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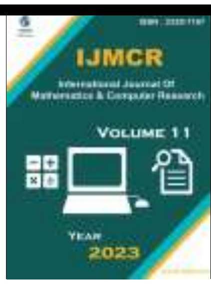

A Note on the Bourbaki-Completeness and the Cofinal Bourbaki-Completeness

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ABSTRACT
In this paper, we review two properties of completeness known as the Bourbaki-completeness and cofinal Bourbaki-completeness in the setting of metric spaces. These notions came from new classes of generalized Cauchy sequences appearing when characterizing the so-called Bourbaki-boundedness in a similar way that Cauchy sequences characterize the totally boundedness. For the clustering of Bourbaki–Cauchy sequences and cofinally Bourbaki–Cauchy sequences, we have respectively what is call Bourbaki-completeness and cofinal Bourbaki-completeness of metric spaces. The topological problem of metrizability by means of a Bourbaki-complete or a cofinally Bourbaki-complete metric has also been considered. Finally, we present detailed review of some relationships and mutual differences between these kinds of completeness.

KEYWORDS: Bourbaki, Cofinal Bourbaki, Bourbaki-Completeness, Bourbaki-Boundedness, Totally Boundedness, Cauchy Sequences, Bourbaki- Cauchy Sequences

1. INTRODUCTION

The concept of two new properties of completeness was introduced by M. Isabel Garrido and Ana S. Meroño **(2014)** and some classes of metric spaces satisfying stronger than completeness but weaker than compactness have been recently studied by many authors. A good reference for this topic is the nice paper by Beer **(2008)**, entitled “Between compactness and completeness”. Examples of these properties are the bounded compactness, the uniform local compactness, the cofinal completeness, the strong cofinal completeness, recently introduced by Beer **(2012),** as well as the so-called UC-ness for metric spaces. The study of all these spaces have shown to be not only interesting by themselves but also in connection with some problems in Convex Analysis, in Optimization Theory and in the setting of Convergence Structures on Hyperspaces, see for instance Beer, G and S. Levi **(2008)**.

First of all, note that a way to achieve a property stronger than completeness for a metric space consists of asking for the clustering of all the sequences belonging to some class bigger than the class of Cauchy sequences. Thus, the class of Bourbaki–Cauchy sequences and the class of cofinally Bourbaki–Cauchy sequences have been defined. These sequences appear in a metric space when the so-called Bourbaki-bounded sets are cosidered. This notion of boundedness was introduced by Atsuji **(1958)** in order to exhibit metric spaces where every real-valued uniformly continuous function is bounded but they are not necessarily totally bounded. The name of Bourbaki-bounded cames from the book of Bourbaki **(1966),** where these subsets in uniform spaces is considered.The Bourbaki-bounded subsets of a metric space have been characterized in terms of sequences in the same way that Cauchy sequences characterize total boundedness. Thus, a new type of sequences appears that is call Bourbaki–Cauchy sequences. Next, studies have shown another class of sequences which are cofinal with respect to the previous ones, in the sense that the residuality of the indexes is replaced by the cofinality. Thus, we present a detailed review of both of these properties showing, in particular, that they are stronger than the usual completeness but weaker than compactness, and also that they are mutually different. The general aim of this paper is to review the relationships and mutual differences between the two properties of completeness in the context of metric spaces.

2. PRELIMINARIES

Definition 1. Let (X, d) be a metric space. A subset $B \subset X$ is said to be bounded if it has finite diameter, $d(B) < \infty$, that is , when it is contained in some open ball.

Definition 2. Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in it. The sequence $\{x_n\}$ is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists a positive integer N such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

Definition 3. A metric space (X, d) is said to be complete if every Cauchy sequence in it converges to an element of it

Definition 4. Let (X, d) be a metric space. A subset $B \subset X$ is said to be totally bounded if, and only if , every sequence in B has a Cauchy subsequence

Definition 5. A subset B of a metric space (X, d) is said to be a Bourbaki – bounded subset of X if for every $\varepsilon > 0$ there exist $m \in N$ and a finite collection of points $p_1, \ldots, p_k \in X$ such that, $B \subset \bigcup_{i=1}^{k} B_\varepsilon^m(p_i)$.

Note that the family $\boldsymbol{B}$ of all Bourbaki- bounded subsets of X forms a bornology in X, that is , B satistfies the following conditions: (i) for every $x \in X$, the set $\{x\} \in \boldsymbol{B}$; (ii) if $B \in \boldsymbol{B}$ and $A \subset B$, then $A \in \boldsymbol{B}$; (iii) if $A, B \in \boldsymbol{B}$ then $A \cup B \in \boldsymbol{B}$.

Definition 6. A let (X, d) be a metric space. A sequence $(x_n)_{n \in N}$ is said to be Bourbaki – Cauchy in X if for every $\epsilon > 0$ there exist $m \in N$ and $n_0 \in N$ such that for some $p \in X$ we have that $x_n \in B_\varepsilon^m(p)$, for every $n \geq n_0$.

Definition 7. A sequence $(x_n)_{n \varepsilon N}$ in a metric space (X, d) is called cofinally Cauchy if for every $\varepsilon > 0$ there exists an infinite subset N_ε of $\boldsymbol{N}$ such that for each $i, j \in N_\varepsilon$ we have $d\left(x_i, x_j\right) < \varepsilon$.

Definition 8. A let (X, d) be a metric space. A sequence $(x_n)_{n \in N}$ is said to be cofinally Bourbaki – Cauchy in X if for every $\epsilon > 0$ there exist $m \in N$ and an infinite subset $N_\varepsilon \subset N$ such that for some $p \in X$ we have that $x_n \in B_\varepsilon^m(p)$, for every $n \in N_\varepsilon$.

Definition 9. A metric space (X, d) is said to be Bourbaki – complete if every Bourbaki – Cauchy sequence in X clusters (i.e., it has some convergent subsequence).

Definition 10. A metric space is said to be cofinally Bourbaki –complete if every cofinally Bourbaki – Cauchy sequence clusters.

3. BOURBAKI – CAUCHY AND COFINALLY BOURBAKI – CAUCHY SEQUENCES

Recall that in a metric space a subset is said to be (metric) bounded if it has finite diameter, that is, when it is contained in some open ball. This notion of boundedness is quite natural but it has some inconvenience. Namely, to be metric bounded is not a uniform invariant, that is, it is not preserved under uniform homeomorphisms. However, in the setting of normed spaces metric boundedness is a uniform property and in fact the bounded subsets can be characterized by means of uniformly continuous functions. Indeed, it is easy to see that a subset B of a normed space X is bounded by the norm if, and only if, for every real-valued uniformly continuous function f on X, we have that $f(B)$ is bounded in R endowed with its usual metric. As we have said in the introduction, it was Atsuji who introduced in **(1958)** the notion of Bourbaki-bounded metric space (under the name of finitely-chainable metric space) and he showed that they are just the metric spaces having bounded image under every real-valued uniformly continuous function defined on it. These spaces were also considered in the frame of uniform spaces by Hejcman in **(1959)** who called them simply as bounded.

Theorem 1. Let (X, d) be a metric space, the following statements are equivalent:
(1) X is a Bourbaki-bounded metric space.
(2) (Atsuji) For every real uniformly continuous function f on X, $f(X)$ is bounded in R.
(3) (Hejcman) X is d'-bounded, for every metric d' uniformly equivalent to d.
(4) (Njastad) Every star-finite uniform cover of X is finite.

Theorem 2. For a metric space (X, d) and $B \subset X$, the following statements are equivalent:

(1) B is a Bourbaki-bounded subset in X.
(2) Every countable subset of B is a Bourbaki-bounded subset in X.
(3) Every sequence in B cofinally Bourbaki-Cauchy in X.
(4) Every sequence in B is cofinally Bourbaki-Cauchy in X.

Proof. (1) ⇒ (2) By definition it is clear that every subset of a Bourbaki-bounded subset of X is also a Bourbaki- bounded subset in X.
(2) ⇒ (3) Let $(x_n)_{n \in N}$ be a sequence in B. By the hypothesis, the set $\{x_n : n \in \boldsymbol{N}\}$ is Bourbaki-bounded in X, and then for $\epsilon = 1$ there exist $m_1 \in N$ and some points $p_1^1, \ldots, p_{j_1}^1 \in X$ such that,

$$\{x_n : n \in N\} \subset \bigcup \{B_1^{m_1}(p_i^1) : i = 1, \ldots, j_1\}.$$

Since the family $\{B_1^{m_1}(p_i^1) : i = 1, \ldots, j_1\}$. is finite, then some $B_1^{m_1}\left(p_{i_1}^1\right)$ contains infinite terms of the sequence. Therefore, there exists a subsequence $(x_n^1)_{n \in N}$ of $(x_n)_{n \in N}$ inside to $B_1^{m_1}(p_{i_1}^1)$.

Repeating this process we have that, for every $k \geq 2$ and $\varepsilon = \frac{1}{k}$, there exist some $m_k \in \boldsymbol{N}$ and points $p_1^k, \ldots, p_{j_k}^k \in X$, such that $\left\{B_{1/k}^{m_k}(p_i^k) : i = 1, \ldots, j_k\right\}$. Is a finite cover of $\{x_n^{k-1} : n \in \boldsymbol{N}\}$. Then there exist some $B_{1/k}^{m_k}\left(p_{i_k}^k\right)$ containing some subsequence $(x_n^k)_{n \in N}$ of $(x_n^{k-1})_{n \in N}$

Finally, choosing the standard diagonal subsequence $(x_n^k)_{n\in N}$ we can check, in an easy way, that it is the required Bourbaki–Cauchy subsequence of $(x_n)_{n\in N}$.
(3) ⇒ (4) Trivial.
(4) ⇒ (1) Suppose that B is not a Bourbaki-bounded subset of X. Then, there exists $\varepsilon_0 > 0$ such that, for every $m \in \boldsymbol{N}$, the family $\{B_{\varepsilon_0}^m(x): x \in X\}$ does not contain any finite subcover of B. Fix $x_0 \in X$ and for every $m \in \boldsymbol{N}$choose $x_m \in B$ such that $x_m \notin B_{\varepsilon_0}^m(x_i), i = 0, \dots, m-1$. Then, the sequence $(x_m)_{m\in N}$ constructed in this way is not a cofinally Bourbaki–Cauchy sequence in X. Otherwise, for this ε_0 there must exist $m_0 \in \boldsymbol{N}$ and an infinite subset $\boldsymbol{N}_{\boldsymbol{\varepsilon 0}} \subset \boldsymbol{N}$ such that for some $p_0 \in X$ we have that $x_n \in B_{\varepsilon_0}^{m_0}(p_0)$, for every $n \in N_{\varepsilon_0}$. Then taking $n_0 \in N_{\varepsilon_0}$, we have that there are infinitely many terms of the sequence $(x_m)_{m\in N}$ in $B_{\varepsilon_0}^{2m_0}(x_{n_0})$, which is a contradiction. ∎

4. BOURBAKI-COMPLETENESS AND COFINAL BOURBAKI-COMPLETENESS

Theorem 3. *The following statements are equivalent for a metric space* (X, d)*:*
(1) X *is compact.*
(2) X *is totally bounded and complete.*
(3) X *is Bourbaki-bounded and Bourbaki-complete.*

Proof. It is well known the equivalence between (1) and (2) On the other hand, as we said before, if X is compact then it is Bourbaki-complete. And since every compact space is totally bounded then it is also Bourbaki-bounded, and hence (1) implies (3). Conversely, in order to see that (3) imply (1), take any sequence of X. By Bourbaki-boundedness this sequence has a Bourbaki–Cauchy subsequence and by Bourbaki-completeness this subsequence clusters. Therefore every sequence in X clusters and then X is compact. ∎

Another useful relation between compactness and Bourbaki-completeness is the following.

Theorem 4. *A metric space is Bourbaki-complete if, and only if, the closure of every Bourbaki-bounded subset is compact.*
Proof. First of all, note that the closure $\bar{B}$, of a Bourbaki-bounded set B is also Bourbaki-bounded. Indeed, for every $\varepsilon > 0$ there exist $m \in N$ and some points$p_1, \dots, p_k \in X$ such that $B \subset \bigcup_{i=1}^k B_\varepsilon^m(p_i)$. Since $\bar{B} \subset \bigcup_{i=1}^k B_\varepsilon^{m+1}(p_i)$, we follows that $\bar{B}$ is also Bourbaki-bounded in X. Now, let (X, d) be Bourbaki-complete and B a Bourbaki-bounded subset of X. In order to see that $\bar{B}$ is compact, let $(x_n)_{n\in N}$ be a sequence of $\bar{B}$. Then, according to Theorem 2, $(x_n)_{n\in N}$ has a Bourbaki–Cauchy subsequence in X. Then by Bourbaki-completeness this subsequence clusters in X. But, $\bar{B}$ is closed and then $(x_n)_{n\in N}$ clusters in $\bar{B}$. Therefore $\bar{B}$ is compact.

Conversely, let $(x_n)_{n\in N}$ be a Bourbaki–Cauchy sequence of X then $\{x_n : n \in N\}$ is a Bourbaki-bounded subset of X, and by hypothesis $\overline{\{x_n : n \in N\}}$ is compact. Hence $(x_n)_{n\in N}$ clusters, and therefore (X, d) is Bourbaki-complete. ∎

Then last result says that a metric space is Bourbaki-complete if, and only if,
every closed and Bourbaki-bounded subset is compact. According to this fact, we are going to see that only finite dimensional Banach spaces can be Bourbaki-complete. In particular, that means that in some sense completeness and Bourbaki-completeness are very far from one another. Recall that an analogous result exists for cofinal completeness (see Beer).

Corollary 1. *A Banach space is Bourbaki-complete if, and only if, it is finite*
dimensional.

Proof. It is clear that every finite dimensional Banach space is Bourbaki-complete since every closed and bounded subset is compact. Conversely, if the Banach space is a Bourbaki-complete metric space, then according last result, its unit closed ball must be compact since in normed spaces bounded subsets are also Bourbaki-bounded. Finally, if the unit ball of a normed space is compact, then it is well known that it must have finite dimension. ∎

Theorem 5. *The following statements are equivalent for a metric space* (X, d)*:*
(1) X *is compact.*
(2) X *is totally bounded and cofinally complete.*
(3) X *is Bourbaki-bounded and cofinally Bourbaki-complete.*

Recall that the equivalence between (1) and (2) was pointed by Beer. On the other hand, according to Theorem 3, it is clear that in above condition (3) cofinal Bourbaki-completeness should be paired with a weaker boundedness notion corresponding to the property that each sequence has a cofinally Bourbaki–Cauchy subsequence. But note that, this weaker notion would be again Bourbaki-boundedness, as we can deduce easily from Theorem 2.

Now, it is interesting to see that the completeness properties are not only weaker than compactness but also weaker that uniform local compactness. Recall that a metric space (X, d) is said to be uniformly locally compact whenever there exists some $\delta > 0$ such that the set $\overline{B_\delta(x)}$ is compact, for every $x \in X$.

Proposition 1. *Every uniformly locally compact metric space is cofinally Bour-*
baki-complete.
Proof. Firstly, let $\delta > 0$ such that, for every $x \in X$, $\overline{B_\delta(x)}$ is compact. We can see that if $K \subset X$ is compact, then $\overline{K^{\delta/2}}$ is also compact. Indeed, from the open cover of $K \subset \bigcup_{y\in K} B_{\delta/2}(y)$, we can take a finite subcover $K \subset \bigcup_{i=1}^n B_{\delta/2}(y_i)$. Since

$K^{\delta/2} \subset \cup_{i=1}^{n}\left(B_{\delta/2}(y_i)\right)^{\delta/2} \subset \cup_{i=1}^{n}\overline{B_{\delta}(y_i)}$ and $\cup_{i=1}^{n}\overline{B_{\delta}(y_i)}$

is compact, it follows that $\overline{K^{\delta/2}}$ is compact, as wanted. And that means that, in particular, for every $x \in X$ and every $m \in N$, the set $\overline{B^{m}_{\delta/2}(y_i)}$is compact.

Now, if $(x_n)_{n\in N}$ is a cofinally Bourbaki–Cauchy sequence, then there exist $x \in X$ and $m \in N$ such that $\{x_n : n \in N\}$ is cofinally contained in $B^{m}_{\delta/2}(y_i)$.Therefore, by compactness of $B^{m}_{\delta/2}(y_i)$, we have that $(x_n)_{n\in N}$ clusters. ■

According to the above, we have the following diagram:
uniformly locally compact =⇒ cofinally Bourbaki-complete =⇒ cofinally complete

Theorem 6. *For a metric space (X, d) the following statements are equivalent:*
(1) *X is uniformly locally compact.*
(2) *X is locally totally bounded and cofinally complete.*
(3) *X is locally Bourbaki-bounded and cofinally Bourbaki-complete.*
Proof. That (1) ⇒ (3) follows at once from Proposition 13. (3) ⇒ (2) This implication can be obtained easily taking into account that, in particular, condition (3) implies that X is in addition locally compact. Indeed, let $x \in X$ and let V be a Bourbaki-bounded neighborhood of x. Take B any closed ball around x contained in V . Now it is easy to check that B is both Bourbakibounded
(since this property is hereditary) and cofinally Bourbaki-complete (since this property is inherited by closed sets). Then, from Theorem 5, B is a compact neighborhood of x.
(2) ⇒ (1) Firstly note that, as in the above implication, we can see that (2) implies also the local compactness of X. Next, suppose by contradiction that X is not uniformly compact, then for every $n \in N$, there exists such that $x_n \in X$ $\overline{B_{1/n}(x_n)}$ is not compact. Then, by local compactness of X, we can assert that the sequence $(x_n)_{n\in N}$ does not cluster. Now, for every $n \in N$, let $(y^n_k)_{k\in N}$ be a sequence in $\overline{B_{1/n}(x_n)}$ without cluster points. Next, consider a partition of N into a countable family of infinite subsets $\{M_n, n \in N\}$. Finally, defining the sequence Z_k=y^n_k, if $k \in M_n$ it is easy to check that $(z_k)_{k\in N}$ is a cofinally Cauchy sequence which does not cluster. ■

Proposition 2. *Every UC metric space is cofinally Bourbaki-complete.*
Proof. Let $(x_n)_n$ be a cofinally Bourbaki–Cauchy sequence, that we can suppose has no constant subsequence. According to the above characterization by Hueber, we are going to see that $(x_n)_n$ has a subsequence along which the isolation functional goes to zero. Indeed, for every $j \in N$ there exist $M_J \in N$ and $p_j \in X$ such that $\{x_n : n \in N\}$ is cofinally contained in $B^{m_j}_{1/j}(p_j)$. Since every $x \in B^{m_j}_{1/j}(p_j)$ satisfies $I(x) < 1/j$, then we can construct a subsequence $\left(x_{n_j}\right)_j$ such that $\lim_{j\to\infty} I\left(x_{n_j}\right) = 0$, as wanted. ■

5. CONCLUSION

We finish this review just linking Bourbaki and cofinal Bourbaki-completeness with the well known class of UC-spaces. In this line, we have seen that every UC metric space is cofinally Bourbaki-complete, and hence Bourbaki-complete. Recall that a metric space (X, d) is called UC or Atsuji when every real continuous function on X is uniformly continuous. There are several characterizations of these spaces, as we can see in the nice paper by Jain and Kundu **(2006)**

REFERENCES

1. Arkhangel ′skii, A.V., and V.I. Ponomarev:(1984) Fundamentals of general topology, problems and exercises. - D. Reidel Publishing Co., Dordrecht.
2. Atsuji, M.: (1958) Uniform continuity of continuous functions of metric spaces. - Pacific J. Math. 8, 11–16. New types of completeness in metric spaces 757
3. Beer, G.: (1981) Which connected metric spaces are compact? - Proc. Amer. Math. Soc. 83, 1981, 807–811.
4. Beer, G.: (2012) Between the cofinally complete spaces and the UC spaces. - Houston J. Math. 38, 999–1015.
5. Beer, G., and S. Levi: (2008) Gap, excess and bornological convergence. - Set-Valued Anal. 16, 1347–1362.
6. Beer, G., and M. Segura: (2009) Well-posedness, bornologies, and the structure of metric spaces. - Appl. Gen. Topol. 10, 131–157.
7. Bourbaki, N.: (1966) Elements of mathematics, general topology, part 1. - Hermann, Paris,
8. Buhagiar, D., and I. Yoshioka: (2001) Ultracomplete topological spaces. - Acta Math. Hungar. 92, 2001, 19–26.
9. Čech, E.: (1937) On bicompact spaces. - Ann. of Math. 38, 823–844.
10. Corson, H.H.: (1958) The determination of paracompactness by uniformities. - Amer. J. Math. 80, 185–190.
11. Garrido, I., and A.S. Meroño: (2013) Uniformly metrizable bornologies. - J. Convex Anal. 20, 285–299.
12. Garrido, I., and A.S. Meroño: (2012) Some classes of bounded sets in metric spaces. - In: Contribuciones matemáticas en homenaje a Juan Tarrés, eds. M. Castrillón et al., Universidad Complutense de Madrid, Madrid, 179–186.

13. Hejcman, J.: (1959) Boundedness in uniform spaces and topological groups. - Czechoslovak Math. J. 9, 544–563.
14. Hohti, A.: (1981) On uniform paracompactness. - Ann. Acad. Sci. Fenn. Ser. A I Math. Diss. 36.
15. Howes, N.R.: (1971) On completeness. - Pacific J. Math. 38, 431–440.
16. Hueber, H.: (1981) On uniform continuity and compactnessin metric spaces.- Amer. Math. Monthly 88, 204–205.
17. Jain, T., and S. Kundu: Atsuji spaces: (2006) equivalent conditions. - Topology Proc. 30. 301–325.
18. M. Isabel Garrido and Ana S. Meroño (2014) New Types of Completeness in Metric Spaces.- Annales Academic Scientiarum Fennicae Mathematica, Volume 39, 733-758.
19. Musaev, D.K.: (2007) Uniformly superparacompact, completely paracompact, and strongly para- compact uniform spaces. - J. Math. Sci. (N.Y.) 144, 2007, 4111–4122.
20. Njåstad, O.: (1965) On uniform spaces where all uniformly continuous functions are bounded. - Monatsh. Math. 69, 167–176. 758 M. Isabel Garrido and Ana S. Meroño
21. Pears, A.R.: (1975) Dimesion theory of general spaces. - Cambridge Univ. Press, Cambridge.
22. Rice, M.D.: (1977) A noteon uniform paracompactness.- Proc. Amer. Math. Soc. 62, 359–362.
23. Romaguera, S.: (1998) On cofinally complete metric spaces. - Questions Answers Gen. Topology 16, 165–170.
24. Smith, J.C.: (1978) Review of “A note on uniform paracompactness” by Michael D. Rice. - Math. Rev. 55.
25. Woods, R.G.: (1995) The minimun uniform compactification of a metric space. - Fund. Math. 147, 39–59.
26. Yasui, Y.: (1968) Unions of strongly paracompact spaces II. - Proc. Japan Acad. 44, 27–31. [37]
27. Yoshioka, I.: (2002) On the subsets of non locally compact points of ultracomplete spaces. - Com- ment. Math. Univ. Carolin. 43, 707–721.