International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 11 Issue 01 January 2023, Page no. – 3190-3194

Index Copernicus ICV: 57.55, Impact Factor: 7.362

DOI: 10.47191/ijmcr/v11i1.11

Summation of the Fourier Coefficients of the Non-Holomorphic Eisenstein Cusp Series of the *PSL*(2,Z) Group

Orchidea Maria Lecian

Sapienza University of Rome, Faculty of Medicine and Pharmacy,
Viale Regina Elena, 324 - 00185 Rome, Italy;
Sapienza University of Rome, Faculty of Medicine and Dentistry,
Piazzale Aldo Moro, 5 - 00185 Rome, Italy;
Kursk State University, Faculty of Physics, Mathematics and Information Sciences,
Chair of Algebra, Geometry and Didactics of Mathematics Theory, ul. Radis'c'eva, 33, aud. 201, Kursk,
Russia.

ARTICLE INFO

Published Online: 25 January 2023

Corresponding Author: **Orchidea Maria Lecian**

ABSTRACT

The Fourier coefficients of the non-holomorphic Eisenstein cusp forms of the Laplace-Beltrami operator are here summed.

The summation is performed after the Dirichlet series related to the divisor function, after the properties of the meromorphic continuability; a new dependence on the Euler's γ constant is also found.

KEYWORDS: Number theory; Summations.

I. INTRODUCTION

The Fourier expansion of the non-holomorphic Eisenstein cusp series was spelled out in [1] pag. 76 and pag.'s 508-540.

In [4], the meromorphic continuability of the $\zeta_k(s)$ functions is analysed, and a power estimate is obtained of the growth of these functions as $s \to \infty$ in $0 < Re(s) \leq 1$. The asymptotic formula of the $\zeta_k(s)$ as $|s| \to \infty$ is discussed in [5].

In [2], the desymmetrized *PSL*(2,Z) group is considered, the parabolic elements are considered, and the remainders for particular divisor functions related to the $\zeta_k(s)$ functions are calculated.

In [6], the number-theory approach of the divisor function is framed.

In [3], the use of the divisor function in the faltungs of the Fourier coefficients in the Eisenstein-Maass series is outlined. In [7], the absence of eigenvalues in the discrete spectrum within the interval $(0,-1/4]$ is controlled.

In the present study, the Fourier coefficients of the Eisenstein cusp series forms of the *PSL*(2,Z) group are summed. The Fourier coefficients are here summed according to the Dirichlet sum of the divisor function; a new dependence on the Euler's γ constant is also found.

The paper is organized as follows.

In Section I, the studies about the Eisenstein cusp series and the discrete divisor faltungs are recapitulated.

In Section II, the Fourier coefficients of the Eisenstein series of the *PSL*(2,Z) group are recalled.

The meromorphic continuability is explained to be an essential tool in the study of the divisor functions and of the spectral analysis in Section III for the Hecke operators.

The summation of the coefficients of the Fourier expansions of the cusp Eisenstein forms is performed in Section IV.

Outlook and perspectives are discussed in Section V.

In the Appendix, the meromorphic continuability is complemented and commented.

II. THE FOURIER COEFFICIENTS OF THE EISENSTEIN SERIES OF THE *PSL*(2,Z) GROUP

The distribution of the Fourier coefficients of the Eisenstein series is found in [1] Ch. 11.4 as from the Fourier series expansion of the Eisenstein series $E(z;s)$

$$E(z,s) = y^s + \phi(s)y^{1-s} + \sum_{1\leq m<+\infty} \sqrt{y}\phi_m(s)K_{s-\frac{1}{2}}(2\pi\mid m\mid y)e^{2\pi i m x} \qquad (1)$$

with the coefficients $\varphi(s)$ and $\varphi_n(s)$ defined ibid. pag. 508 as

$$\phi(s) \equiv \sqrt{\pi}\frac{\Gamma\ s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} = \pi^{2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}\frac{\zeta(2-2s)}{\zeta(2s)}, \qquad (2)$$

and pag. 76

$$\phi_m(s) \equiv 2\pi^s\frac{m^{s-\frac{1}{2}}}{\Gamma(s)}\sum_{j=1}^{j=\infty}\frac{c_j(m)}{k^2} \qquad (3)$$

where the latter is summed for the considered grouppal structure as

$$\phi_m(s) = 2\pi^s\frac{m^{s-\frac{1}{2}}}{\Gamma(s)}\frac{\sigma_{1-2s}[\mid m\mid]}{\zeta(2s)} \qquad (4)$$

$$\phi_m(s) = 2\pi^s\frac{m^{s-\frac{1}{2}}}{\Gamma(s)}\frac{\sigma_{1-2s}[\mid m\mid]}{\zeta(2s)} \qquad (4)$$

The non-holomorphic Eisenstein series $E(z,s)$ becomes

$$E(z,s) = \sum_{n=-\infty}^{n+\infty} a_n(y,s)e^{2\pi inx}, \qquad (5)$$

$$a_n(y,s) = 2\mid n\mid^{s-\frac{1}{2}}\sigma_{1-2s}(\mid n\mid)\sqrt{y}K_{s-\frac{1}{2}}(2\pi\mid n\mid y) \qquad (7)$$

III. THE DIVISOR FUNCTION AND THE SPECTRAL DECOMPOSITION

Some properties of the spectral decomposition of the automorphic Laplacian are recalled in [4].

The meromorphic continuability of the zeta functions $\zeta_k(s)$ to the complete complex plane is obtained after the powers estimate of the growth of the $\zeta_k(s)$'s in the limit $s \to \infty$ in $0 < Re(s) \leq 1$. In particular, the ζ functions $\zeta_k(s)$

$$\zeta_k(s) = \sum_{n=1}^{n=\infty}\frac{\sigma(n)\sigma(n+k)}{n^s} \qquad (8)$$

in $Re(s) > 1$ are analysed according to the asymptotic formula as $\mid s\mid \to \infty$ in the critical strip $0 < Re(s) \leq 1$ as from [5].

In [6] a demonstration is provided with, of the fact that $\zeta_k(s)$ can be meromorpihcally continued into the critical strip $0 < Re(s) \leq 1$.

From Th. 1 in [4], ζ_k can be meromorphically continued into the whole *x,iy* plane for $Re(s) > 0$.

According to the proper representation, the continuation is represented in terms of $\Gamma(s)$ and of the 'polar terms' of $\zeta_k(s)$.

A reformulation of the division function $\sigma_w(k)$ is provided as

$$\sigma_w(k) = k^{w-\frac{1}{2}}\sigma_{1-2w}(k). \qquad (9)$$

From Th. 2, at the critical strip exterior of a neighbourhood δ of its poles, there exists an ϱ as small as leisure such that the $\zeta_k(s)$ functions admit the majorization

$$\mid \zeta_k(s) <<_{\epsilon} \frac{1}{\delta}(s)log^4(\mid s\mid +1)\,k^{1+\epsilon}; \qquad (10)$$

for this, in the interval $\frac{1}{2}+\delta < \sigma < 1+\delta$, in the limit $\mid s\mid \to \infty$, the following majorization holds

$$\zeta_k(s) <<_{\epsilon} \frac{k^{1+\epsilon}}{\delta^3}s^{2-2\sigma+\delta}log^4(\mid s\mid). \qquad (11)$$

Therefore, from Lemma 1.1, one has that an Eisenstein-Maass series admits a meromorphic continuation to the whole *x,iy* plane as

$E^\star(z,s) = \xi(2s)$; (12)

$E(z,s)$ is integral.

$E(z,s)$ can be expanded in a Fourier series, which is absolutely convergent $\forall s \in C$ as

$$E^\star(z,s) = \xi(2s)y^s + \xi(2-2s)y^{1-s} + 2\sqrt{y}\sum_{n\neq0,\ n=-\infty}^{n=+\infty}\sigma_s(\mid n\mid)K_{s-\frac{1}{2}}(2\pi\mid n\mid y)e^{2\pi inx}; \qquad (13)$$

it satisfies the functional equation

$E^\star(z,1-s) = E^\star(z,s)$. (14)

One demonstrates that at $y \to \infty$ the infinite series safely does not exceed e^{-y}.

From the estimate of Kuznetsov for the sum of Kloosterman sums, the constant c is defined i.e. the particular case $s = 0$ can also be studied; more in detail, one has that

$$E^\star(z)=E^\star\left(z,\frac{1}{2}\right)=\sqrt{y}ln[y-c]+2\sqrt{y}\sum_{n\neq 0}\sum_{n=-\infty}^{n=+\infty}\sigma(|n|)K_0(2\pi|n|y)e^{2\pi inx} \quad (15)$$

and

$$|E^\star(z)-yln[y-c]|<<e^{-y} \quad (16)$$

at $y\rightarrow\infty$.

a. The discrete faltungs In [3], the convolution of the divisor function is defined. The sums in the faltung are to be expressed in the forms of combinations of the corresponding Fourier coefficients of the eigenfunctions of the automorphic Laplace operator.

The trivial representation of a modular group is taken into account.

The analogue of the discrete faltung identities of the real analytic Eisenstein series of weight zero is looked for.

The spectrum of the Hecke operators as far as parabolic forms are concerned is considered.

The Hecke operators are denoted and normalized according to the eigenvalues n according to the weight of the form. The multiplicativity Hecke relations are outlined not to depend on the weight of the form.

The Hecke series are obtained after making correspond to the eigenvalue of the $n-th$ eigenfunction of the Hecke operator the $n-th$ coefficient of Dirichlet series.

IV. SUMMATION OF THE FOURIER COEFFICIENTS OF THE NON-HOLOMORPHIC EISENSTEIN CUSP SERIES OF THE *PSL*(2,Z) GROUP

The Eisenstein series Eq.(1) is summed after summing the Fourier coefficients Eq. (4) with the tools developped in Section III applied the Dirichlet formula, for which one now here therefore for the terms following Eq.'s (5) has

$$\sigma_{1-2s}[|m|]=|m|^{\frac{1}{2}-s}\sigma_s[|m|] \quad (17)$$

and, for $s\geq 1$,

$$\sigma_{1-2s}[|m|]=|m|^{\frac{1}{2}-s}\left[s\log(s)+(2\gamma-1)s+O(\sqrt{s})\right] \quad (18)$$

γ being the Euler's γ constant.

Formula Eq. (4) is therefore now summed as

$$\phi_\gamma(s)=2\pi^s\frac{1}{\Gamma(s)}\frac{[s\log(s)+(2\gamma-1)s+O(\sqrt{s})]}{\zeta(2s)} \quad (19)$$

i.e. the dependence of $\varphi_m(s)$ in Eq. (4) is now summed as a dependence on the Euler's γ constant.

It is crucial to remark that the result obtained in Eq. (19) allows one to sum the Fourier coefficients of the Eisenstein series of Eq. (1) for each s, i.e. to resolve the dependence on the summation value m of the Eisenstein series only to the modified Bessel functions of the second kind $K_{2\pi|m|y}$ and on the trigonometric function of the chosen boundary conditions, and a new dependence on the Euler's γ constant is found.

V. OUTLOOK AND PERSPECTIVES

The present works contains the result of the summation of the Fourier coefficients of the cusp Eisenstein series forms of *PSL*(2,Z). The Fourier coefficients are majorized and summed at all the n orders of he series expansion. A new dependence on the Euler's γ constant is found.

The results here obtained can be extended to the subgroups of *PSL*(2,Z).

The analysis of the Monster group, and, more in particular, of the Modular Monster group, ca be therefore improved. The study of the boundary conditions is to be implemented.

The Hamiltonian problem associated to that of the Laplace-Beltrami operator on the hyperbolic plane can therefore now be considered [15], [14], according to the geodesics lines, on which there lye the solutions of such Hamiltonian problem.

Acknowledgments

The Programme Education in Russia for Foreign Citizens of the Ministry of Science and of Higher Education of the Russian Federation is thanked.

Appendix A: COMPLEMENTS OF THE MEROMORPHIC CONTINUABILITY

In [7], the Fourier coefficients of the eigenfunctions of the discrete spectrum are investigated within the tool of the Kloosterman sums. The Kloosterman sums are performed after the Hecke operators. The eigenfunction $\psi_j(z)$ with positive eigenvalue $\lambda>1/4$ admit a Fourier decomposition

$$\psi_j=\sum_{n\neq 0}e^{2\pi inx}\sqrt{y}\tilde{K}_{ix_j}(2\pi|n|y)\rho_j(n) \quad (A\,1)$$

with $K^{\sim}$ the Hankel function of the I kind.
The Fourier coefficients ρ_j are written after the Hankel functions of the I kind. Be T_n the Hecke operator, the equation is obtained

(A2)

$T(n)\psi_j = \mu_j(n) psi_j$
$\forall n \geq 1$

$\mu_j(n)\rho_j(1) = \rho_j(n),$ (A3)

$\forall j > 0, \rho_j(1) =6 0$. Eq. (A3) and the explanations in the following are motivated because, differently, $\forall n \geq 1$, Eq. (A3) would demand $\rho_j(n) = \rho_h(^{-}-n) = 0$, where the latter demand cannot be true $\forall n \geq 1$ because ψ_j in not identically zero.
In [8], f being normalized cuspidal Hecke eigenform of integral weight k on the full modular group $SL(2,Z)$ are controlled. Given the function D^*_f, $s \in \mathbb{C}$, the D^*_f functions are symmetric square L-functions of f, endowed with the properties elcidated after [9] and [10].
In [9], for a cusp form of a certain weight, an Euler product is taken into account, of common eigenfucntions of Hecke operators of level M, M being the modulo of a Dirichlet character of the considered cusp form.
For each s, the Euler product and the Dirichlet series are absolutely convergent for sufficiently large $Re(s)$. In [10], Dirichlet series of modular forms f are given, whose Fourier coefficients are sums of zeta functions of quadratic fields at an arbitrary complex argument. Expressions of the modular forms are found as linear combinations of Hecke eigenfunctions of the Hecke operators $T(n)$. The coefficients are Rankin zeta functions $\zeta_R(s)$ as

$$\zeta_R(s) - \sum_{n-1}^{n=\infty} \frac{(a(n))^2}{n^2} \quad \text{(A 4)}$$

such that $f \mid T(n) = a(n)f$.
The $\zeta_R(s)$ are considered at integral value within the critical strip.
Convolutions of L-series associated with modular forms are considered.The function $D_f(s)$ is related with $\zeta_R(s)$, ad has a meromorphic continuation. A new zeta function $\zeta(s,\Delta)$ is posed, Δ being the discriminant involving sums over all Γ equivalence classes of forms of discriminants Δ, and sums over inequivalent pairs of integers with respect to the group of units of the form. If Δ is the discriminant over quadratic fields K_Q, $\Rightarrow \zeta_R(s,\Delta)$ coincides with the Dedekind zeta function $\zeta_K(s)$.

A kernel function can be defined, whose series converges absolutely, and a function holomorphic in both variables is defined, which transforms as a modular form of weight K.
From Proposition 1 in [10], the kernel function of the $m-t$ Hecke operator is defined after representation of linear combination of Hecke eigenforms. It is thus possible to set the convolution of L-series associated with modular forms. From [10] formula (03), the series $D_f(s+k-1)$ are finite, such that there $\exists$ a cusp form $\forall$ eigenforms in a domain S_K containing only one cusp.
It is possible to express the $D_f(s)$ in terms of Rankin zeta functions.

Therefore, there $\exists$ a holomorphic continuation of D_f^* to C, and D_f^* is invariant under $s \to 2k-1-s$.
In [11], a relation between automorphic forms on $GL(2)$ and $GL(3)$ is established.
In [13], the matrices which are decomposed into a discrete sum of irreducible representation are classified, which each one is happening with multiplicity 1.
After [11], it is possible to state that D_f^* is a zeta function.
After [13], it is possible to state that the zeros of D^*_f can happen only in the critical strip $k-1 < Re(s) < k$. 1

The generalised Riemann hypothesis is revisited. The zeros of D^*f should all happen on the critical line $Re(s) = k-2$; $\forall s : k-1 < Re(s) < k,\ Re(s) \neq k - \frac{1}{2},\ \forall k$ large enough, $\exists$ Hecke eigenvalues f of weight k such that $D^*f =6 0$. The kernel function of $D^*f(s)$ can be defined after [10].
As from [12],be f a non-zero cusp eigenform of integral weight k on the complete modular group $SL()$, and be $L^*(f,s)$, $s \in C$, the associated Hecke function completed with the natural Γ factor. The zeros of $L^*(f,s)$ van happen only inside the critical strip $\frac{k-1}{2} < Re(s) < \frac{k+1}{2}$; according to the generalized Riemann hypothesis, the zeros should all happen on the line $Re(s) < \frac{k}{2}$.
Non-vanishing results for L-functions on the average can be envisaged are envisaged in [12] without the Riemann hypothesis.
Given $t_0 \in R$, and given $\varrho > 0, \varrho \in R$, $\forall k$ large enough, the sum of the functions $L^*(f,s)$ does not vanish on the line segment $Im(s) = t_0,\ \frac{k-1}{2} < Re(s) < \frac{k}{2} - \epsilon,\ \frac{k}{2} + \epsilon < Re(s) < \frac{k+1}{2}$, with f composed over a basis of properly-normalized Hecke eigenforms of weight k.
Therefore, from [8]form [12], the corresponding result hold for Hecke L-functions, $s \in C$, $s = \sigma + it$, with $\sigma, t \in R$, k an even integer.

REFERENCES

1. Dennis A. Hejhal, The Selberg Trace Formula for PSL (2,R): Volume 2 (Lecture Notes in Mathematics), Springer nature, Hemsbach (Germany) (1983).
2. V. A. Bykovskii, On a summation formula in the spectral theory of automorphic functions and its applications in analytic number theory, Dokl. Akad. Nauk SSSR, 264:2 (1982), 275-277.
3. N. V. Kuznetsov, Convolution of Fourier coefficients of Eisenstein-Maass series, in: Automorphic Functions and Number Theory. I, J. Soy. Math., 29, No. 2 (1985).
4. A. I. Vinogradov and L. A. Takhtadzhyan,A. I. Vinogradov and L. A. Takhtadzhyan, The Zeta function of the additive divisor problem ans spectral decomposition of the automorphic laplacian, Journal of Soviet Mathematics volume 36 , pages57-78 (1987)
5. J.-M. Deshouillers and H. Iwaniec, An additive divisor problem, J. London Math. Soc. (2), 26, 1-14 (1982).
6. N. V. Kuznetsov, Spectral methods in arithmetical problems, in: Analytic Number Theory and Theory of Functions, J. Sov. Math., 18, No. 3 (1982).
7. N. V. Kuznetsov, The Petersson hypothesis for parabolic forms of weight zero and the Linnik hypothesis. I. A sum of Kloosterman sums, Math. Sb., , No. 3, 334-383 (1980).
8. Winfried Kohnen, Jyoti Sengupta, Nonvanishing of Symmetric Square L-Functions of Cusp Forms inside the Critical Strip,Proceedings of the American Mathematical Society, Vol. 128, No. 6 (Jun., 2000), pp. 1641-1646
9. G. Shimura: On the holomorphy of certain Dirichlet series, Proc. London Math. Soc. 31, 75-98 (1975) MR 52:3064
10. D. Zagier: Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields, in 'Modular Functions of One Variable VI' (eds. J.-P. Serre and D. Zagier), pp. 105-169, LNM no. 627, Springer, Berlin-Heidelberg-New York, 1976. MR 58:5525
11. S. Gelbart and H. Jacquet: A relation between automorphic representations of GL2 and GL3, Ann. Sci. E.N.S. IV ser.,11, 471-542 (1978) MR 81e:10025
12. W. Kohnen: Nonvanishing of Hecke L-functions associated to cusp forms inside the critical strip, J. of Number Theory vol. 67, no. 2, 182-189 (1997) MR 98j:11037.
13. H. Jacquet and J. A. Shalika: A nonvanishing theorem for zeta functions of GLn, Invent. Math. 38, 1-16 (1976/77) MR 55:5583
14. Ralph Phillips, Peter Sarnak, Geodesics in homology classes, Duke Math. J. 55(2): 287-297 1987.
15. R. Miatello, N. Wallach, The resolvent of the Laplacian on locally symmetric spaces, J. Differential Geom. 36(3): 663-698 (1992).