinear coupled system of korteweg-de vries equations. *European Journal of Mathematical Analysis*, 1: 133–150, 2021.

[23] Owuor, J. Conserved quantities of a nonlinear coupled system of korteweg-de vries equations. *International Journal of Mathematics And Computer Research*, 10(5):2673–2681, 2022.

[24] Owuor, J. Exact symmetry reduction solutions of a nonlinear coupled system of korteweg-de vries equations. *International Journal of Advanced Multidisciplinary Research and Studies*, 2(3):76–87, 2022.

[25] P.E, H. *Symmetry methods for differential equations: a beginner's guide*, volume 22. Cambridge University Press, 2000.

[26] Wazwaz, A.-M. *Partial differential equations and solitary waves theory*. Springer Science & Business Media, 2010.