[Volume 4 issue 06 June 2016]
PageNo.1429-1440 ISSN : 2320-7167

INTERNATIONAL JOURNAL OF MATHEMATICS AND COMPUTER RESEARCH

Existence, Uniqueness and Asymptotic Behaviour of Solutions for a class of Parabolic Systems involving $(p_1(x), p_2(x))-$Laplacian

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Abstract

This paper presents the existence and uniqueness of the initial and boundary value problem to a system of evolution $(p_1(x), p_2(x))-$Laplacian systems

$$\begin{cases} u_{1t} - \operatorname{div}(|\nabla u_1|^{p_1(x)-2}\nabla u_1) + a_1(x)\,|u_1|^{p_1(x)-2}\,u_1 = f_1(x,t,u_1,u_2), \\ u_{2t} - \operatorname{div}(|\nabla u_2|^{p_2(x)-2}\nabla u_2) + a_2(x)\,|u_2|^{p_2(x)-2}\,u_2 = f_2(x,t,u_1,u_2). \end{cases}$$

With general assumptions on a_i, f_i and the exponent $p_i(x), (i = 1, 2)$, we prove the existence and the uniqueness in propre spaces. The asymptotic behaviour of solutions is also discussed. The results are proved by using a method to construct a sequence of approximations solutions and use a standard limiting process.

Keywords : p(x)-laplacian operator; Existence; Uniqueness; Variable exponents; Parabolic System; Asymptotic Behaviour.

A.M.S. Subject Classification 35A01, 35A02, 35G55

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with Lipshitz continuons boundary $\partial\Omega$. We consider the following $(p_1(x), p_2(x))$ − laplacian systems :

$$\begin{cases} \frac{\partial u_1}{\partial t} - \Delta_{p_1(x)} u_1 + a_1(x)\,|u_1|^{p_1(x)-2}\,u_1 = f_1(x,t,u_1,u_2), & \text{in } \Omega \times (0,T), \\ \frac{\partial u_2}{\partial t} - \Delta_{p_2(x)} u_2 + a_2(x)\,|u_2|^{p_2(x)-2}\,u_2 = f_2(x,t,u_1,u_2), & \text{in } \Omega \times (0,T), \\ u_1 = u_2 = 0, & \text{in } \partial\Omega \times (0,T), \\ (u_1(.,0), u_2(.,0)) = (\varphi_1, \varphi_2), & \text{on } \Omega. \end{cases} \tag{1.1}$$

where $p_i(x) \in C(\overline{\Omega})$ is a function, $(i = 1, 2)$. The operator $-\Delta_{p(x)} w = -\operatorname{div}\left(|\nabla w|^{p(x)-2}\,\nabla w\right)$ is called $p(x) - Laplacian$, which will be reduced to the $p - Laplacian$ when $p(x) = p$ a constant.

The $(p_1(x), p_2(x))-$laplacian systems (1.1) can be viewed as a generalization of $(p,q)-$laplacian system

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u &= f(x,u,v), \qquad \text{in } \ Q_T, \\ \frac{\partial v}{\partial t} - \Delta_q v &= g(x,u,v) \qquad \text{in } \ Q_T, \\ u = v = 0 & \qquad \text{on } \partial\Omega \times (0,T), \\ (u(.,0), v(.,0)) = (\varphi_1, \varphi_2) & \qquad \text{in } \ \Omega. \end{cases} \tag{1.2}$$

For the case $p_i(x) = p_i > 2, i = 1,2,$ systems (1.2) models as non-Newtonian fluids [27, 2] and nonlinear filtration [2], etc. In the non-Newtonian fluids theory, (p_1, p_2) is a characteristic quantity of the fluids, there have been many results about the existence, uniqueness of the solutions. We refer the readers to the bibliography given in [11, 13, 8, 9, 12, 14, 26] and the references therein.

In recent years, the research of nonlinear problems with variable exponent growth conditions has been an interesting topic. $p(\cdot)$-growth problems can be regarded as a kind of nonstandard growth problems and these problems possess very complicated nonlinearities, for instance, the $p(x)$-Laplacian operator $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is inhomogeneous. And these problems have many important applications in nonlinear elastic, electrorheological fluids and image restoration. The reader can find in ([22, 31]) several models in mathematical physics where this class of problem appears.

The case of a single equation of the type (1.1) has been studied in [5, 6, 7, 25] and the authors established the existence and uniqueness results, in [25], the authors use the difference scheme to transform the parabolic problem to a sequence of elliptic problems and then obtain the existence of solutions with less constraint to $p_i(x)$.

The more intersting question concerning parabolic systems of $(p_1(x), p_2(x))$-Laplacian type is to understand the asymptotic behavior of solutions when time gows to infinity. The study of the asymptotic behaviour of the system is giving us relevant information about the structure of the phenomenon described in the model.

Concerning the elliptic systems with variable exponents, the results about existence and non-existence are proved in [10, 30, 33, 18].

In this paper, we consider the existence and uniqueness for the system of the type (1.1) under some assumptions. The proof consists of two steps. First, we prove that the approximating problem admits a global solution; then we do some uniform estimates for these solutions. We mainly use skills of inequality estimation and the method of approximation solutions. By a standard limiting process, we obtain the existence to system of the type (1.1).

The outline of this paper is the following: In Section 2, we introduce some basic Lebesgue and Sobolev spaces and state our main theorems. In Section 3, we give the existence and uniqueness of weak solutions. The asymptotic behaviour of solution is established in Section 4.

2 Basic spaces and the main results

To consider problems with variable exponents, one needs the basic theory of spaces $L^{r(x)}(\Omega)$ and $W^{1,r(x)}(\Omega)$. For the convenience of readers, let us review them briefly here. The détails and more properties of variable-exponent Lebesgue-Sobolev spaces can be found in [19, 20].

Denote

$$r^- := \min_{\overline{\Omega}} r(x), \quad r^+ = \max_{\overline{\Omega}} r(x),$$

Let $r(x) \in C(\overline{\Omega})$. When $r^- > 1$, one can introduce the variable-exponent Lebesgue space

$$L^{r(x)}(\Omega) = \left\{ w : \Omega \to \mathbb{R};\ u \text{ is measurable and } \int_\Omega |w|^{r(x)}\, dx < \infty \right\},$$

endowed with the Luxemburg norm.

$$\|w\|_{r(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{w}{\lambda} \right|^{r(x)} dx \leq 1 \right\}.$$

Thanks to results in [19], the following inequality holds :

$$\min \left\{ \|w\|_{r(x)}^{+} \|w\|_{r(x)}^{-} \right\} \leq \int_\Omega |w|^{r(x)}\, dx \leq \max \left\{ \|w\|_{r(x)}^{r^+}, \|w\|_{r(x)}^{r^-} \right\}.$$

Moreover, let $r_i(x) \in C(\overline{\Omega})$, with $r_i^- > 1, i = 1, 2$. Then, if $r_1(x) \leq r_2(x)$ for any $x \in \Omega$, the imbedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$ is continuous, the norm of the imbedding does not exceed $|\Omega| + 1$. As $r^- > 1$, and the space is reflexive Banach space with dual $L^{r^*(x)}(\Omega)$, where $\frac{1}{r(x)} + \frac{1}{r^*(x)} = 1$. Beside, for any $v \in L^{r^*(x)}(\Omega)$, we have the following Hölder-type inequality :

$$\int_\Omega |uv|\, dx \leq \left(\frac{1}{r^-} + \frac{1}{(r^-)^*} \right) \|u\|_{r(x)} \|v\|_{r^*(x)}.$$

For positive integer k, the generalized Lebesgue-Sobolev space is defined as

$$W^{k,r(x)}(\Omega) = \left\{ u \in L^{r(x)}(\Omega) : D^\alpha u \in L^{r(x)}(\Omega), |\alpha| \leq k \right\}.$$

endowed with the norm

$$\|u\|_{W^{k,r(x)}(\Omega)} = \sum_{\alpha \leq k} \|D^\alpha u\|_{r(x)}.$$

Such spaces are separable and reflexive Banach spaces.

Besides, $r(x)$ is log-Hölder continuous, i.e., there exists a constant C such that

$$|r(x) - r(y)| \leq -\frac{C}{\log|x-y|}, \text{ for any } x, y \in \Omega \text{ with } |x-y| < \frac{1}{2}. \tag{2.1}$$

Under assumption (2.1), the smooth functions are dense in Sobolev spaces with variable exponents, and we can define $W_0^{k,r(x)}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ in $W^{k,r(x)}(\Omega)$ with respect to the norm $\|.\|_{W^{k,p(x)}(\Omega)}$; see [20]. For $u \in W_0^{1,r(x)}(\Omega)$, the Poincaré-type inequality holds, i.e.

$$\|u\|_{r(x)} \leq C \|\triangledown u\|_{r(x)}, \tag{2.2}$$

where the positive constant C depend on r and Ω. So $\|\triangledown u\|_{r(x)}$ is an equivalent norm in $W_0^{1,r(x)}(\Omega)$.

Proposition 2.1 *([20]) If we denote*

$$\rho(w) = \int_{\Omega} |w|^{r(x)}\, dx, \ \forall w \in L^{r(x)}(\Omega),$$

then

(i) $|w|_{r(x)} < 1 \ (= 1; > 1) \Leftrightarrow \rho(w) < 1 \ (= 1; > 1)$;
(ii) $|w|_{r(x)} > 1 \Rightarrow |w|_{r(x)}^{r^-} \le \rho(w) \le |w|_{r(x)}^{r^+}$; $|w|_{r(x)} < 1 \Rightarrow |w|_{r(x)}^{r^+} \le \rho(w) \le |w|_{r(x)}^{r^-}$;
(iii) $|w|_{r(x)} \to 0 \Leftrightarrow \rho(w) \to 0$; $|w|_{r(x)} \to \infty \Leftrightarrow \rho(w) \to \infty$.

Problem (1.1) does not admit classical solutions in general. So, we introduce weak solutions in the following sence.

Definition 2.2 *A function $u = (u_1, u_2)$ is said to be a weak solution of system (1.1), if u satisfies the following: $u_i \in L^{\infty}(0, T; W_0^{1,p_i(x)}) \cap L^{p_i(x)}(Q_T)$ such that :*

$u_{it} \in L^2(Q_T)$ and for $\phi_i \in C_0^{\infty}(Q_T)$

$$\int_0^T \int_{\Omega} (u_{it}\phi_i + |\nabla u_i|^{p_i(x)-2} \nabla u_i . \nabla \phi_i + a_i(x) |u_i|^{p_i(x)-2} u_i \phi_i = \int_0^T \int_{\Omega} f_i(x, u)\phi_i dxdt$$

and $u_i(0, .) = \varphi_i$ a.e in Ω.

In the study of the global existence of solutions, we need the following hypotheses (H):

(H1) The exponents $p_i(x)$ satisfies the local logarithmic Hölder continuity condition (2.1) and $p_1(x) \le p_2(x)$ with $2 < p_1^- \le p_1(x) \le p_1^+ < +\infty$ and $2 < p_2^- \le p_2(x) \le p_2^+ < +\infty$.

(H2) $(\varphi_1, \varphi_2) \in (L^{\infty}(\Omega) \cap W_0^{1,p_1(x)}(\Omega)) \times (L^{\infty}(\Omega) \cap W_0^{1,p_2(x)}(\Omega))$.

(H3) $\exists\ K_i \in \mathbb{R}$ such that $0 < K_i \le a_i(x) \in L^{\infty}(\Omega)$, $i = 1, 2$.

(H4) $H(x, t, s_1, s_2) \in C^2(\overline{\Omega} \times [0, T] \times \mathbb{R}^2)$

3 Main results

Our main existence result is the following:

Theorem 3.1 *Assume that hypothesis (H1)-(H4) are satisfied. Then system (1.1) admits a unique solution $u = (u_1, u_2) \in \left(C([0, T) ; L^2(\Omega))\right)^2$. Moreover, the mapping $(\varphi_1, \varphi_2) \to (u_1(t), u_2(t))$ is continuous in $\left(L^2(\Omega)\right)^2$.*

Proof of the main results.

a) Existence.

The proof of Theorem 3.1 is based on a priori estimates.

Starting from a suitable initial iteration

$$(u_1^{(0)}(x, t), u_2^{(0)}(x, t)) = (\varphi_1(x), \varphi_2(x)),$$

we construct a sequence $\left\{(u_1^{(n)}(x), u_2^{(n)}(x)\right\}_{n=1}^{\infty}$ from the iteration process

$$\begin{cases} u_{it}^{(n)} - \operatorname{div}\left(\left|\nabla u_i^{(n)}\right|^{p_i(x)-2}\nabla u_i^{(n)}\right) + a_i(x)\left|u_i\right|^{p_i(x)-2}u_i = f_i(x,t,u_1^{(n-1)},u_2^{(n-1)}), \text{in } Q_T, \\ u_i^{(n)} = 0, \qquad\qquad \text{on } \partial\Omega\times(0,T), \\ u_i^{(n)}(x,0) = (\varphi_1(x),\varphi_2(x)), \end{cases} \tag{3.1}$$

where $i = 1, 2$. It is clear that for each $n = 1, 2, ...,$ the above systems consists of two uncoupled initial boundary-value problems. By resuls (see, [5]) for fixed n the probem has a solution $u_i^{(n)} \in L^\infty(Q_T)\cap L^\infty(0,T;W_0^{1,p_i(x)})$. In the following, we prove that $u_i^{(n)} \to u_i$, as $n\to\infty$, $(i = 1, 2)$.

Remark 3.2 *In this paper, we shall denote by c, C_i differents constants, depending on $p_i(x), T, \Omega$, but not on n, which may vary from line to line. Sometimes we shall refer to a constant depending on specific parameters $C_i(T)$, etc.*

(i) Multiplying the first equation in (3.1) by $|u_i^{(n)}|^k u_i^{(n)}$ and using the growth condition on a_i and f_i , we deduce that

$$\frac{1}{k+2}\frac{d}{dt}\int_\Omega |u_i^{(n)}|^{k+2}dx + K_i\int_\Omega |u_i^{(n)}|^{k+p_i(x)}dx \le m_i\int_\Omega |u_i^{(n)}|^{k+1}dx. \tag{3.2}$$

Setting $y_{n,k}(t) = \|u_i^{(n)}\|_{L^{k+2}(\Omega)}$ and using Hölder's inequality on both sides of (3.2), there exist two constants $c > 0$ and $c' > 0$ (independent of k and n) such that

$$\frac{dy_{n,k}(t)}{dt} + cy_{n,k}^{p_i^- - 1}(t) \le c';$$

which implies from Ghidaglia's lemma [32] that

$$y_{n,k}(t) \le \Big(\ c + \frac{c'}{\left[(p_i^- - 2)t\right]^{\frac{1}{p^- - 2}}}\ , \forall t > r, r > 0, \tag{3.3}$$

as $k\to+\infty$, and for all $t \ge r > 0$, we have

$$\|u_i^{(n)}\|_{L^\infty(\Omega)} \le \Big(\ c + \frac{c'}{\left[(p_i^- - 2)t\right]^{\frac{1}{p^- - 2}}}\ , \forall t > r, r > 0. \tag{3.4}$$

(ii) Multiplying the first equation in (3.1) by $u_i^{(n)}$ and integrating over Q_T,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|u_i^{(n)}\right|^2 dx + \int_\Omega \left|\nabla u_i^{(n)}\right|^{p_i(x)} dx + \int_\Omega a_i(x)|u_i^{(n)}|^{p_i(x)}dx = \int_\Omega f_i(x,t,u_1^{(n-1)},u_2^{(n-1)})u_i^{(n)}dx. \tag{3.5}$$

If we take assumption (H3), we have

$$\int_\Omega a_i(x)|u_i^{(n)}|^{p_i(x)}dx \ge K_i\int_\Omega \left|u_i^{(n)}\right|^{p_i(x)} dx. \tag{3.6}$$

By Young's inequality and the Poincaré inequality, we obtain

$$\int_\Omega \left|u_i^{(n)}\right|^2 dx \le \epsilon \int_\Omega \left|u_i^{(n)}\right|^{p_i(x)} dx + C_i \le \frac{1}{2}\int_\Omega \left|\nabla u_i^{(n)}\right|^{p_i(x)} dx + C_i, \tag{3.7}$$

$$\int_\Omega f_i(x,t,u_1^{(n-1)},u_2^{(n-1)})u_i^{(n)} dx \le c_i \int_\Omega \left|f_i(x,u_1^{(n-1)},u_2^{(n-1)})\right|^2 dx + k_i \int_\Omega \left|u_i^{(n)}\right|^2 dx + C_i. \tag{3.8}$$

Taking the inequality above into (3.4), we obtain :

$$\frac{d}{dt}\int_\Omega \left|u_i^{(n)}\right|^2 dx + \frac{1}{2}\int_\Omega \left|\nabla u_i^{(n)}\right|^{p_i(x)} dx + K_i \int_\Omega \left|u_i^{(n)}\right|^{p_i(x)} dx \le C_i. \tag{3.9}$$

iii) Now, multiplying the first equation in (3.1) by $u_{it}^{(n)}$, we get

$$\begin{aligned} &\int_\Omega \left|u_{it}^{(n)}\right|^2 dx + \frac{d}{dt}\int_\Omega \frac{1}{p_i(x)}\left|\nabla u_i^{(n)}\right|^{p_i(x)} dx + \frac{d}{dt}\int_\Omega \frac{a_i(x)}{p_i(x)}\left|u_i^{(n)}\right|^{p_i(x)} dx \\ &\le \int_\Omega f_i(x,t,u_1^{(n-1)},u_2^{(n-1)})u_{it}^{(n)} dx. \end{aligned} \tag{3.10}$$

By Young's inequaltiy, we deduce

$$\int_0^T\int_\Omega \left|u_{it}^{(n)}\right|^2 dx + \int_0^T\int_\Omega \frac{1}{p_i(x)}\left|\nabla u_i^{(n)}\right|^{p_i(x)}(x,T)dx + c_i\int_\Omega \left|u_i^{(n)}\right|^{p_i(x)}(x,T))dx \le C_i(T). \tag{3.11}$$

By (3.4) and (3.11), there exists a subsequence of $u_i^{(n)}$ (denoted again by itself, $i=1,2$) and a function u_i such that as $n\to+\infty$:

$$u_i^{(n)} \to u_i, \qquad weak \;\; in \;\; L^{p_i(x)}(0,T;W_0^{1,p_i(x)}(\Omega)) \quad and \quad in \quad L^{p_i(x)}(Q_T),$$

$$u_{it}^{(n)} \to u_{it}, \quad in \quad L^2(Q_T),$$

$$\left|\nabla u_i^{(n)}\right|^{p_i(x)-2}\nabla u_i^{(n)} \overset{weak}{\to} \chi_i \quad in \quad L^{\frac{p_i(x)}{p_i(x)-1}}(Q_T),$$

By the same argument as that in [35] , we have that $\chi_i = |\nabla u_i|^{p_i(x)-2}\nabla u_i$. To conclude that $u=(u_1,u_2)$ is a weak solution of system (S) it is enough to observe that $f_i(x,t,u_1^{(n-1)},u_2^{(n-1)})$ converges to $f_i(x,t,u_1,u_2)$ strongly in $L^1(Q_T)$ and in $L^s(0,T;L^s(\Omega))$ for all $s\ge 1$, thanks to Vitali's theorem.

b) Uniqueness.

Assume that $u=(u_1,u_2)$ and $v=(v_1,v_2)$ are two solutions of (1.1).

Let $w_i = u_i - v_i, i=1,2$, then we have

$$\int_0^t\int_\Omega (-u_i(w_i)_t) + |\nabla u_i|^{p_i(x)-2}\nabla u_i \nabla w_i + a_i(x)|u_i^{(n)}|^{p_i(x)} w_i \, dxdt$$
$$+\int_\Omega u_i(x,t)w_i(x,t) - \int_\Omega u_{i0}w_i(x,0)dx$$
$$=\int_0^t\int_\Omega f_i(x,t,u_1,u_2)w_i dxdt, \text{ a.e. } \quad t\in(0,T).$$

$$\int_0^t\int_\Omega (-v_i(w_i)_t) + |\nabla v_i|^{p_i(x)-2}\nabla v_i \nabla w_i + a_i(x)|v_i^{(n)}|^{p_i(x)} w_i \, dxdt$$
$$+\int_\Omega v_i(x,t)w_i(x,t) - \int_\Omega v_{i0}w_i(x,0)dx$$
$$=\int_0^t\int_\Omega f_i(x,t,v_1,v_2)w_i dxdt, \text{ a.e. } \quad t\in(0,T).$$

Subtracing the 2 equations, we get

$$\int_\Omega (u_i-v_i)^2dx + \int_0^t\int_\Omega \left(|\nabla u_i|^{p_i(x)-2}\nabla u_i - |\nabla v_i|^{p_i(x)-2}\nabla v_i\right)\nabla(u_i-v_i)dxdt$$
$$+\int_0^t\int_\Omega a_i(x)\left(|u_i^{(n)}|^{p_i(x)} - |v_i^{(n)}|^{p_i(x)}\right)(u_i-v_i)dxdt$$
$$=\int_0^t\int_\Omega \left(f_i(x,t,u_1,u_2)-f_i(x,t,v_1,v_2)\right)(u_i-v_i)dxdt.$$

Applying the folowing basic inequality, for any $y,z\in\mathbb{R}^N$

$$\left(|y|^{r(x)-2}y - |z|^{r(x)-2}z\right).(y-z) \geq 2^{2-r^+}|y-z|^{r(x)}, \quad \text{if } r(x)\geq 2$$

Note that

$$\left(|\nabla u_i|^{p_i(x)-2}\nabla u_i - |\nabla v_i|^{p_i(x)-2}\nabla v_i\right)\nabla(u_i-v_i)\geq 0, \quad i=1,2.$$

Using the previous inequality and the Lipschitz condition, a simple calculation shows that

$$\int_\Omega (|u_1-v_1|^2+|u_2-v_2|^2)dx \leq c\int_0^t\int_\Omega (|u_1-v_1|^2+|u_2-v_2|^2)dxdt,$$

Set

$$H(T) = \int_0^T \int_\Omega (|u_1 - v_1|^2 + |u_2 - v_2|^2) dxdt,$$

then the above inequality can be written as

$$H'(T) \le cH(T).$$

A standard argument shows that $H(T) = 0$ since $H(0) = 0$, and hence $u_i = v_i,\ i = 1, 2$.
Thus the solution is unique. The continuity of the the mapping $(\varphi_1, \varphi_2) \to (u_1(t), u_2(t))$ can be obtained similarly. ■

4 ASYPMTOTIC BEHAVIOUR

This section is devoted to the asymptotic behaviour of solutions. In order to prove the asymptotic behaviour, we assume

(H5) $f_1(x, t, u_1, u_2)u_1 + f_2(x, t, u_1, u_2)u_2 \le 0$

Theorem 4.1 *The weak solution $u = (u_1(t), u_2(t))$ obtained in Theorem 3.1, satifies :* $\int_\Omega |u_1(x,t)|^2 dx + \int_\Omega |u_2(x,t)|^2 dx \le \frac{C_1}{(C_2 t + C_3)^\alpha}$, *where $C_i > 0$ (i=1,2,3), $\alpha = \frac{2}{\beta - 2}, \beta = p_1^-$ or p_2^+ or p_2^-.*

Proof. Let u_i be solution of (1.1)
Multiplying the first equation in (1.1) by u_1 and integrating over Q_T,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_1|^2 dx + \int_\Omega |\nabla u_1|^{p_1(x)} dx + \int_0^T \int_\Omega a_1(x)|u_1|^{p_1(x)} dx = \int_0^T \int_\Omega f_1(x, u_1, u_2)u_1 dx, \quad (4.1)$$

Multiplying the second equation in (1.1) by u_2 and integrating over Q_T,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_2|^2 dx + 1\int_\Omega |\nabla u_2|^{p_2(x)} dx + \int_0^T \int_\Omega a_2(x)|u_2|^{p_2(x)} dx = \int_0^T \int_\Omega f_2(x, u_1, u_2)u_2 dx. \quad (4.2)$$

Summing up (4.0) and (4.1), we have from hypothes (H5) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_1|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |u_2|^2 dx = \int_\Omega |\nabla u_1|^{p_1(x)} dx + \int_\Omega |\nabla u_2|^{p_2(x)} dx \le 0. \quad (4.3)$$

By $u_i \in W_0^{1,p_i(x)}(\Omega)$, using Poincaré inequality, we obtain

$$\|u_i\|_{L^2}^2 \le c\,\|\nabla u_i\|_{L^2}^2 \le c\,\|\nabla u_1\|_{p_i(x)}^2. \quad (4.4)$$

If $|\nabla u_1|_{p_1(x)} > 1$ and $|\nabla u_2|_{p_2(x)} > 1$, by Proposition 2.1,

$$|\nabla u_1|_{p_1(x)}^{p_1^-} \le \int_\Omega |\nabla u_1|^{p_1(x)} dx \text{ and } |\nabla u_2|_{p_2(x)}^{p_2^-} \le \int_\Omega |\nabla u_2|^{p_2(x)} dx. \quad (4.5)$$

According to the assumption that $p_1(x) \le p_2(x)$, Then $2 < p_1^- \le p_1^+ \le p_2^- \le p_2^+$.
Hence, we get from (4.2) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_1|^2\,dx+\frac{1}{2}\frac{d}{dt}\int_\Omega |u_2|^2\,dx+C_1\left(\int_\Omega |u_1|^2\,dx\right)^{\frac{p^-}{2}}+C_2\left(\int_\Omega |u_2|^2\,dx\right)^{\frac{p^-}{2}}\le 0,\ \text{a.e},\ t\ge 0. \tag{4.6}$$

By the formula $\left(\frac{a+b}{2}\right)^\alpha \le a^\alpha + b^\alpha,\ \forall a,b>0,\ \alpha>1,$ we have

$$\left(\frac{1}{2}\int_\Omega \left[|u_1|^2\,dx+|u_2|^2\right]dx\right)^{\frac{p^-}{2}} \le C\left(\int_\Omega |\nabla u_1|^{p_1(x)}\,dx\right)^{\frac{p^-}{2}}+\left(\int_\Omega |\nabla u_2|^{p_2(x)}\,dx\right)^{\frac{p^-}{2}}, \tag{4.7}$$

this implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_1|^2\,dx+\frac{1}{2}\frac{d}{dt}\int_\Omega |u_2|^2\,dx+C_3\left(\int_\Omega \left[|u_1|^2\,dx+|u_2|^2\right]dx\right)^{\frac{p^-}{2}}\le 0,\ a.e, t\ge 0 \tag{4.8}$$

where $C_3=\min(C_1,C_2)$.
Denote

$$H(t)=\int_\Omega \left[|u_1|^2\,dx+|u_2|^2\right]dx.$$

Then, we obtain from (4.8) that

$$H'(t)+CH(t)^{\frac{P^-}{2}}\le 0. \tag{4.9}$$

If $|\nabla u_1|_{p_1(x)}<1$ and $|\nabla u_2|_{p_2(x)}<1$, by Proposition 2.1,

$$|\nabla u_1|_{p_1(x)}^{p_1^+}\le \int_\Omega |\nabla u_1|^{p_1(x)}\,dx \text{ and } |\nabla u_2|_{p_2(x)}^{p_2^+}\le \int_\Omega |\nabla u_2|^{p_2(x)},$$

Then we get (4.4) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_1|^2\,dx+\frac{1}{2}\frac{d}{dt}\int_\Omega |u_2|^2\,dx+C_1\left(\int_\Omega |u_1|^2\,dx\right)^{\frac{p_2^+}{2}}+C_2\left(\int_\Omega |u_2|^2\,dx\right)^{\frac{p_2^+}{2}}\le 0,\ \text{a.e},\ t\ge 0. \tag{4.10}$$

That is

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_1|^2\,dx+\frac{1}{2}\frac{d}{dt}\int_\Omega |u_2|^2\,dx+C_3\left(\int_\Omega \left[|u_1|^2\,dx+|u_2|^2\right]dx\right)^{\frac{p_2^+}{2}}\le 0,\ a.e, t\ge 0. \tag{4.11}$$

Again we have

$$H'(t)+CH(t)^{\frac{p_2^+}{2}}\le 0.$$

Similarly, if $|\nabla u_1|_{p_1(x)}>1$ and $|\nabla u_2|_{p_2(x)}<1$, or $|\nabla u_1|_{p_1(x)}<1$ and $|\nabla u_2|_{p_2(x)}>1$, we can also obtain the similar results

$$H'(t)+CH(t)^{\frac{p_1^+}{2}}\le 0,\ \text{or}\ \ H'(t)+CH(t)^{\frac{p_2^-}{2}}\le 0.$$

Hence

$$\int_\Omega \left[|u_1|^2\,dx+|u_2|^2\right]dx\le \frac{C_1}{(C_2t+C_3)^\alpha},\quad \alpha=\frac{2}{\beta-2},\quad \beta=p_1^-\ or\ \ p_2^+\ or\ p_2^-,\ C_i>0,\quad i=1,2,3.$$

The proof is complete. ■

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