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Common Fixed Point Theorems for Pair of Maps of Integral Type in Modular Metric Spaces

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ABSTRACT

In this paper we study and prove the existence of some common fixed point theorems for two pairs of compatible, weakly compatible and occasionally weakly compatible mappings of integral type in modular metric spaces. Our results extend and generalize the results of Azadifar et al. [2], Rashwan and Hammad [28], Rahimpoor et.al. [27], and Pariya Aklesh et. al. [25], Abdou, A. and Khamsi,[1], Chirasak [5], Nawab [24], Hanna, et al [10].

KEYWORDS: Common fixed point, modular space, compatible, weakly compatible, occasionally weakly compatible mappings, integral type inequality.

AMS Mathematics Subject Classification (2010): 47H09, 47H10, 54H25.

1. INTRODUCTION

In 1922, Banach, S., proved a contraction principle, which ensures the existence and uniqueness of a fixed point of a self map on complete metric space, under some appropriate conditions. This principle is known as 'Banach Fixed Point Theorem'. This theorem states that 'if f be a self-mapping of a complete metric space (X,d) and if there exist a number k, with $0 \leq k < 1$, such that $d(Tx,Ty) \leq cd(x,y)$ for all $x, y \in X$, then f has a unique fixed point in X . During the last 80 years, this result was extended and generalize through a lot of fixed point and common fixed point theorems which have been established by many authors in different spaces by taking more general contractive conditions.

In the year 1950, The notion of modular space, as a generalization of a metric space, was introduced by Nakano [21]. This interesting result have been extensively generalized and developed by Chistyakov,V.,[7, 8, 9], Yamamuro [31], Koshi and Shimogaki [15], Mongkolkeha at el. [16], Musielak [18] and others. The main idea behind this concept is the physical interpretation of the modular. Informally speaking whereas a metric on a set represents finite non-negative distance between two points of the set, a module on a set attribute a non-negative (possibly, infinite valued) 'field of (generalized) velocities' such that to each 'time' $\lambda > 0$ (the absolute value of), an average velocity $\omega(x, y)$ is associated in such way that in order to cover the 'distance' between points $x, y \in X$, it takes time λ to move from x to y with velocity $\omega(x, y)$. Later in 1959, this concept was further redefined and generalized by Musielak and Orlicz [19]. Later in 2010 Chistyakov [8, 9] defined the notion of modular on an arbitrary set and develop the theory of modular metric spaces generated by modular and proved some fixed point theorems for maps which are related to contracting 'generalized average velocities' rather than metric distances.

In 1982, Sessa [29] generalized the notion of commutativity to that of pairwise weakly commutativity. Jungck [11] weakened the condition of weak commutativity to that of pairwise compatible and then [12] pairwise compatible maps. In 2006, Jungck and Rhoades [13] was introduced the concept of occasionally weakly compatible maps. This concept has been frequently employed to prove the existence of common fixed points.

In 2002, Branciari [4] gave an analogue of Banach's contraction principle for an integral type inequality, which is stated as follows :

Let (X, d) be a complete metric space, $k \in [0, 1)$, $f : X \rightarrow X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(f(x)-f(y))} \varphi(t)dt \leq k \int_0^{d(x,y)} \varphi(t)dt,$$

Where, $\varphi : R^+ \rightarrow R^+$ be a Lebesgue integrable mapping which is summable, non-negative and for all $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. Then f has a unique fixed point $u \in X$, such that for each $x \in X$, $\lim_{n \to \infty} f^n x = u$.

Recently, Rahimpoor et al. [26], establish some fixed point theorems in modular metric spaces for weakly compatible mappings. Azadifar et al. [2] proved some fixed point results for compatible mappings of integral type in modular metric spaces. Rashwan and Hammad [28], Aklesh Pariya et al. [25] and Hanna at el. [10] proved some common fixed point results for integral type mappings in modular metric spaces. In this paper we prove some common fixed point theorems for compatible, weakly compatible and occasionally weakly compatible pair of mappings of integral type in modular metric spaces.

We start with a brief recollection of basic definitions and facts in modular spaces and modular metric spaces from [2], [5], [7], [8], [9], [10], [14], [15], [19], [25] and [26].

2. PRELIMINARIES

Definition 2.1. Let X be a vector space over the field R (or C). A functional $\rho : X \to [0, \infty]$ is called a modular if for any arbitrary x and y in X, these three conditions are satisfied:

(i) $\rho(x) = 0$ if and only if $x = 0$,

(ii) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$,

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, whenever $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. If one replaces (iii) by (iv)

(iv) $\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$, for $\alpha, \beta \geq 0$ and $\alpha^s + \beta^s = 1$, where $s \in (0, 1]$ then, the modular ρ is called s–convex modular, and if $s = 1$, then ρ is called convex modular.

If ρ is modular in X, then the set defined by

(2.1.1) $X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0^+\}$

is called a modular space. Clearly, the modular space X_ρ is a subspace of space X.

Let X be a non-empty set and $\lambda \in (0, \infty)$ for the sake of convenience, function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ will be written as $\omega(x, y)$ instead of $\omega(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

Definition 2.2. Let X be a non-empty set . A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a metric modular on X if it satisfies the following axioms:

(i) $\omega(x, y) = 0$ if and only if $x = y$, for all $\lambda > 0$ and $x, y \in X$;

(ii) $\omega(x, y) = \omega(y, x)$, for all $\lambda > 0$ and $x, y \in X$;

(iii) $\omega_{+\mu}(x, y) \leq \omega(x, z) + \omega_\mu(z, y)$, for all $\lambda, \mu > 0$ and $x, y, z \in X$. If instead of (i), we only have the condition

(i)(a) $\omega(x, x) = 0$ for all $\lambda > 0$ and $x \in X$, then ω is said to be a (metric) pseudo modular on X and if ω satisfies (i)(a) and

(i)(b) for x, y in X, if there exists a number $\lambda > 0$, possibly depending on x and y, such that $\omega(x, y) = 0$, then $x = y$, with this condition ω is said to be a strict modular on X.

A modular (pseudo modular, strict modular) ω is said to be convex if the condition (iii) is replaced by the condition

(iv) $\omega_{\lambda+\mu}(x, y) = \frac{\lambda}{\lambda+\mu}\omega_\lambda(x, z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z, y)$, for all $\lambda, \mu > 0$ and $x, y, z \in X$.

Clearly, if ω is a strict modular, then ω is a modular, which is turn implies ω is a pseudo modular on X, and similar implications hold for convex modular ω.

The essential property of a (pseudo) modular ω on a set X is, for x, y in X and $\lambda > 0$, the function $\lambda \to \omega(x, y) \in [0, \infty]$ is non-increasing on $(0, \infty)$. In fact, if $0 < \mu < \lambda$, then (iii), (i)(a) and (i)(b) implies that

(2.2.1) $\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y)$ for all $\lambda > 0$ and $x, y \in X$;

It follows that at each $\lambda > 0$ both the limits, the right-hand limit $\omega_{+0}(x, y) = \lim_{s\to 0+} \omega_{+s}(x, y)$ and the left-hand limit $\omega_{-0}(x, y) = \lim_{s\to +0} \omega_{-s}(x, y)$ exists in $[0, \infty]$ and the following inequality holds:

(2.2.2) $\omega_{+0}(x, y) \leq \omega_\lambda(x, y) \leq \omega_{\lambda-0}(x, y)$.

From (2.2.1) and (2.2.2), we know that if $x_0 \in X$, the set

$X_\omega = \{x \in X : \lim_{\lambda\to\infty} \omega_\lambda(x, x_0) = 0\}$ be a metric space, called a modular metric space. with metric given by

$d^0_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, x_0) \leq \lambda\}$ for all $x, y \in X_\omega$

Moreover, if ω is convex, the modular set X_ω is equal to the set

$X^*_\omega = \inf\{x \in X : \exists\, \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x, x_0) \leq \infty\}$

and metrizable by

$d^*_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq 1\}$ for all $x, y \in X^*_\omega$

We know that if X is a real linear space, $\rho : X \to [0, \infty]$ and

(2.2.3) $\omega_\lambda(x, y) = \rho\left(\frac{x-y}{\lambda}\right)$ for all $\lambda > 0$ and $x, y \in X$.

Then ρ is modular (convex modular) on X in the sense of conditions (i), (ii), (iii) and (iv) of the definition 2.1, if and only if ω is metric modular (convex metric modular) on X.

On the other hand, if ω satisfy the following two conditions :

(i) $\omega_\lambda(\mu x, 0) = \omega_{\frac{\lambda}{\mu}}(x, 0)$ for all $\lambda, \mu > 0$ and $x \in X$.

(ii) $\omega_\lambda(x+z, y+z) = \omega_\lambda(x, y)$ for all $\lambda > 0$ and $x, y, z \in X$.

If we set,

$$\rho(x) = \omega_1(x, 0) \text{ for } x \in X$$

(I) $X_\rho = X_\omega$ is a linear subspace of X and the functional $\|x\|_\rho = d^0_\omega(x, 0)$, $x \in X_\rho$, is an F- norm on X_ρ, and

(II) If ω is convex, $X^*_\rho = X^*_\omega(0) = X_\rho$ be a linear subspace and the functional $\|x\|_\rho = d^*_\omega(x, 0)$, $x \in X^*_\rho$, be a norm on X^*_ρ.

Similar assertions hold if we replace the word modular by pseudo modular. If ω metric modular in X, the set X_ω is said to be a modular metric space.

Definition 2.3.[21] Let X_ω be a modular metric space. Then

The sequence $\{x_n\}_{n\in N}$ is said to be convergent to $x \in X_\omega$, if $\omega_\lambda(x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$.

(i) The sequence $\{x_n\}_{n\in N}$ in X_ω is said to be Cauchy, if $\omega_\lambda(x_m, x_n) \to 0$, as $m, n \to \infty$ for all $\lambda > 0$.

(ii) A subset S of X_ω is said to be complete, if each Cauchy sequence in S is convergent in S.

Now, we recall the following definitions in metric spaces.

Definition 2.4. Let (X,d) be a metric space, then two self-maps S and T on X are called compatible, if $d(STx_n - TSx_n) \to 0$, as $n \to \infty$, whenever $\{x_n\}_{n\in N}$ be a sequence in X, such that $Sx_n \to z$ and $Tx_n \to z$ for some $z \in X$.

Definition 2.5. Let X be a non-empty set and S, T : X → X. Then a point x∈X is said to be a coincidence point of S and T if and only if Sx = Tx. If u = Sx = Tx, then u is called a point of coincidence of S and T.

Definition 2.6. Let X be a non-empty set and S, T : X → X. Then S and T are said to be weakly compatible if they commute at coincidence point.

Definition 2.7. [11] Let X be a non-empty set and S, T : X → X. Then S and T are said to be occasionally weakly compatible (owc) if and only if there is a point x∈X which is coincidence point of S and T at which S and T commute.

Lemma 2.1. [11] Let X be a non-empty set and S, T : X → X are occasionally weakly compatible (OWC) maps. If, S and T have a unique point of coincidence u = Sx = Tx, then u be a unique fixed point of S and T.

In the modular metric space, above definitions are defined as follows:

Definition 2.8. [25] Let X_ω be a modular metric space induced by metric modular ω. Two mappings S, T : $X_\omega \to X_\omega$ are said to be ω –compatible, if $\omega_\lambda(STx_n, TSx_n) \to 0$, as $n \to \infty$, whenever $\{x_n\}_{n\in N}$ be a sequence in X_ω, such that $Sx_n \to z$ and $Tx_n \to z$ for some $z \in X_\omega$.

Definition 2.9. [25] Let X_ω be a modular metric space and S, T : $X_\omega \to X_\omega$. Then a point $x \in X_\omega$ is said to be a coincidence point of S and T iff Sx = Tx. If u = Sx = Tx, then u is called a point of coincidence of S and T.

Definition 2.10. [25] Let X_ω be a modular metric space and S, T : $X_\omega \to X_\omega$. Then S and T are said to be weakly compatible if they commute at coincidence point.

Definition 2.11. [25] Let X_ω be a modular metric space and S, T : $X_\omega \to X_\omega$. Then S and T are said to be occasionally weakly compatible (owc) if and only if there is a point $x \in X_\omega$ which is coincidence point of S and T at which S and T commute.

Definition 2.12.[24] A function $\psi : R^+ \to R^+$ is said to be a comparison function if it satisfies the following conditions:

(i) ψ is monotone increasing, $\psi(t) < t$ for some $t > 0$,

(ii) $\psi(0) = 0$,

(iii) $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$.

Lemma 2.2. [25] Let X_ω be a modular metric space and S, T : X → X are occasionally weakly compatible (owc) maps. If, S and T have a unique point of coincidence u = Sx = Tx, then u be a unique fixed point of S and T.

3. MAIN RESULT

Theorem 3.1. Let X_ω be a complete modular space and f, g, S, T : $X_\omega \to X_\omega$ are mappings satisfying the conditions

(3.1.1) $S(X_\omega) \subseteq g(X_\omega)$ and $T(X_\omega) \subseteq f(X_\omega)$

(3.1.2) If $\psi : R^+ \to R^+$ be a non-decreasing right continuous function, $\psi(0) = 0$ and $\psi^n(t) < t$ for every $t > 0$, such that for all $x, y \in X_\omega$

$$\int_0^{\omega_\lambda(Sx,\ Ty)} \varphi(t)dt \leq \psi \int_0^{m(x,y)} \varphi(t)dt \quad \text{for all } x, y \in X_\rho$$

Where,

$$m(x,y) = \max\{\omega_\lambda(fx, gy), \omega_\lambda(fx, Sx), \omega_\lambda(gy, Ty), \tfrac{\omega_\lambda(gy,Sx) + \omega_{2\lambda}(fx,Ty)}{2},$$

$\frac{\omega_\lambda(gy,Sx)[1+\omega_\lambda(fx,Ty)]}{1+\omega_\lambda(fx,gy)}, \frac{\omega_\lambda(gy,Ty)[1+\omega_\lambda(fx,Sx)]}{1+\omega_\lambda(fx,gy)}\}$

Where, $\lambda > 0$ and $\varphi : R^+ \to R^+$ be a Lebesgue integrable mapping which is summable, non-negative and

$\int_0^c \varphi(t)dt > 0$, for all c > 0.

If the pair (f, S) is compatible and (g, T) is weakly compatible on X_ω and if one of the mappings f or S is continuous, then f, g, S and T have a unique common fixed point in X_ω.

Proof : Let x_0 be an arbitrary point in X_ω.

Since $S(X_\omega) \subseteq g(X_\omega)$, we choose a point $x_1 \in X_\omega$ such that $Sx_0 = gx_1$ and since $T(X_\omega) \subseteq f(X_\omega)$, let x_2 be a point in X_ω such that $Tx_1 = fx_2$.

Continuing in this manner, we construct a sequence $\{y_n\}$ in X_ω such that

$Sx_n = gx_{n+1} = y_n$ and $Tx_{n+1} = fx_{n+2} = y_{n+1}$, for all $n \geq 0$.

Now, we have from (3.1.2), on taking $x = x_n$ and $y = x_{n+1}$

(3.1.3) $\int_0^{\omega_\lambda(y_n, y_{n+1})} \varphi(t)dt = \int_0^{\omega_\lambda(Sx_n, Tx_{n+1})} \varphi(t)dt \leq \psi \int_0^{m(x_n, x_{n+1})} \varphi(t)dt,$

Where,

$$m(x_n, x_{n+1}) = \max\{\omega_\lambda(fx_n, gx_{n+1}), \omega_\lambda(fx_n, Sx_n), \omega_\lambda(gx_{n+1}, Tx_{n+1}), \frac{\omega_\lambda(gx_{n+1},Sx_n) + \omega_{2\lambda}(fx_n,Tx_{n+1})}{2}, \frac{\omega_\lambda(gx_{n+1},Sx_n)[1+\omega_\lambda(fx_n,Tx_{n+1})]}{1+\omega_\lambda(fx_n, gx_{n+1})}, \frac{\omega_\lambda(gx_{n+1}, Tx_{n+1})[1+\omega_\lambda(fx_n,Sx_n)]}{1+\omega_\lambda(fx_n,gx_{n+1})}\}$$

$$= \max\{\omega_\lambda(y_{n-1}, y_n), \omega_\lambda(y_{n-1}, y_n), \omega_\lambda(y_n, y_{n+1}), \frac{\omega_\lambda(y_n, y_n) + \omega_{2\lambda}(y_{n-1}, y_{n+1})}{2}, \frac{\omega_\lambda(y_n, y_n)[1+\omega_\lambda(y_{n-1}, y_{n+1})]}{1+\omega_\lambda(y_{n-1}, y_n)}, \frac{\omega_\lambda(y_n, y_{n+1})[1+\omega_\lambda(y_{n-1}, y_n)]}{1+\omega_\lambda(y_{n-1}, y_n)}\}$$

$$= \max\{\omega_\lambda(y_{n-1}, y_n), \omega_\lambda(y_n, y_{n+1}), \frac{\omega_\lambda(y_{n-1}, y_n) + \omega_\lambda(y_n, y_{n+1})}{2}\}$$

$$= \max\{\omega_\lambda(y_{n-1}, y_n), \omega_\lambda(y_n, y_{n+1})\}$$

So that from (3.1.3)

$$\int_0^{\omega_\lambda(y_n, y_{n+1})} \varphi(t)dt \leq \psi \int_0^{\omega_\lambda(y_{n-1},y_n)} \varphi(t)dt$$

By induction, we have

$$\int_0^{\omega_\lambda(y_n, y_{n+1})} \varphi(t)dt \leq \psi \int_0^{\omega_\lambda(y_{n-1},y_n)} \varphi(t)dt$$
$$\leq \psi^2 \int_0^{\omega_\lambda(y_{n-2},y_{n-1})} \varphi(t)dt$$
$$\dots \quad \dots \quad \dots$$

(3.1.4) $\int_0^{\omega_\lambda(y_n, y_{n+1})} \varphi(t)dt \leq \psi^n \int_0^{\omega_\lambda(y_0,y_1)} \varphi(t)dt$

On taking the limit $n \to \infty$, and using the definition of ψ, we get

$$\psi^n \int_0^{\omega_\lambda(y_0,y_1)} \varphi(t)dt \to 0.$$

Hence, $\{y_n\}$ be a Cauchy sequence in X_ω.

By the completeness of X_ω, there exists a point $z \in X_\omega$ such that the sequence $\{y_n\}$ and its subsequences $\{y_{n+1}\}$ and $\{y_{n+2}\}$ are converges to z in X_ω.

Now, on assuming the continuity of f, we have

$$f^2x_n \to fz \text{ and } fSx_n \to fz.$$

Also, in view of compatibility of the pair (f, S),

$$Sfx_n \to fz$$

From (3.1.2), we have

(3.1.5) $\int_0^{\omega_\lambda(Sfx_n, Tx_{n+1})} \varphi(t)dt \leq \psi \int_0^{m(fx_n, x_{n+1})} \varphi(t)dt$

Where,

$$m(fx_n, x_{n+1}) = \max\{\omega_\lambda(f^2x_n, gx_{n+1}), \omega_\lambda(f^2x_n, Sfx_n), \omega_\lambda(gx_{n+1}, Tx_{n+1}), \frac{\omega_\lambda(gx_{n+1},Sfx_n) + \omega_{2\lambda}(f^2x_n,Tx_{n+1})}{2}, \frac{\omega_\lambda(gx_{n+1},Sfx_n)[1+\omega_\lambda(f^2x_n,Tx_{n+1})]}{1+\omega_\lambda(f^2x_n, gx_{n+1})}, \frac{\omega_\lambda(gx_{n+1}, Tx_{n+1})[1+\omega_\lambda(f^2x_n,Sfx_n)]}{1+\omega_\lambda(f^2x_n,gx_{n+1})}\}$$

On taking the limit $n \to \infty$,we get

$$\lim_{n\to\infty} m(fx_n, x_{n+1}) = \max\{\omega_\lambda(fz, z), \omega_\lambda(fz, fz), \omega_\lambda(z, z), \frac{\omega_\lambda(z,fz) + \omega_{2\lambda}(fz,z)}{2}, \frac{\omega_\lambda(z,fz)[1+\omega_\lambda(fz,z)]}{1+\omega_\lambda(fz,z)}, \frac{\omega_\lambda(z,z)[1+\omega_\lambda(fz,fz)]}{1+\omega_\lambda(fz,z)}\}$$

$= \max\{\omega_\lambda(fz,z), 0, 0, \frac{\omega_\lambda(z,fz) + \omega_{2\lambda}(fz,z)}{2}, \quad \omega_\lambda(fz,z), 0\}$

Therefore, from (3.1.5) on taking the limit $n \to \infty$,we get

$\int_0^{\omega_\lambda(fz,z)} \varphi(t)dt \leq \psi \int_0^{\omega_\lambda(fz,z)} \varphi(t)dt < \int_0^{\omega_\lambda(fz,z)} \varphi(t)dt.$

This implies that, fz = z.

Again from (3.1.2), we have

(3.1.6) $\int_0^{\omega_\lambda(Sz,Tx_{n+1})} \varphi(t)dt \leq \psi \int_0^{m(z,\ x_{n+1})} \varphi(t)dt$

Where,

m(z, x_{n+1}) = $\max\{\omega_\lambda(fz, gx_{n+1}), \omega_\lambda(fz,Sz), \omega_\lambda(gx_{n+1},Tx_{n+1}), \frac{\omega_\lambda(gx_{n+1},Sz) + \omega_{2\lambda}(fz,Tx_{n+1})}{2},$
$\frac{\omega_\lambda(gx_{n+1},Sz)[1+\omega_\lambda(fz,Tx_{n+1})]}{1+\omega_\lambda(fz,gx_{n+1})}, \frac{\omega_\lambda(gx_{n+1},\ Tx_{n+1})[1+\omega_\lambda(fz,Sz)]}{1+\omega_\lambda(fz,gx_{n+1})}\}$

On taking the limit $n \to \infty$,we get from (3.1.6)

$\int_0^{\omega_\lambda(Sz,z)} \varphi(t)dt \leq \psi \int_0^{\max\{\omega_\lambda(z,z),\omega_\lambda(z,Sz),\omega_\lambda(z,z),\frac{\omega_\lambda(z,Sz)+\omega_{2\lambda}(z,z)}{2},\frac{\omega_\lambda(z,Sz)[1+\omega_\lambda(z,z)]}{1+\omega_\lambda(z,z)},\frac{\omega_\lambda(z,z)[1+\omega_\lambda(z,Sz)]}{1+\omega_\lambda(z,z)}\}} \varphi(t)dt$

Hence,

$\int_0^{\omega_\lambda(Sz,z)} \varphi(t)dt \leq \psi \int_0^{\omega_\lambda(z,Sz)} \varphi(t)dt < \int_0^{\omega_\lambda(z,Sz)} \varphi(t)dt$

This implies that, Sz = z.

Now, since S(X_ω) $\subseteq$ g(X_ω), then there exists another point u in X_ω such that

z = Sz = gu.

Now we prove that Tu = z.

From (3.1.2), we have

$\int_0^{\omega_\lambda(z,Tu)} \varphi(t)dt = \int_0^{\omega_\lambda(Sz,Tu)} \varphi(t)dt \leq \psi \int_0^{m(z,u)} \varphi(t)dt$

Where,

m (z, u) = $\max\{\omega_\lambda(fz,gu), \omega_\lambda(fz,Sz), \omega_\lambda(gu,Tu), \frac{\omega_\lambda(gu,Sz)+\omega_{2\lambda}(fz,Tu)}{2},$
$\frac{\omega_\lambda(gu,Sz)[1+\omega_\lambda(fz,Tu)]}{1+\omega_\lambda(fz,gu)}, \frac{\omega_\lambda(gu,Tu)[1+\omega_\lambda(fz,Sz)]}{1+\omega_\lambda(fz,gu)}\}$

$= \max\{\omega_\lambda(z,z), \omega_\lambda(z,z), \omega_\lambda(z,Tu), \frac{\omega_\lambda(z,z)+\omega_{2\lambda}(z,Tu)}{2}, \frac{\omega_\lambda(z,z)[1+\omega_\lambda(z,Tu)]}{1+\omega_\lambda(z,z)}, \frac{\omega_\lambda(z,Tu)[1+\omega_\lambda(z,z)]}{1+\omega_\lambda(z,z)}\}$

Hence,

$\int_0^{\omega_\lambda(z,Tu)} \varphi(t)dt \leq \psi \int_0^{\omega_\lambda(z,Tz)} \varphi(t)dt < \int_0^{\omega_\lambda(z,Tz)} \varphi(t)dt$

This implies that, Tu = z.

Since, the pair (g, T) is weakly compatible on X_ω and Tu = gu = z, so that

Tgu = gTu and Tz = Tgu = gTu = gz,

From (3.1.2), we have

(3.1.7) $\int_0^{\omega_\lambda(z,gz)} \varphi(t)dt = \int_0^{\omega_\lambda(Sz,Tz)} \varphi(t)dt \leq \psi \int_0^{m(z,z)} \varphi(t)dt$

Where,

m(z, z) = $\max\{\omega_\lambda(fz,gz), \omega_\lambda(fz,Sz), \omega_\lambda(gz,Tz), \frac{\omega_\lambda(gz,Sz)+\omega_{2\lambda}(fz,Tz)}{2},$
$\frac{\omega_\lambda(gz,Sz)[1+\omega_\lambda(fz,Tz)]}{1+\omega_\lambda(fz,gz)}, \frac{\omega_\lambda(gz,Tz)[1+\omega_\lambda(fz,Sz)]}{1+\omega_\lambda(fz,gz)}\}$

$= \max\{\omega_\lambda(z,gz), \omega_\lambda(z,z), \omega_\lambda(gz,gz), \frac{\omega_\lambda(gz,z)+\omega_{2\lambda}(z,gz)}{2}, \frac{\omega_\lambda(gz,z)[1+\omega_\lambda(z,gz)]}{1+\omega_\lambda(z,gz)}, \frac{\omega_\lambda(gz,gz)[1+\omega_\lambda(z,z)]}{1+\omega_\lambda(z,gz)}\}$

Hence, from (3.1.7)

$\int_0^{\omega_\lambda(z,gz)} \varphi(t)dt \leq \psi \int_0^{\omega_\lambda(z,gz)} \varphi(t)dt < \int_0^{\omega_\lambda(z,gz)} \varphi(t)dt$

This implies that, z = gz.

Thus, z is a common fixed point of f, g, S and T.

To prove the uniqueness, let w ≠ z be another common fixed point of f, g, S and T.

From (3.1.2), we have

(3.1.8) $\int_0^{\omega_\lambda(w,z)} \varphi(t)dt = \int_0^{\omega_\lambda(Sw,Tz)} \varphi(t)dt \leq \psi \int_0^{m(w,z)} \varphi(t)dt$

Where,

m (w, z) = max $\{\omega_\lambda(fw,gz), \omega_\lambda(fw,Sw), \omega_\lambda(gz,Tz), \frac{\omega_\lambda(gz,Sw)+\omega_{2\lambda}(fw,Tz)}{2}, \frac{\omega_\lambda(gz,Sw)[1+\omega_\lambda(fw,Tz)]}{1+\omega_\lambda(fw,gz)}, \frac{\omega_\lambda(gz,Tz)[1+\omega_\lambda(fw,Sw)]}{1+\omega_\lambda(fw,gz)}\}$

= max $\{\omega_\lambda(w,z), \omega_\lambda(w,w), \omega_\lambda(z,z), \frac{\omega_\lambda(z,w)+\omega_{2\lambda}(w,z)}{2}, \frac{\omega_\lambda(z,w)[1+\omega_\lambda(w,z)]}{1+\omega_\lambda(w,z)}, \frac{\omega_\lambda(z,z)[1+\omega_\lambda(w,w)]}{1+\omega_\lambda(w,z)}\}$

Hence, from (3.1.8)

$\int_0^{\omega_\lambda(w,z)} \varphi(t)dt \leq \psi \int_0^{\omega_\lambda(w,z)} \varphi(t)dt < \int_0^{\omega_\lambda(w,z)} \varphi(t)dt$

Which is a contradiction. Hence z is a unique common fixed point of f, g, S and T in X_ω.

Theorem 3.2. Let X_ω be a complete modular space and f, g, S, T : $X_\omega \to X_\omega$ are mappings such that $S(X_\omega) \subseteq g(X_\omega)$ and $T(X_\omega) \subseteq f(X_\omega)$ and one of the spaces $f(X_\omega)$ or $g(X_\omega)$ be a ω-complete subspace of X_ω. Suppose there exists numbers $a_1, a_2, \ldots a_6 \in [0,1)$ with at least one of $a_i > 0$ (i = 1,2, . . . 6) such that for all x, y $\in X_\omega$ and $\lambda > 0$, the following assertion hold:

(3.2.1) $a_1 + 2a_4 + a_5 < 1$ for all $0 \leq a_1, a_2, a_3\ a_4, a_5, a_6 < 1$.

(3.2.2) $\int_0^{\omega_\lambda(Sx,Ty)} \varphi(t)dt \leq a_1 \int_0^{\omega_\lambda(fx,gy)} \varphi(t)dt + a_2 \int_0^{\omega_\lambda(fx,Sx)} \varphi(t)dt + a_3 \int_0^{\omega_\lambda(gy,Ty)} \varphi(t)dt$

$+ a_4 \int_0^{\omega_\lambda(gy,Sx)+\omega_{2\lambda}(fx,Ty)} \varphi(t)dt + a_5 \int_0^{\frac{\omega_\lambda(gy,Sx,)[1+\omega_\lambda(fx,Ty)]}{1+\omega_\lambda(fx,gy)}} \varphi(t)dt + a_6 \int_0^{\frac{\omega_\lambda(gy,Ty)[1+\omega_\lambda(fx,Sx)]}{1+\omega_\lambda(fx,gy)}\}} \varphi(t)dt$

(3.2.3) $\omega_\lambda(Sx, Ty) < \infty$

Then f, g, S and T have a coincidence point. If the pairs (f, S) and (g, T) are occasionally weakly compatible then f, g, S and T have a common fixed point in X_ω.

Proof: Since the pairs (f, S) and (g, T) are occasionally weakly compatible then there exists points u and v in X_ω, such that

(3.2.4) Su = fu and gv = Tv

Now, from (3.2.2) we have

$\int_0^{\omega_\lambda(Su,Tv)} \varphi(t)dt \leq a_1 \int_0^{\omega_\lambda(fu,gv)} \varphi(t)dt + a_2 \int_0^{\omega_\lambda(fu,Su)} \varphi(t)dt + a_3 \int_0^{\omega_\lambda(gv,Tv)} \varphi(t)dt$

$+ a_4 \int_0^{\omega_\lambda(gv,Su)+\omega_{2\lambda}(fu,Tv)} \varphi(t)dt + a_5 \int_0^{\frac{\omega_\lambda(gv,Su,)[1+\omega_\lambda(fu,Tv)]}{1+\omega_\lambda(fu,gv)}} \varphi(t)dt$

$+ a_6 \int_0^{\frac{\omega_\lambda(gv,Tv)[1+\omega_\lambda(fu,Su)]}{1+\omega_\lambda(fu,gv)}\}} \varphi(t)dt$

$= a_1 \int_0^{\omega_\lambda(fu,gv)} \varphi(t)dt + a_2 \int_0^{\omega_\lambda(fu,fu)} \varphi(t)dt + a_3 \int_0^{\omega_\lambda(gv,gv)} \varphi(t)dt$

$+ a_4 \int_0^{\omega_\lambda(gv,fu)+\omega_{2\lambda}(fu,gv)} \varphi(t)dt + a_5 \int_0^{\frac{\omega_\lambda(gv,fu,)[1+\omega_\lambda(fu,gv)]}{1+\omega_\lambda(fu,gv)}} \varphi(t)dt$

$+ a_6 \int_0^{\frac{\omega_\lambda(gv,gv)[1+\omega_\lambda(fu,fu)]}{1+\omega_\lambda(fu,gv)}\}} \varphi(t)dt$

$\int_0^{\omega_\lambda(Su,Tv)} \varphi(t)dt \leq (a_1 + a_5) \int_0^{\omega_\lambda(fu,gv)} \varphi(t)dt + a_4 [\int_0^{\omega_\lambda(gv,fu)} \varphi(t)dt + \int_0^{\omega_{2\lambda}(fu,gv)} \varphi(t)dt]$

By the definition of metric modular and the inequality (2.2.1), we have

$\int_0^{\omega_\lambda(Su,Tv)} \varphi(t)dt \leq (a_1 + a_4 + a_5) \int_0^{\omega_\lambda(fu,gv)} \varphi(t)dt + a_4 \int_0^{\omega_{2\lambda}(fu,gv)} \varphi(t)dt$

$= (a_1 + a_4 + a_5) \int_0^{\omega_\lambda(fu,gv)} \varphi(t)dt + a_4[\int_0^{\omega_\lambda(fu,fu)} \varphi(t)dt + \int_0^{\omega_\lambda(fu,gv)} \varphi(t)dt]$

$= (a_1 + 2a_4 + a_5) \int_0^{\omega_\lambda(fu,gv)} \varphi(t)dt$

$(1-a_1 - 2a_4 - a_5) \int_0^{\omega_\lambda(Su,Tv)} \varphi(t)dt \leq 0$

Hence, Su = Tv

So that

(3.2.5) Su = fu = Tv = gv

Moreover, let z be another point of coincidence of f and S in X_ω, i.e., fz = Sz.

Now, from (3.2.2), we have

$\int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt \leq a_1 \int_0^{\omega_\lambda(fz,gv)} \varphi(t)dt + a_2 \int_0^{\omega_\lambda(fz,Sz)} \varphi(t)dt + a_3 \int_0^{\omega_\lambda(gv,Tv)} \varphi(t)dt$

$+ a_4 \int_0^{\omega_\lambda(gv,Sz)+\omega_{2\lambda}(fz,Tv)} \varphi(t)dt + a_5 \int_0^{\frac{\omega_\lambda(gv,Sz,)[1+\omega_\lambda(fz,Tv)]}{1+\omega_\lambda(fz,gv)}} \varphi(t)dt$

$+ a_6 \int_0^{\frac{\omega_\lambda(gv,Tv)[1+\omega_\lambda(fz,Sz)]}{1+\omega_\lambda(fz,gv)}} \varphi(t)dt$

$$\int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt \le a_1 \int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt + a_2 \int_0^{\omega_\lambda(Sz,Sz)} \varphi(t)dt + a_3 \int_0^{\omega_\lambda(Tv,Tv)} \varphi(t)dt + a_4 \int_0^{\omega_\lambda(Tv,Sz) + \omega_{2\lambda}(Sz,Tv)} \varphi(t)dt + a_5 \int_0^{\frac{\omega_\lambda(Tv,Sz,)[1+\omega_\lambda(Sz,Tv)]}{1+\omega_\lambda(Sz,Tv)}} \varphi(t)dt + a_6 \int_0^{\frac{\omega_\lambda(Tv,Tv)[1+\omega_\lambda(Sz,Sz)]}{1+\omega_\lambda(Sz,Tv)}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt \le a_1 \int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt + a_2 \int_0^{\omega_\lambda(Sz,Sz)} \varphi(t)dt + a_3 \int_0^{\omega_\lambda(Tv,Tv)} \varphi(t)dt + a_4 \int_0^{\omega_\lambda(Tv,Sz) + \omega_{2\lambda}(Sz,Tv)} \varphi(t)dt + a_5 \int_0^{\frac{\omega_\lambda(Tv,Sz,)[1+\omega_\lambda(Sz,Tv)]}{1+\omega_\lambda(Sz,Tv)}} \varphi(t)dt + a_6 \int_0^{\frac{\omega_\lambda(Tv,Tv)[1+\omega_\lambda(Sz,Sz)]}{1+\omega_\lambda(Sz,Tv)}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt \le (a_1 + a_4 + a_5) \int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt + a_4 \int_0^{\omega_{2\lambda}(Sz,Tv)} \varphi(t)dt$$

$$= (a_1 + a_4 + a_5) \int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt + a_4[\int_0^{\omega_\lambda(Sz,Sz)} \varphi(t)dt + \int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt]$$

$$= (a_1 + 2a_4 + a_5) \int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt$$

$$(1-a_1 - 2a_4 - a_5) \int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt \le 0$$

Hence, Sz = Tv.

So that,

(3.2.6) Sz = fz = Tv = gv

Therefore, from (3.2.5) and (3.2.6) we have

Sz = Su

This implies that z = u.

Thus, z = fu = Su is the coincidence point of f and S.

Then by Lemma 2.2, z be the unique common fixed point of f and S.

Similarly, there is another fixed point v in X_ω, such that v = gv = Tv.

Suppose, v ≠ z, then by (3.2.2), we have

$$\int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt \le a_1 \int_0^{\omega_\lambda(fz,gv)} \varphi(t)dt + a_2 \int_0^{\omega_\lambda(fz,Sz)} \varphi(t)dt + a_3 \int_0^{\omega_\lambda(gv,Tv)} \varphi(t)dt + a_4 \int_0^{\omega_\lambda(gv,Sz) + \omega_{2\lambda}(fz,Tv)} \varphi(t)dt + a_5 \int_0^{\frac{\omega_\lambda(gv,Sz,)[1+\omega_\lambda(fz,Tv)]}{1+\omega_\lambda(fz,gv)}} \varphi(t)dt + a_6 \int_0^{\frac{\omega_\lambda(gv,Tv)[1+\omega_\lambda(fz,Sz)]}{1+\omega_\lambda(fz,gv)}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(z,\ v)} \varphi(t)dt \le a_1 \int_0^{\omega_\lambda(z,v)} \varphi(t)dt + a_4[\int_0^{\omega_\lambda(v,z)} \varphi(t)dt + \int_0^{\omega_{2\lambda}(z,v)} \varphi(t)dt] + a_5[\int_0^{\omega_\lambda(v,z)} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(z,v)} \varphi(t)dt \le (a_1 + 2a_4 + a_5) \int_0^{\omega_\lambda(z,v)} \varphi(t)dt .$$

Which is a contradiction.

Hence, z = v.

Thus, z is a unique common fixed point of f, g, S and T in X_ω.

Remark 3.1. Theorem 3.2 remains true, if we take $\varphi(t)$ =1. Hence, we have the following corollary:

Corollary 3.3. Let X_ω be a complete modular space and f, g, S, T : $X_\omega \to X_\omega$ are mappings such that S(X_ω) ⊆ g(X_ω) and T(X_ω) ⊆ f(X_ω) and one of the spaces f(X_ω) or g(X_ω) be a ω-complete subspace of X_ω. Suppose there exists numbers $a_1, a_2, \ldots a_6 \in [0,1)$ with at least one of $a_i > 0$ (i = 1,2, . . . 6) such that for all x, y ∈ X_ω and λ > 0, the following assertion hold:

(3.3.1) $a_1 + 2a_2 + a_5 < 1$ for all $0 \le a_1, a_2, a_3\ a_4, a_5, a_6 < 1$.

(3.3.2) $\omega(Sx,\ Ty) \le a_1\omega_\lambda(fx,gy) + a_2\omega_\lambda(fx,Sx) + a_3\omega_\lambda(gy,Ty) + a_4[\omega_\lambda(gy,\ Sx) + \omega_{2\lambda}(fx,Ty)] + a_5 \frac{\omega_\lambda(gy,Sx,)[1+\omega_\lambda(fx,Ty)]}{1+\omega_\lambda(fx,gy)} + a_6 \frac{\omega_\lambda(gy,Ty)[1+\omega_\lambda(fx,Sx)]}{1+\omega_\lambda(fx,gy)}$

(3.3.3) $\omega(Sx,\ Ty) < \infty$

Then f, g, S and T have a coincidence point. If the pairs (f, S) and (g, T) are occasionally weakly compatible then f, g, S and T have a common fixed point in X_ω.

Remark 3.2. If we put S = T = Ix_ω, where Ix_ω be an identity mapping on X_ω in the theorem 3.2, then we have the following corollary:

Corollary 3.4. Let X_ω be a complete modular space and f, g : $X_\omega \to X_\omega$ are mappings such that g(X_ω) ⊆ f(X_ω) and one of the spaces f(X_ω) or g(X_ω) be a ω-complete subspace of X_ω. Suppose there exists numbers a_1, a_2, a_3, a_4 ∈ [0,1) with at least one of a_i > 0 (i = 1,2, 3,4) such that for all x, y ∈ X_ω and λ > 0, the following assertion hold:

(3.4.1) $a_1 + 2a_2 < 1$ for all $0 \le a_1, a_2, a_3, a_4 < 1$.

(3.4.2) $$\int_0^{\omega_\lambda(gx,gy)} \varphi(t)dt \le a_1 \int_0^{\omega_\lambda(fx,fy)} \varphi(t)dt + a_2 \int_0^{\omega_\lambda(fx,gx)} \varphi(t)dt + a_3 \int_0^{\omega_\lambda(fy,gy)} \varphi(t)dt + a_4 \int_0^{\omega_\lambda(fx,gy) + \omega_{2\lambda}(fx,gy)} \varphi(t)dt$$

(3.4.3) $\omega_\lambda(fx,\ gy) < \infty$

Then f and g have a coincidence point. If the pair (f, g) is occasionally weakly compatible then f and g have a common fixed point in X_ω.

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