International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 10 Issue 12 December 2022, Page no. – 3068-3073

Index Copernicus ICV: 57.55, Impact Factor: 7.362

DOI: 10.47191/ijmcr/v10i12.12

Commutativity and Cancellability of Finite Semi-Multigroups

Gambo Jeremiah Gyam[1], Tella Yohanna[2], Umar A.M.[3]

[1,3]Nassarawa State University, Keffi, Nigeria.

[2]Kaduna State University, Kaduna, Nigeria.

ORCID: 0000-0002-6295-9802[1], 0000-0002-1542-6554[2]

ARTICLE INFO

Published Online: 22 December 2022

Corresponding Author: **Gambo Jeremiah Gyam**

ABSTRACT

In this paper, the concept of Semi-group in multiset context is introduced. The condition for a sub multiset of a semi-multigroup to be a sub semi-multigroup is established and a study of the closure of multiset operationson the class of finite semi-multigroups is carried out.Commutative andcancellative properties of multiset operations on semi-multigroups are studied. We also studiedthe closure of multiset operations on the class of finite commutative and cancellative semi-multigroups.

KEYWORDS: Semi-group, Multiset,Root set,Multiset Operations, Semi-multigroup,Cancellative, Commutative

1. INTRODUCTION

A multiset (mset for short) is a collection of objects, unlike a standard Cantorian set, in which the elements are notallowed to repeat. Here repetitions are allowed. For the various applications of msets the reader is referred to article [1], [4,], [7], [9], and [11]. It is observed from the survey of available literature on msets and applications that the idea of mset was hinted by R. Dedikind in 1888. The msettheory which generalizes set theory as a special case was introduced by Cerf et al.[2]. The term mset, as noted by Knuth [4] was first suggested by N.G de Bruijn in a private communication to him. Further study was carried out by Yager [14], Blizard [1]. Other researchers ([5], [7], [8]) gave a new dimension to the multiset theory.

From a practical point of view msets are very useful structures arising in many areas of mathematics and computer science. Mset Topological space has been studied by Shravan and Tripathy [10]. Research on the mset theory has been gaining grounds. The research carried out so far shows a strong analogy in the behaviour of msets. It is possible to extend some of the main notion and result of sets to the setting of msets. In 2009, Girish and Sunil [3], introduced the concepts of relations, function, composition, and equivalence in msets context. Tella and Daniel ([12], [13]) have considered sets of mappings between msets and studied about symmetric groups under mset perspective. Nazmul et al. [6] improved on Tella and Daniel's work and added two axioms which marks the foundations of studying group theory in mset perspective. In this paper we present the study of semi-groups in mset context. From the literatures, there may be some variations in the definition of semigroup depending on the point of view of the different authors. However, in this paper we consider definitions in [15] and [16].

In addition to this section, we present some preliminary definitions in section two to make the paper self-contained and some fundamental results are presented in section three while the entire paper is summarized in section four.

2. PRELIMINARIES

2.1 Definitions and notations

Definition 2.1.1[15, 16]: Let S be a set and $\mu: S \times S \to S$ a binary operation that maps each ordered pair (x, y) of S to an element $\mu(x, y)$of S. The pair (S, μ)(or just S, if there is no fear of confusion) is called a **groupoid**. The mapping μ is called the product of (S, μ). We shall mostly write simply xy Instead of $\mu(x, y)$. If we want to emphasize the place of the operation then we often write$x.y$. The element $xy(= \mu(x, y))$ is the product of x and $y$in S.

Definition 2.1.2[15, 16]: A groupoid$S$ is a Semigroup if the operation μ is associative; for all $x\mu(y\mu z) = (x\mu y)\mu z$. Thus a semigroup is a pair (S, μ) where S is a non empty set and μ is its binary operation on μ which satisfied two axioms:

(i) The closure property

(ii) The associativity property.

Definition 2.1.3[1]. An mset$A$ over the set X can be defined as a function $C_A: X \to \mathbb{N} = \{0,1,2, \dots\}$ where the value $C_A(x)$

denote the number of times or multiplicity or count function of x in A. For example, Let $A = [x, x, x, y, y, y, z, z]$, then $C_A(x) = 3, C_A(y) = 3, C_A(z) = 2$. $[C_A(x) = 0 \Leftrightarrow x \notin A]$. The mset M over the set X is said to be empty if $C_M(x) = 0$ for all $x \in X$. We denote the empty mset by $\emptyset$. Then $C_\emptyset(x) = 0, \forall\, x \in X\,.(C_A(x) > 0 \Leftrightarrow x \in A)$. If $C_A(x) = n$ then the membership of x in A can be denoted by $x \in^n A$, meaning x belong to A exactly n times.

Definition2.1.4[1]: The cardinality of amset$M$ denoted $|M|$ or $card(M)$ is the sum of all the multiplicities of its elements given by the expression $|M| = \sum_{x \in X} c_A(x)$.

Note: Anmset$M$ is said to be finite if $|M| < \infty$.

We denote the class of all finite msets$A$ over the set X by $M(X)$

Note: Presentation of mset on paper work became a challenged as every researcher has his thought in that aspect. However the use of square brackets was adopted in ([1], [9],[11]) to represent an mset and ever since then it has become a standard. For example if the multiplicity of the elements x, y and z in an mset M are 2,3 and 2 respectively, then the mset$M$ can be represented as $M = [x, x, y, y, y, z, z,]$, others may put it like $[x, y, z]_{2,3,2}$ or $[x^2, y^3, z^2]$ or $[x2, y3, z2]$ or $[2/x, 3/y, 2/z]$ depending on one's taste or expediencies. But for conveniences sake, curly bracket can be used instead of the square bracket.

Definition 2.1.5[2]: Let M be an mset drawn from a set X. The support set of Mdenoted byM^* is a subset of X given by $M^* = \{x \epsilon X: C_M(x) > 0\}$. $M^*$is also called root set.

Definition 2.1.6[1](Equal msets): Two msets$A, B \in M(X)$ are said to be equal, denoted $A = B$ if and only if for any objects $x \in X, C_A(x) = C_B(x)$. This is to say that $A = B$ if the multiplicity of every element in A is equal to its multiplicity in B and conversely.

Note that $A = B \Longrightarrow A^* = B^*$, though the converse need not hold. For example, let

$A = [a, a, b, b, c]$ and $B = [a, a, b. b, b, c, c]$ $where$ $A^* = B^* = \{a, b, c\}$ but $A \neq B$.

Definition 2.1.7[1](Submultiset): Let $A, B \in M(X)$. $A$is a submultiset (submset for short) of B, denoted by $A \subseteq B$ or $B \supseteq A$, if $C_A(x) \leq C_B(x)$ for all $x \in X$. Also if $A \subseteq B$ and $A \neq B$, then A is called proper submset of B denoted by $A \subset B$. In other words $A \subset B$ if $A \subseteq B$ and there exist at least an $x \in X$ such that $C_A(x) < C_B(x)$. We assert that a mset$B$ is called the parent mset in relation to the mset A.

Definition.2.1.8[1]:(Regular or Constant mset): An mset$A$ over the set X is called regular or constant if all its elements are of the same multiplicities, i.e for any $x, y \in A$ $such$ $that$ $x \neq y, C_A(x) = C_A(y)$.

Definition 2.1.9 [9] ($\wedge$ and $\vee$ notations): The notations $\wedge$ and $\vee$ denote the minimum and maximum operator respectively, for instance;

$C_A(x) \wedge C_A(y) = min\{C_A(x), C_A(y)\}$ and $C_A(x) \vee C_A(y) = max\{C_A(x), C_A(y)\}$.

2.2 mset Operations.

Definition2.2.1[9] (msets union): Let $A, B \in M(X)$. The union of A and B denoted$A \cup B$is the mset defined by $C_{A \cup B}(x) = C_A(x) \vee C_B(x)\ \forall\, x \in X$

Definition 2.2.2[9] (msets intersection) Let$A, B \in M(X)$.The intersection of two mset A and B denoted by $A \cap B$, is the mset for which

$C_{A \cap B}(x) = C_A(x) \wedge C_B(x) \forall\, x \in X$.

Definition 2.2.3[9](mset addition): Let $A, B \in M(X)$. The direct sum or arithmetic addition of A and B denoted by $A + B$ or $A \uplus B$ is the mset defined by

$C_{A \uplus B}(x) = C_A(x) + C_B(x) \forall\, x \in X$.

Note that $|A \uplus B| = |A \cup B| + |A \cap B|$.

Definition 2.2.4[9] (mset difference): Let $A, B \in M(X)$, then the difference of $B$from A, denoted by $A - B$ is the mset such that $C_{A-B}(x) = \big(C_A(x) - C_B(x)\big) \vee 0 \forall\, x \in X$. If $B \subseteq A$, then $C_{A-B}(x) = C_A(x) - C_B(x)$.

It is sometimes called the arithmetic difference of $B$from A. If $B \nsubseteq A$ this definition still holds. It follows that the deletion of an element x from an mset A give rise to a new mset $A' = A - x$ such that $C_{A'}(x) = (C_A(x) - 1) \vee 0 \forall\, x \in X$

Definition 2.2.5[8] (mset symmetric difference): Let X be a set and $A, B \in M(X)$Then the symmetric difference, denoted $A \Delta B$, is defined by $C_{A \Delta B}(x) = |C_A(x) - C_B(x)|$.

Note that $A \Delta B = (A - B) \cup (B - A)$.

Definition2.2.6[8] (mset complement): Let $G = \{A_1, A_2, \ldots, A_n\}$ be a family of finite msets generated from the set X. Then, the maximum mset Z is defined by $C_Z(x) = max_{A \in G} C_A(x)$ for all $A \in G$and $x \in X$. The Complement of an mset A, denoted by $\bar{A}$, is defined:

$\bar{A} = Z - A$ $such$ $that$ $C_{\bar{A}}(x) = C_Z(x) - C_A(x), for$ all $x \in X$.

Note that $A_i \subseteq Z$ for all i and $\bar{A} \cap A \neq \emptyset$

Definition2.2.7[8] (Multiplication by Scalar): Let $A \in M(X)$, then the scalar multiplication denoted by $b.A$ is defined by $C_{b.A}(x) = b.C_A(x), \forall\, x \in X$ and $b \in \{1,2,3, \ldots\}$.

Definition 2.2.8[8] (Arithmetic Multiplication): Let $A, B \in M(X)$, then the Arithmetic Multiplication denoted by $A.B$ is defined by $C_{A.B}(x) = C_A(x).C_B(x)\ \forall\, x \in X$.

Definition2.2.9[7] (Raising to an Arithmetic Power): Let $A \in M(X)$, then A raised to power n denoted by A^n**is** defined:

$C_{A^n}(x) = \big(C_A(x)\big)^n$ for $n \in \{0,1,2,3, \ldots\}$.

Theorem 2.2.10[19]: Let X be a set and let $A \in M(X)$. Then

(i) $A^* = A^0$.

(ii) $A^n\,.A^m = A^{n+m}$, and

(iii) $(A.B)^n = A^n\,.B^n$ for any $n, m \in \{0,1,2, \ldots\}$

Theorem 2.2.11[19]: For any $A \neq \emptyset$ such that $A \in M(X)$, then $(A^n)^* = A^*$ for $n \in \{0,1,2 \ldots\}$

Definition 2.3.12[19]: Let X be a groupoid, and $A \in M(X)$. $A$is said to be a multi-groupoid (mgroupoid for short) if the following condition is satisfied.

$$C_A(xy) \geq C_A(x) \wedge C_A(y), \forall\, x, y \in X.$$

We denote the class of all mgroupoids over $X$by $MGP(X)$.

Definition 2.3.13[19](Composition of mgroupoids): Let $A, B \in MGP(X)$, then the composition of A and B denoted $A \circ B$ is defined:

$$C_{A\circ B}(x) = \vee\{C_A(y) \wedge C_B(z): y, z \in X \ni yz = x\}$$

Definition 2.3.14[19]: Let $A \in MGP(X)$ and let B be a submset of A. Then B can be said to be a sub mgroupoid of A, if $B \in MGP(X)$

Theorem 2.3.15[19]: For any $A \in MGP(X)$, then

(i).$A^* = A^0 \in MGP(X)$

(ii). $A^n \in MGP(X)$

(iii) $kA \in MGP(X), k \in \mathbb{N} = \{1,2,3, \dots\}$

Note that A^* is a subgroupoid of X[19]

Theorem 2.3.16[19]:Let $A, B \in MGP(X)$. Then

(i) $A \cap B \in MGP(X)$

(ii) $A.B \in MGP(X)$

(iii) $AoB \in MGP(X)$

Note that $A \cup B$, $A + B$, $A - B$, $A\Delta B$, and $\hat{A}$ need not be mgroupo

Definition 2.3.17[19]: Let $A \in MGP(X)$ an element $a \in A$ is said to be cancellable if

$C_A(ax) = C_A(ay),\ and\ C_A(xa) = C_A(ya),$

implies $C_A(x) = C_A(y)$.

Definition 2.3.18[19]: Let $A \in MGP(X)$. Then A is said to be cancellable if a is cancellable for all $a \in A$..

Theorem2.3.19[19]: Let $A \in MGP(X)$, then A is regular if and only if A is cancellable.

Definition (2.3.20)2.4.4[19]: Let $A \in MGP(X)$, then A is said to be a commutative mgroupoid if

$C_A(xy) = C_A(yx)\ \ \forall\, x, y \in X.$

Commutative mgroupoid can also be called Abelianmgroupoid.

Theorem2.3.21[19]: Let $A \in MGP(X)$. Then A is commutative if and only if A is regular.

Proposition2.3.22: Let $A \in MGP(X)$. Then A is commutative if and only if A is cancellable.

Proof: Let $A \in MGP(X)$ be commutative. Then A is regular (Theorem 2.3.21) and cancellable (Theorem 2.3.19)

Conversely, Let$A \in MGP(X)$ be cancellable. Then A is regular (Theorem2.3.19) and commutative (Theorem 2.3.21)

Theorem2.3.23[19]: Let $A \in MGP(X)$.If A is a commutative mgroupoid, then A^* is a commutative sub mgroupoid.

3.1Semi-group in mset Context.

Definition 3.1.1: Let $A \in MGP(X)$, then $A$is said to be a semi –multigroup (semi-mgroup for short) if X is a semi-group.

Example 3.1.2: Let $X = \{e, a, b, c\}$, such that $a^2 = b^2 = c^2 = e^2 = e\ and$

$ab = ba = c, ac = ca = b, bc = cb = a$.Where e is the identity element. If $A = \{e, a, b, c\}_{3,2,3,2}$ is an mset over$X$.

Clearly X is a semi-group and

$$C_A(ea) = C_A(a) = 2 \geq [C_A(e) \wedge C_A(a)] = min[C_A(e), C_A(a)] = min[3,2] = 2$$

$$C_A(aa) = C_A(e) = 3 \geq [C_A(a) \wedge C_A(a)] = min[C_A(a), C_A(a)] = min[2,2] = 2$$

$$C_A(bc) = C_A(a) = 2 \geq [C_A(b) \wedge C_A(c)] = min[C_A(b), C_A(c)] = min[3,2] = 2$$

$$C_A(bb) = C_A(e) = 3 \geq [C_A(b) \wedge C_A(b)] = min[C_A(b), C_A(b)] = min[3,3] = 3$$

$$C_A(ac) = C_A(b) = 3 \geq [C_A(a) \wedge C_A(c)] = min[C_A(a), C_A(c)] = min[2,3] = 2$$

$$C_A(cc) = C_A(e) = 3 \geq [C_A(c) \wedge C_A(c)] = min[C_A(c), C_A(c)] = min[3,3] = 3$$

$$C_A(ab) = C_A(c) = 2 \geq [C_A(a) \wedge C_A(b)] = min[C_A(a), C_A(b)] = min[2,3] = 2$$

$$C_A(eb) = C_A(b) = 3 \geq [C_A(e) \wedge C_A(b)] = min[C_A(e), C_A(b)] = min[3,3] = 3$$

$$C_A(ec) = C_A(c) = 2 \geq [C_A(e) \wedge C_A(c)] = min[C_A(e), C_A(c)] = min[3,2] = 2$$

$$C_A(ee) = C_A(e) = 3 \geq [C_A(e) \wedge C_A(e)] = min[C_A(e), C_A(e)] = min[3,3] = 3$$

Thus $C_A(xy) \geq C_A(x) \wedge C_A(y), \forall\, x, y \in X$.

We denote the collection of all finite semi-mgroups over X by *SMG(X)*

Definition 3.1.3: Let $A \in SMG(X)$ and let$B$ be a submset of$A$. Then B can be said to be a sub semi-mgroup of A, if $B \in SMG(X)$.

Proposition 3.1.4:$SMG(X) \subset MGP(X)$

Proof: Let $A \in SMG(X)$. Then A is an mgroupoid (since X is a groupoid)

In particular, $A \in MG(X)$. But not all mgroupoids are semi-mgroupoids.

Thus, $SMG(X) \subset MGP(X)$.

Proposition 3.1.5: Let X be a semi-group and$A \in SMG(X)$. Then

(i)$A^*$is a sub semi-group of$X$ and

(ii) $A^* \in SMG(X)$

Proof: (i) Supposing $A \in SMG(X)$. Then $A \in MGP(X)$ (proposition 3.1.4)

andA^* is a subgroupoid of X[19].

But $A^* \subseteq X$ and X is associative (by definition).

Thus A^* is associativeandA^* is a sub semi-group

(ii)Let $A \in SMG(X)$. Then $A \in MG(X)$(proposition 3.1.4) and $A^* \in MGP(X)$(Theorem 2.3.15).

Since X is a semi-group, we have $A^* \in SMG(X)$ (by definition)

Proposition 3.1.6:Let X be a semi-group and $A \in SMG(X)$. Then $A^0 \in SMG(X)$

Proof: The prove follows from theorem 2.3.15(i) and propositions 3.1.4,3.1.5(ii)

Proposition 3.1.7: Let X be a semi-group and let $A, B \in SMG(X)$, Then

(i)$A \cap B \in SMG(X)$.

(ii) $k.A \in SMG(X), k \in \{1,2 \dots.\}$

(iii) $A.B \in SMG(X)$

(iv)$A^n \in SMG(X), n \in \{0,1,2,\dots\}$
(v) $AoB \in SMG(X)$
Proof:
(i) Let $A, B \in SMG(X)$. Then $A, B \in MGP(X)$ (Proposition 3.1.4)
and$A \cap B \in MGP(X)$ (Theorem 2.3.16(i)).
Since X is a semi-group, then $A \cap B \in SMG(X)$.
(ii) Since $A \in SMG(X)$, then $A \in MGP(X)$(Proposition3.1.4) and
$k.A \in MGP(X), k \in \{1,2 \dots .\}$ (Theorem 2.3.15(iii))
Since X is a semi-group, then$k.A \in SMG(X), k \in \{1,2 \dots .\}$
(iii) Since $A, B \in SMG(X)$, Then $A, B \in MGP(X)$ (Proposition 3.1.4) and
$A.B \in MGP(X)$ (Theorem2.3.16(ii)
Since X is a semi-group, then $A.B \in SMG(X)$
(iv) Since $A \in SMG(X)$, then $A \in MGP(X)$ (Proposition3.1.4) and
$A^n \in MGP(X), n \in \{0,1,2,\dots\}$(Theorem2.3.15(ii))
Since X is a semi-group, then$A^n \in SMG(X), n \in \{0,1,2,\dots\}$
(v) Let $A, B \in SMG(X)$. Then $A, B \in MGP(X)$ (Proposition 3.1.4) and
$AoB \in MGP(X)$ (Theorem 2.3.16(iii)
Since X is a semi-group, then $AoB \in SMG(X)$
Note that $A \cup B, A + B, A - B, A\Delta B$, and $\hat{A}$ need not be semi-mgroups since
$A \cup B, A + B, A - B, A\Delta B$, and $\hat{A}$ need not bemgroupoids

Definition 3.1.8: Let $A \in SMG(X)$, then A is said to be a commutative semi-mgroup if it is a commutative mgroupoid Commutative semi- mgroup can also be called Abelian semi-mgroup.

Example:**3.1.9:**Let $X = \{e, a, b, c\}$, with $a^2 = b^2 = c^2 = e^2 = e \text{ and } ab = ba = c$,
$ac = ca = b, bc = cb = a$. Where e is the identity element.If $A = \{e, a, b, c\}_{3,2,3,2}$ is an mset over X.Clearly A is a commutative semi-mgroup.

3.2 Commutative and cancellative Expressions

Proposition 3.2.1: Let $A, B \in SMG(X)$ such that A and B are commutative. Then the followingexpressions are commutative:
(i) $A \cap B$
(ii) $A \cup B$
(iii) $A + B$
(iv)$A - B$
(v) $A\Delta B$
(vi)$A \cdot B$
(vii) $kA, k \in \{1,2,3,\dots\}$
(viii) $A^n,\ n \in \{0,1,2,\dots\}$
(ix) AoB

Proof:
(i) Let $x, y \in X$. We show that $C_{A\cap B}(xy) = C_{A\cap B}(yx)$
Now $C_{A\cap B}(xy) = C_A(xy) \wedge C_B(xy)$ (by definition) (1)
But $C_A(xy) = C_A(yx)$ and $C_B(xy) = C_B(yx)$ (by hypothesis) (2)
substituting (2) in (1) above, we have:
$$C_{A\cap B}(xy) = C_A(xy) \wedge C_B(xy) = C_A(yx) \wedge C_B(yx) = C_{A\cap B}(yx)$$
(ii) Let $x, y \in X$. We show that $C_{A\cup B}(xy) = C_{A\cup B}(yx)$
Now $C_{A\cup B}(xy) = C_A(xy) \vee C_B(xy)$ (by definition) (3)
But $C_A(xy) = C_A(yx)$ and $C_B(xy) = C_B(yx)$ (by hypothesis) (4)
substituting (4) in (3) above, we have:
$$C_{A\cup B}(xy) = C_A(xy) \vee C_B(xy) = C_A(yx) \vee C_B(yx) = C_{A\cup B}(yx)$$
(iii) Let $x, y \in X$. We show that $C_{A+B}(xy) = C_{A+B}(yx)$
Now $C_{A+B}(xy) = C_A(xy) + C_B(xy)$ (by definition) (5)
But $C_A(xy) = C_A(yx)$ and $C_B(xy) = C_B(yx)$ (by hypothesis) (6)
substituting (6) in (5) above, we have:
$$C_{A+B}(xy) = C_A(xy) + C_B(xy) = C_A(yx) + C_B(yx) = C_{A+B}(yx)$$
(iv) Let $x, y \in X$. We show that $C_{A-B}(xy) = C_{A-B}(yx)$
Now $C_{A-B}(xy) = (C_A(xy) - C_B(xy)) \vee 0$ (by definition) (7)
But $C_A(xy) = C_A(yx)$ and $C_B(xy) = C_B(yx)$ (by hypothesis) (8)
substituting (8) in (7) above, we have:
$$C_{A-B}(xy) = (C_A(xy) - C_B(xy)) \vee 0 = (C_A(yx) - C_B(yx)) \vee 0 = C_{A-B}(yx)$$
(v) Let $x, y \in X$. We show that $C_{A\Delta B}(xy) = C_{A\Delta B}(yx)$
Now $C_{A\Delta B}(xy) = |C_A(xy) - C_B(xy)|$ (9)
But $C_A(xy) = C_A(yx)$ and $C_B(xy) = C_B(yx)$ (by hypothesis) (10)
substituting (10) in (9) above, we have:
$$C_{A\Delta B}(xy) = |C_A(xy) - C_B(xy)| = |C_A(yx) - C_B(yx)| = C_{A\Delta B}(yx)$$
(vi) Let $x, y \in X$. We show that $C_{A\cdot B}(xy) = C_{A\cdot B}(yx)$
Now $C_{A\cdot B}(xy) = C_A(xy)C_B(xy)$ (by definition) (11)
But $C_A(xy) = C_A(yx)$ and $C_B(xy) = C_B(yx)$ (by hypothesis) (12)
substituting (12) in (11) above, we have:
$$C_{A\cdot B}(xy) = C_A(xy)C_B(xy) = C_A(yx)C_B(yx) = C_{A\cdot B}(yx)$$
(vii) Let $x, y \in X$ and $k \in \{1,2,3,\dots\}$.We show that $C_{kA}(xy) = C_{kA}(yx)$
Now $C_{kA}(xy) = kC_A(xy)$ (by definition) (13)
But $C_A(xy) = C_A(yx)$ (by hypothesis) (14)
substituting (14) in (13) above, we have:
$$C_{kA}(xy) = kC_A(xy) = kC_A(yx) = C_{kA}(yx)$$
(viii) Let $x, y \in X$and $n \in \{0,1,2,\dots\}$. We show that $C_{A^n}(xy) = C_{A^n}(yx)$
Now $C_{A^n}(xy) = (C_A(xy))^n$ (by definition) (15)
But $C_A(xy) = C_A(yx)$ (by hypothesis) (16)
substituting (16) in (15) above, we have:
$$C_{A^n}(xy) = (C_A(xy))^n = (C_A(yx))^n = C_{A^n}(yx)$$

(ix) Let $x, y \in X$. We show that $C_{AoB}(xy) = C_{AoB}(yx)$

Now $C_{A\circ B}(xy) = \vee\{C_A(w) \wedge C_B(z): y, z \in X \ni wz = xy\}$ (17)

Let $w = ab$ and $z = cda, b, c, d \in$ X. From (17) we have

$C_{A\circ B}(xy) = \vee\{C_A(ab) \wedge C_B(cd): a, b, c, d \in X \ni (ab)(cd) = xy\}$ (18)

But $C_A(ab) = C_A(ba)$ and $C_B(cd) = C_B(dc)$ (by hypothesis) (19)

substituting (19) in (18) above, we have:

$$C_{A\circ B}(xy) = \vee\{C_A(ab) \wedge C_B(cd): a, b, c, d \in X \ni (ab)(cd) = xy\}$$

$$= \vee\{C_A(ba) \wedge C_B(dc): a, b, c, d \in X \ni (ba)(dc) = xy\}$$

But $(ab)(cd) = xy$ and $(ba)(dc) = xy$ implies that $ab = ba$ for all $a, b \in X$

In particular, $xy = yx$ (20)

Substituting (20) in (18) and hence (17) we have:

$$C_{A\circ B}(xy) = \vee\{C_A(w) \wedge C_B(z): y, z \in X \ni wz = xy = yx\} = C_{A\circ B}(yx)$$

Proposition 3.2.2: Let $A, B \in SMG(X)$ such that A and B are cancellable. Then the following expressions are cancellable:

(i) $A \cap B$

(ii) $A \cup B$

(iii) $A + B$

(iv)$A - B$

(v) $A\Delta B$

(vi)$A \cdot B$

(vii) $kA, k \in \{1,2,3, \dots\}$

(viii) A^n, $n \in \{0,1,2, \dots\}$

(ix) AoB

Proof: Since $A, B \in SMG(X)$ are cancellable, then $A, B \in SMG(X)$ are commutative

(Proposition 2.3.22)

Thus all the above expressions are commutative (Proposition 3.2.1)

and cancellable (Proposition 2.3.22).

We denote the class of finite cancellable semi-mgroupas $\mathbb{C}SMG(X)$. That is,

$$\mathbb{C}SMG(X) = \{A \in SMG(X) | A \text{ is cancellable}\}$$

Proposition 3.2.3: Let $A, B \in \mathbb{C}SMG(X)$. Then

(i) (i)$A \cap B \in \mathbb{C}SMG(X)$.

(ii) $k.A \in \mathbb{C}SMG(X)$, $k \in \{1,2 \dots.\}$

(iii) $A.B \in \mathbb{C}SMG(X)$

(iv)$A^n \in \mathbb{C}SMG(X), n \in \{0,1,2, \dots\}$

(v) $AoB \in \mathbb{C}SMG(X)$

Proof:

(i) Since $A, B \in \mathbb{C}SMG(X)$, then $A, B \in SMG(X)$ (by definition) and

$A \cap B \in SMG(X)$ (Proposition3.1.7 (i))

But $A \cap B$ is cancellable (Proposition 3.2.2)

Thus $A \cap B \in \mathbb{C}SMG(X)$

(ii) Since $A \in \mathbb{C}SMG(X)$, then $A \in SMG(X)$ (by definition) and

$kA \in SMG(X)$ (Proposition3.1.7 (ii))

But kA is cancellable (Proposition 3.2.2)

Thus $k.A \in \mathbb{C}SMG(X), k \in \{1,2 \dots.\}$

(iii) Since $A, B \in \mathbb{C}SMG(X)$, then $A, B \in SMG(X)$ (by definition) and

$A.B \in SMG(X)$ (Proposition3.1.7 (iii))

But $A.B$ is cancellable (Proposition 3.2.2)

Thus $A.B \in \mathbb{C}SMG(X)$

(iv) Since $A \in \mathbb{C}SMG(X)$, then $A \in SMG(X)$ (by definition) and

$A^n \in SMG(X)$ (Proposition3.1.7 (iv))

But A^n is cancellable (Proposition 3.2.2)

Thus $A^n \in \mathbb{C}SMG(X), n \in \{0,1,2 \dots.\}$

(v) Since $A, B \in \mathbb{C}SMG(X)$, then $A, B \in SMG(X)$ (by definition) and

$AoB \in SMG(X)$ (Proposition3.1.7 (v))

But AoB is cancellable (Proposition 3.2.2)

Thus $AoB \in \mathbb{C}SMG(X)$

We denote the class of finite commutative semi-mgroupsas $CSMG(X)$. That is,

$$CSMG(X) = \{A \in SMG(X) | A \text{ is commutative}\}$$

Proposition 3.2.4: Let $A, B \in CSMG(X)$. Then

(i) (i)$A \cap B \in CSMG(X)$.

(ii) $k.A \in CSMG(X)$, $k \in \{1,2 \dots.\}$

(iii) $A.B \in CSMG(X)$

(iv)$A^n \in CSMG(X), n \in \{0,1,2, \dots\}$

(v) $AoB \in CSMG(X)$

Proof:

(i) Since $A, B \in CSMG(X)$, then $A, B \in SMG(X)$ (by definition) and

$A \cap B \in SMG(X)$ (Proposition3.1.7 (i))

But $A \cap B$ is commutative (Proposition 3.2.1)

Thus $A \cap B \in CSMG(X)$

(ii) Since $A \in CSMG(X)$, then $A \in SMG(X)$ (by definition) and

$kA \in SMG(X)$ (Proposition3.1.7 (ii))

But kA is commutative (Proposition 3.2.1)

Thus $k.A \in CSMG(X), k \in \{1,2 \dots.\}$

(iii) Since $A, B \in CSMG(X)$, then $A, B \in SMG(X)$ (by definition) and

$A.B \in SMG(X)$ (Proposition3.1.7 (iii))

But $A.B$ is commutative (Proposition 3.2.1)

Thus $A.B \in CSMG(X)$

(iv) Since $A \in CSMG(X)$, then $A \in SMG(X)$ (by definition) and

$A^n \in SMG(X)$ (Proposition3.1.7 (iv))

But A^n is commutative (Proposition 3.2.1)

Thus $A^n \in CSMG(X), n \in \{0,1,2 \dots.\}$

(v) Since $A, B \in CSMG(X)$, then $A, B \in SMG(X)$ (by definition) and

$AoB \in SMG(X)$ (Proposition3.1.7 (v))

But AoB is commutative (Proposition 3.2.1)

Thus $AoB \in CSMG(X)$.

Proposition 3.2.5: $\mathbb{C}SMG(X) = CSMG(X)$.

Proof: We show that $\mathbb{C}SMG(X) \subseteq CSMG(X)$and $\mathbb{C}SMG(X) \supseteq CSMG(X)$.

Now let $A \in \mathbb{C}SMG(X)$. Then A is cancellable and commutative.Thus $A \in CSMG(X)$.(proposition 3,2,1).
In particular,$\mathbb{C}SMG(X) \subseteq CSMG(X)$ (1)
Let $B \in CSMG(X)$. Then B is commutative and cancellable.Thus $B \in \mathbb{C}SMG(X)$.(proposition 3,2,1).
In particular,$CSMG(X) \subseteq \mathbb{C}SMG(X)$ (2)
Compairing (1) and (2) above.we have
$\mathbb{C}SMG(X) = CSMG(X)$.

4.0 CONCLUSION

We have introduced and studied the concepts of semi-mgroup. In the study,we have established the closure of some msetoperations over the class of finite semi-mgroups such as ;mset intersection, arithmetic multiplication, raising to arithmetic power, scalar multiplication and composition of semi-mgroups.. Cancellation law was introduced and studied and we established that a semi-mgroup is cancellable if and only if is commutative.We also studied the commutativity and cancellability of all expressions involving mset operations and established that these expressions are commutative and cancellable..Then the closure properties of commutative and cancellativesemi-mgroups.onmset operations were studied. We established that the msetopertions such as intersection, arithmetic multiplication, raising to arithmetic power, scalar multiplication and composition of semi-mgroups. were closed under the commutative and cancellativeproperties of semi-mgroups.

REFERENCES

1. Blizard, W..(1989); Multiset Theory, Notre Dame, J. Formal Logic 30, 36-66.
2. Cerf V., Fenandez E., Gostelow K. and Volausky S., Formal control and low properties of a model of computation, Report ENG 7178, Computer Science Department, University of California, Los Angeles, CA, December, pp. 81, (1971).
3. K.P. Girish and Sunil Jacob John, (2012); On Multiset Topologies; Theory and Application of Mathematical and Computer Sciences(2)(2012),37-52
4. Knuth, D.E. (1981); The art of computer programming, Vol. 2 (semi numerical Algorithms), Second Edition, Addison-Wesley, Reading, Massachusetts.
5. M. Raafat and A.M.Abd El-Latif,2010; On Multi Weak Structures; International Journal of Advances in Mathematics; Volume 2017,Number 6, 42-50.
6. Nazmul, S.K., Majumbar, P. and Samantha, S.K. (2013); Onmultiset and multigroups, Annals of Fuzzy Mathematics and Informatics, 6(3), 643-656.
7. Parker – Rhodes, A.F. (1981); The Theory of Indistinguishables, D. Reidel, Dordrecht.
8. Petrovsky, A. B, (1997); Structuring Techniques in Multiset Spaces. Springer. 174-184
9. Wildberger, N.J. (2003); A new look at multiset, School of Mathematics, UNSW, Sydney.
10. Shravan K. and Tripathy B.C., (2018); Generalised closed Sets in multiset topological spaces, Proyecciones Jour. Math. Vol. 37, 2.pp. 223-237.
11. Singh, D., Ibrahim, A,M., Yohanna, T. and Singh, J.N. (2011); Complemention in multiset Theory, International Mathematica Forum, 6(38), 1877.
12. Tella, Y. and Daniel, S. (2013); A Study of group theory in the context of Multiset Theory. International Journal of Science and Technology, 2(8), 609-615.
13. Tella, Y. and Daniel, S. (2013); Symmetric groups under multiset perspective, Journal of Mathematics, 7(5), 47-52.
14. Yager, R. R, (1986); On the theory of bags. International Journal of Genaral System, 13; 23-37
15. TeroHarja, (1996); Lecture note on Semigroups, Department of Mathematics, University of Turku, FIN-20014 Turku, Finland.
16. Benjamin Baumslag and Bruce Chanler, (1968); Theory and Problems of group theory, Schaums Outline Seriries, McGRAW-HILL BOOK COMPANY, New York, St. Louis Francisco, Toronto, Sydney.
17. K.P. Girish and Sunil Jacob John, (2009); Relatioins and Functions in Multiset Context; Information Sciences 179(2009),756-768.
18. Gambo JeremiahGyam and TellaYohanna, (2022); On Multiset Functions, International Journals of Sciences and Engineering Research(IJSER). Volume 13, Issue (6).
19. Gambo Jeremiah Gyam and TellaYohanna, (2022); A new look at Groupoid and Cancellattion law; International Journal of Software and Hardware Research in Engeneering (IJSHRE), Volume 10, Issue 10.