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Serendipity Fixed Point with Respect to PGA Contraction

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ABSTRACT
P L Powar and GRK Sahu [14] noticed that the fixed point may not exist for some self maps T defined on a set X. To deal with such cases the idea of Serendipity fixed point had been introduced by P L Powar in 2015. Considering the lighter concept of completeness viz. weak completeness, the existence and uniqueness of Serendipity fixed point had been established under contraction condition. In this paper we established the result for existence of serendipity fixed point under the PGA contraction [15] defined by Akhilesh Pathak in 2018. This idea may be useful in solving or simplifying the problems which are totally based on fixed point theory.

KEYWORDS: Serendipity fixed point, Weak convergence, Weak Cauchy sequence, Weak complete metric space, Dual contraction, Altering distance function, Dual weak contraction, PGA contraction.
AMS Classification: 47H10, 54H25.

1. INTRODUCTION

Henri Poincare [13] in 1886, proved a result which is equivalent to the Brower's fixed point theorem [5]. The field of Fixed point theory has completed its century in 2012.Many of the researchers contributed a high level work in this particular area. Most of the problems in pure and applied mathematics reduce to a problem of common fixed point of some self mapping defined on metric space. The study of a fixed point and common fixed point satisfying different contractive conditions has been explored by many mathematicians. Recently, the fixed point theorems involving the concept of altering distance functions have been initiated by Delbosco in 1967(cf. [7]) and this was further studied by Khan et al. in 1984 (cf. [10]).The initial concept of altering distance function was generalized by Choudhary [6] and then the concept was extensively used by many researchers (cf. [2], [3], [8], [9], [14]-[18]).

The idea of weak complete metric space involving the weak convergence of weak Cauchy sequence has been described in [11]. The authors have studies the concept of dual space (cf. [11]) and applied it on the fixed point theory. If f is real or complex valued function, then it was interesting to observe that even if for some $x \in X$, $Tx \neq x$ but $f(Tx) = f(x)$. Idea of Serendipity fixed point had been introduced by P L Powar in 2015. In this paper we established the result for existence of serendipity fixed point under the PGA contraction defined by Akhilesh Pathak in 2018.

2. PRELIMINARIES

In order to establish our results, we require the following definitions and results:

2.1 Definition Let $\{x_n\}$ be a sequence in a normed linear space X. $\{x_n\}$ is said to converge weakly to x in X if for every linear functional $f \in X^*$ (dual space of X)

$$f(x_n) \to f(x), \text{ as } n \to \infty$$

i.e. for every $\in > 0$ there exists a natural number $n_o \in N$ such that

$$\left| f(x_n) - f(x) \right| < \in, \ \forall \ n \geq n_o \text{ and for all } f \in X^*$$

In this case, we write $x_n \xrightarrow{w} x$ and x is called the weak limit of the sequence $\{x_n\}$ (cf. [11]).

2.1 Remark Weak convergence does not imply strong convergence in general. In the finite dimensional space the distinction between strong and weak convergence disappears completely (Theorem 4.8-4(c) of [11]).

2.1 Example Consider the Hilbert space $L_2[0, 2\pi\ \ ]$ which is the space of the square-integrable functions on the interval [0,2□].The inner product on the space is defined by

$$\prec f, g \succ = \int_0^{2\pi} f(x).\, g(x)dx$$

The sequence of functions f_1, f_2, f_3,… defined by

$$f_m\,(x) \;=\; sin(mx)$$

converges weakly to the zero function in $L_2[0, 2\pi\ \ ]$, as the integral

$$\int_0^{2\pi} \sin(mx)\,.\,g(x)dx$$

tends to zero for any square-integrable function g on $[0, 2\pi]$ when m tends to infinity,

i.e. $\prec f_n, g \succ \longrightarrow \prec 0, g \succ = \ 0$

However, $sin(mx)$ does not tends to 0 , as $m \to \infty$.

2.2 Definition Weak Cauchy sequence in a real or complex normed space X is a sequence $\{x_n\}$ in X such that for every $f \in X^*$ the sequence $\{f(x_n)\}$ is Cauchy in R or C, respectively. i.e. if for every $\in > 0$, there exists a natural number $n_o \in N$ such that

$$\left| f\left(x_n\right) - f\left(x\right) \right| < \in, \text{ for all n, m} \geq n_o \text{ and for all } f \in X^* \text{ (cf. [11]).}$$

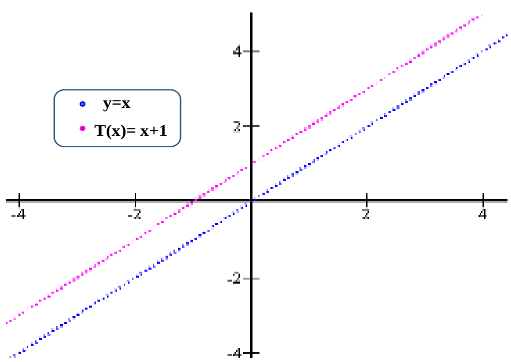
2.3 Definition A normed space X is said to be weakly complete if each weak Cauchy sequence in X converges weakly in X (cf. [11]).

2.4 Definition A fixed point of a mapping T: X→X of a set X into itself is an $x \in X$ which is mapped onto itself, that is,

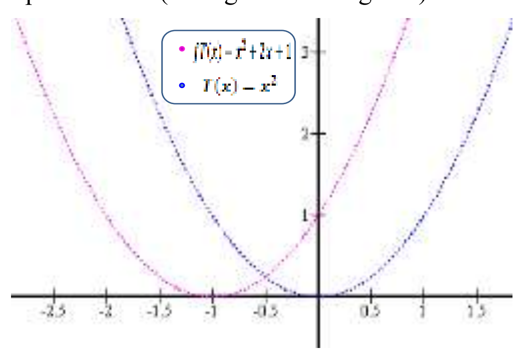
Tx = x.

2.5 Definition A Serendipity fixed point of a mapping T: X→X of a set X into itself is a point $x \in X$ such that there exists real or complex valued function f on X satisfying the condition f(Tx) = f(x).

2.2 Example Let X be the set of all rational numbers, let T: X→X defined by T(x) = x+1 and f: X→R defined by $f(x) = x^2$.
Clearly T has no fixed point but x = − ½ is the only Serendipity fixed point of T (cf. Fig. 2.1 and Fig. 2.2).



T has no fixed point
Fig. 2.1



$x = -1/2$ is Serendipity fixed point of T
Fig. 2.2

3. MAIN RESULTS

In this section , the basic result of fixed point theorem (cf.[4]) has been proved by considering the idea of dual contraction map. Throughout our discussion, we consider X as an infinite dimensional normed linear space.

3.1 Theorem Let X be a normed linear space, T be a selfmap on X and 'd' be the metric defined on R. If $f \in$ X*(real dual space of X) and f is a bijection map satisfying the condition

$$d(fTx, fTy) \leq \ c.\max\{G(fx, fy), d(fx, fy)\}.$$

$$Where\ G(x, y) = \left\{\frac{d(x, Tx)d(x, Ty) + \ d(y, Ty)d(y, Tx)}{\max\{d(x, Ty), d(Tx, y)}\right\}\ if\ max\{d(x, Ty), d(Tx, y)\}\ 0$$

$$0\ if\ max\{d(x, Ty), d(Tx, y)\} \ = \ 0, for\ all\ x, y\ in\ X \qquad (3.1)$$

then T has a unique Serendipity fixed point.

Proof Let x be an arbitrary point in X, and $\{x_n\}$ be a sequence of point in X such that

$$Tx_n = x_{n+1}, \quad \forall \; n \in \mathrm{N} \text{ (set of all natural numbers)} \tag{3.2}$$

Now putting $x = x_n$ and $y = x_{n+1}$ in (3.1.1), yields

Case1. If $\max\{G(fx, fy), d(fx, fy)\} = d(fx, fy)$

$$d(f\,T(x_n), f\,T(x_{n+1})) \le c.\max\{G(fx, fy), d(f(x_n), f(x_{n+1}))\} \quad \text{(cf. (3.1))}$$

$$\le c.\max\left\{\frac{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_n) + d(x_n, Tx_n)d(x_n, Tx_{n-1})}{\max\{d(xn-1, Txn), d(Txn-1, xn)\}}, d(x_{n-1}, x_n)\right\}$$

$$\le c.\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n)\}$$

$$\le c.\,d(x_{n-1}, x_n),$$

Let $\alpha_n = d(y_n, y_{n+1})$, where, $y_n = f(x_n)$, $\forall$ n = 1, 2, …, n,…

Clearly , $\alpha_{n+1} \le c\, \alpha_n \le \alpha_n.$

This implies that $\{\alpha_n\}$ is decreasing sequence of positive real numbers. Since R is complete, hence, $\{\alpha_n\}$ converges and since $\alpha_n \le c^n \alpha_0$, $\alpha_n \to 0$, as $n \to \infty$.

Now if $m < n$, then

$$\begin{aligned}
d(y_m, y_n) &= d(fx_m, fx_n) \\
&= d(fTx_{m-1}, fTx_{n-1}) \\
&\le c\, d(y_{m-1}, y_{n-1}) \\
&< c^m\, d(y_0, y_{n-m}) \\
&< c^m\, (d(y_0, y_1) + d(y_1, y_2) + \ldots\, d(y_{n-m-1}, y_{n-m})) \\
&< c^m\, d(y_0, y_1)\, (1 + c + \ldots + c^{n-m-1}) \\
&< \frac{c^m}{1-c}\, d(y_o, y_1) \longrightarrow 0 \quad \text{as} \quad m \to \infty \quad (\because\ c < 1)
\end{aligned}$$

Hence $\{y_n\}$ is a Cauchy sequence in R, since R is complete, there exists a point y in R such that $y_n \to y$ i.e. $f(x_n) \to y$. Since, f is onto, this implies that there exists $x \in$ X such that $f(x) = y$. Therefore $f(x_n) \to f(x)$. Hence $\{x_n\}$ converges weakly to x in X. We now show that x is Serendipity fixed point of T.

Since
$$\begin{aligned}
d(f\,T(x), y_n)) &= d(f\,T(x), fx_n)) \\
&= d(fTx, fTx_{n-1})) \\
&\le c d(fx, fx_{n-1}))
\end{aligned}$$

Letting $n \to \infty$, we get $d(f\,T(x), f(x)) \le c d(fx, fx)$

This implies that $d(f\,T(x), f(x)) = 0$, therefore $f(Tx) = f(x)$. Hence, x is a Serendipity fixed point of T.

Claim x is unique.

Let, if possible, there exists $p \in$ X such that, $p \ne x$ and p is also a Serendipity fixed of T, i.e. $fT(p) = f(p)$

$$\text{Now} \quad d(fx, fp) = d(fTx, fTp) \le c.\max\{G(fx, fp), d(fx, fp)\}$$

Since $c < 1$ this implies that $d(f(x), f(p)) = 0$, hence $f(x) = f(p)$, since f is one-one

So that $x = p$. Hence, x is unique Serendipity fixed point of T.

4. CONCLUSION

The new concept may have a high potential of applications. For example, if we consider a classical method viz. Picard's method of successive approximation, for finding the solution of linear differential equations in which the sequence of solutions converges to a fixed point which is the unique solution of the equation, this concept of Serendipity fixed point may be applied in case when such types of approximate solutions do not converge to any unique limit.

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