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The Desymmetrized *PSL*(2,Z) Group, The 'Square Box' Congruence Subgroup, The Congruence Subgroups and Leaky Tori

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ABSTRACT

The desymmetrized *PSL*(2,Z) group is considered. The congruence subgroups of the desymmetrized *PSL*(2,Z) group are analysed according to the hyperbolic reflections contained. The Laplace-Beltrami problem and the Hamiltonian problem associated with the Laplace-Beltrami operator are discussed according to the boundary conditions for the grouppal structures. New constructions of the leaky tori are found; in particular, they are obtained after the generators of the *PSL*(2,Z) group, from the desymmetrized-triangle *PSL*(2,Z) group, and from the 'square-box' Γ_0 congruence subgroup of the desymmetrized *PSL*(2,Z) group.

KEYWORDS: Group Theory; Leaky Torus

I. INTRODUCTION

The group *PSL*(2,Z) is introduced in [1], [2], [3]. It is studied in [4] and [5].

The construction of leaky tori is proposed in [6], [1] and [7]. Further results are obtained in [8].

The correlations among *SL*(2,Z), its congruence subgroup $\Gamma_0(N)$ and its congruence subgroup $\Gamma(N)$ and the Monster group(s) are revisited in [9].

The analysis of the desymmetrized *PSL*(2,Z) group is motivated in [10].

In the present work, the structures associated with the desymmetrized *PSL*(2,Z) group are developped.

A new construction of the leaky torus after the folding of the desymmetrized-triangle group *PSL*(2,Z) is accomplished. The congruence subgroups of the desymmetrized *PSL*(2,Z) group are studied.

Furthermore, the Γ_0 congruence subgroup of the desymmetrized *PSL*(2,Z) group is constructed. Moreover, the subgrouppal structure of the leaky torus is newly achieved from the congruence subgroup Γ_N of $\Gamma 0$ of the desymmetrized *PSL*(2,Z) group in the limit $N \to \infty$.

The Laplace-Beltrami problem of the grouppal structures is discussed, and compared with those of the *PSL*(2,Z) case.

The Hamiltonian problem associated with the Laplace-Beltrami operator is faced as well, and the results are compared with those known in the *PSL*(2,Z) case.

The main differences are outlined.

New constructions for tori are found and motivated after the interrogations of the cited literature.

New constructions of tori is obtained after the folding of the triangle symmetric group under the suitable side identifications of generators.

Another newly-found constructions allows one to construct tori after the sides identifications of 'square-box' congruence subgroups.

The role of the cusp is outlined as providing one with the needed side identification. The solutions of the LaplaceBeltrami problems and those of the associated Hamiltonian problem are newly analysed.

The manuscript is organized as follows.

In Section I, the main motivations are recapitulated.

In Section II, the group *PSL*(2,Z) is revised.

In Section III, the desymmetrized *PSL*(2,Z) group is analysed.

In Section IV, the congruence subgroup Γ_0 of $PSL(2,Z)$ is constructed.
In Section V, new constructions of the leaky tori are found after the symmetric group $PSL(2,Z)$ according to new definitions of the generators (which define the sides identifications).
In Section VI, a new construction of the leaky torus after the desymmetrized (triangular) group $PSL(2,Z)$ is accomplished.
In Section VII, a new construction of a leaky torus after the ’square-box’ subgrouppal structure of the desymmetrized $PSL(2,Z)$ group is achieved.
In Section VIII, the outlook and the perspectives are proposed.

II. THE *PSL*(2,Z) GROUP AND THE BOUNDARY CONDITIONS OF DIFFERENT PROBLEMS

The $PSL(2,Z)$ group is generated after the three (hyperbolic) reflections

$$R_0 : z \to z' = -\frac{1}{\bar{z}}, \quad (1a)$$
$$T_{\frac{1}{2}} : z \to z' = -\bar{z} - 1, \quad (1b)$$
$$T_{-\frac{1}{2}} : z \to z' = -\bar{z} + 1 \quad (1c)$$

on the domain

$$b : x = -\frac{1/2}{}, \quad (2a)$$
$$d : x = \frac{1}{2}, \quad (2b)$$
$$c : y = \sqrt{1 - x^2}. \quad (2c)$$

The domain of the symmetric $PSL(2,Z)$ group is drawn in Fig. 1.
Dirichlet boundary conditions can be eligible for the analysis of the Hamiltonian problem associated with the LaplaceBeltrami operator.To this end, one recalls that the eigenforms $f(x,y)$ solution of the problem obey the rules

$$f(u=0,y) = T_{-\frac{1}{2}}^{-1} f(x = -\frac{1}{2}, y) = T_{\frac{1}{2}}^{-1} f(x = \frac{1}{2}, y) \quad (3)$$

which is demonstrated by construction. ✷.

III. THE DESYMMETRIZED *PSL*(2,Z) GROUP

The desymmetrized $PSL(2,Z)$ group is one defined on the (desymmetrized[1]) domain delimited after the sides

$$a : x = 0, \quad (4a)$$
$$b : x = -\frac{1}{2}, \quad (4b)$$
$$c : y = \sqrt{1 - x^2}, \quad (4c)$$

generated after the three (hyperbolic) reflections

$$T_0 : z \to z' = -\bar{z}, \quad (5a)$$
$$T_{-\frac{1}{2}}^{-\frac{1}{2}} : z \to z' = -\bar{z} + 1, \quad (5b)$$
$$R_0 : z \to z' = -\frac{1}{\bar{z}} \quad (5c)$$

The domain of the desymmetrized $PSL(2,Z)$ group is sketched in Fig. (2).
It is possible to demonstrate the boundary conditions (which undergo T_p) for the solution of the Laplace-Beltrami problem of the triangular group. The solutions of the Laplace-Beltrami problem on the upper hyperbolic half plane are geodesics, i.e. either half-circumferences centered on the x axis, or ’degenerate’ geodesics, i.e. ’vertical’ lines $x = const$.
Newmann boundary conditions can imposed for the solutions of the Hamiltonian problem associated with the LaplaceBeltrami operator,i.e. after the choice of the $\cos(n\pi x)$ in the Fourier expansion of the eigenforms. To do so, one calculates that

$$f(u=0,y) = T_{-\frac{1}{2}}^{-1} f(x = -\frac{1}{2}, y) = R_0^{-1} f(x, \sqrt{1-x^2}) \quad (6)$$

A different approach is requested for the Hamiltonian problem associated with the Laplace-Beltrami problem, of which in the above. The solutions of the Hamiltonian problem associated with the Laplace-Beltrami operator are oriented geodesics with (suitable) constraints on the phase space (i.e. such as normalization of velocity).
It is therefore necessary to distinguish between the boundary conditions of the solution of the Laplace-Beltrami operator and those of the associated Hamiltonian problem.
The Laplace-Beltrami-operator problem is solved after the eigenfunctions, which are Fourier-expanded on the considered domain according to the Fourier coefficients, under Neumann boundary conditions.
As far as the Hamiltonian problem associated with the Laplace-Beltrami operator, due to the spectral rigidity of the Laplace-Beltrami operator [11] on the Upper Poincar´e Half Plane, the same Neumann boundary conditions are applied.

IV. THE CONGRUENCE SUBGROUP Γ0 OF *PSL*(2,Z)

It is possible to construct the congruence subgroup Γ_0 of $PSL(2,)$ on the rectangular domain

$a : x = 0$, (7 a)
$b_1 : x = -1,$ (7 b)
$c: x^2 + y^2 = 1,$ (7 c)
$d_1 : (x+1)^2 + y^2 = 1,$ (7 d)

consisting of the (hyperbolic) reflections

[1] with respect to the modular-group domain

$$R_1 : z \to z' \equiv -\bar{z}, \quad (8a)$$
$$T_{-1} : z \to z' \equiv -\bar{z} + 2, \quad (8b)$$
$$T_0 : z \to z' \equiv -\bar{z}, \quad (8c)$$
$$R_2 : z \to z' = T_{\frac{1}{2}} R_1 T_{-\frac{1}{2}} z \quad (8d)$$
$$(8e)$$

Also the congruence subgroup contains a cusp at $y = \infty$.

The 'square-box' domain of the congruence subgroup is schematized in Fig. (3).

For the later purposes, it is interesting to remark that the congruence subgroup can be obtained after 'gluing' one (hyperbolically-reflected with respect to the *b* side of the desymmetrized domain) copy of the desymmetrized domain of *PSL*(2,Z). The congruence-subgroup domain Eq.'s (7) contains a cusp at $y = \infty$, and a cusp form defined on the cusp, as the parabolic element T_p, i.e. $T_p z \longrightarrow z' = z + 1$, sends (i.e. identifies) the side b_1 to the side *a*.

Boundary conditions are studied, i.e. also with the tool just described.

Dirichlet boundary conditions can be chosen for the solutions of the Hamiltonian problem associated with the LaplaceBeltrami operator, for which the eigenforms *f* would vanish on the boundaries, i.e. after the setting of the $sin(n\pi x)$ components in the Fourier expansion. For this purpose, one demonstrates that

$$f(u = 0, y) = T_{-1} f(x = -1, y) = R_1 f(x, \sqrt{1 - x^2}) = R_2 f(x, \sqrt{1 - (x-1)^2}) \quad (9)$$

by construction. ✷.

The attempt to impose Neumann boundary conditions does not succeed, as the congruence subgroup is generated after an odd number of reflections.

V. A NEW GUTZWILLER-TERRAS LEAKY TORUS CONSTRUCTION

A leaky torus is proposed in [6]; it can be obtained after unfolding the *PSL*(2,Z) [1] according to the triangles of the *PSL*(2,Z) domain in the congruence subgroup of *PSL*(2,Z) domain. The domain of the leaky torus from [6] is reported in Fig. 4. The leaky torus from [1] pag. 181 equivalent (also with respect to the generators) to that of [6] is recalled in Fig. 5. The leaky torus in [8] is equivalent to that of [1] pag. 181 with respect to both the domain and the generators.

The above-mentioned leaky torus is here newly constructed; its new generators are the reflections

$$R_n : z \to z' \equiv T_{-n} R_1 T_n z, \quad x > 0, \quad (10a)$$
$$R_{-n} : z \to z' = T_n R_1 \quad (10b)$$

with

$$T_n : T_n z = -\bar{z} - 2n, \quad x > 0, \quad (11a)$$
$$T_n : T_n z = -\bar{z} + 2n, \quad x < 0, \quad (11b)$$

and

$$R_1 : R_1 z = \frac{1}{\bar{z}} \quad (12)$$

on the domain delimited after the sides C_n defines as

$$y \geq \sqrt{1 - (x-n)^2}, \quad n - \frac{1}{2} < x \leq n + \frac{1}{2}, \quad (13)$$

in the limit $n \to \infty$.

The domain of the leaky torus constructed after the generators Eq.'s (10) is pictured in Fig. **??**; the symmetry lines with respect to which the new reflections Eq.'s (11) act are delineated. It is here crucial to remark that the generators identify the arcs of circumferences which are the sides of the tori.

The new sides identification here proposed is different from that induced in [7] after [1] pag. 181 after [6]. As interrogated in [1] pag. 181, different discrete groups may share the same fundamental domain, with different identifications of the sides, which define the subgroups.

VI. THE LEAKY TORUS FROM THE DESYMMETRIZED TRIANGLE GROUP

The Markoff uniqueness property [12] states that there ∃ an isometry between any two simple closed geodesics of equal length on a torus. Form the property, and from the rigidity of the Laplace-Beltrami operator in Riemann surfaces of constant negative curvature, it is possible to present new constructions of leaky tori.

It is possible to construct a leaky torus from the desymmetrized domain of the *PSL*(2,Z) group. The leaky torus is thus constructed after the unfolding of the chosen trajectory according to the desymmetrized (triangular) *PSL*(2,Z) group; the leaky torus is generated after the generators

$$T_{1,n} : z \to z' = T_{-\frac{1}{2}+n} R_1 T_{n-\frac{1}{2}}, \quad n - \frac{1}{2} < x \leq n, \quad (14a)$$

$$T_{2,n} z \to z' = T_{-\frac{1}{2}+n} R_2 T_{n-\frac{1}{2}}, \quad n < x \leq n + \frac{1}{2}, \quad (14b)$$

on the domain delimited after the sides C_n defined as

$$y = \sqrt{1 - \left(x - (n - \frac{1}{2})\right)^2}, \quad n - \frac{1}{2} < x \leq n, \quad (15a)$$

$$y = \sqrt{1 - \left(x - (n + \frac{1}{2})\right)^2}, \quad n < x \leq n + \frac{1}{2}. \quad (15b)$$

The domain Eq.'s (15) is depicted in Fig. 6; the symmetry lines with respect to which the hyperbolic reflections *T* in the generators Eq. (14) are outlined.

VII. THE LEAKY TORUS FROM THE 'SQUARE-BOX' SUBGROUP

It is possible to construct a new leaky torus from the congruence subgroups of Γ_0 of *PSL*(2,Z).

From the Markoff property [12], and from the rigidity of the Laplace-Beltrami operator on surfaces of constant negative curvature, it is possible to construct another scheme of leaky torus. The leaky torus is obtained here after unfolding

[13], [14] the 'square box' of the Γ_0 congruence subgroup of the desymmetrized *PSL*(2,Z) group domain into the congruence subgroup $\Gamma_0(N)$ in the limit $N \to \infty$ whose domain is generated after the reflections

$$T_{1,n+1} : z \to z' = T_{n+1}^{-1} R_1 T_{n+1} z, \quad n + \frac{1}{2} < x \leq n + 1, \tag{16a}$$

$$T_{1,n+1} : z \to z' = T_{n+1}^{-1} R_2 T_{n+1} z, \quad n < x \leq n + \frac{1}{2} \tag{16b}$$

(which contain, of course, also the case R_1, R_2), on the domain delimited after the sides c_n defined as

$$y \geq \sqrt{1 - (x - n)^2}, \quad n < x \leq n + \frac{1}{2}, \tag{17a}$$

$$y \geq \sqrt{1 - (x - (n+1))^2}, \quad n + \frac{1}{2} < x \leq n + 1 \tag{17b}$$

It is here momentous to stress that the new generators Eq.'s (16) identify the arcs of circumferences which are the sides of two different 'square boxes' delimiting the domain of the congruence subgroups of the desymmetrized *PSL*(2,Z) group; the symmetry lines with respect to which the reflections *T* from Eq.'s (8) act are delineated.

VIII. OUTLOOK AND PERSPECTIVES

The aim of the present paper is to analyse the desymmetrized *PSL*(2,Z) group and its congruence subgrouppal structures.

The boundary conditions of the solutions of the Laplace-Beltrami problem and those of the associated Hamiltonian problem are discussed.

New construction of the leaky tori are here achieved. The leaky torus proposed in [6], [1] pag. 181, and [7] is here newly constructed after new generators. One new type of construction of leaky tori, provided with in Section V, is obtained after unfolding the triangular symmetric *PSL*(2,Z) group. Another new construction of the leaky torus is expressed after unfolding the desymmetrized-triangular *PSL*(2,Z) group. The other new construction of the leaky torus, exposed in Section VII, is accomplished after unfolding the 'square boxes' constituting the domain of the examined subgrouppal structure of the desymmetrized *PSL*(2,Z) group.

The study of the desymmetrized *PSL*(2,Z) groups and its congruence subgrouppal structures are therefore relevant in the study of the boundary conditions of the Laplace-Beltrami problem and in that of the Hamiltonian problem associated with the Laplace-Beltrami operator. Indeed, as far the Laplace-Beltrami operator is concerned, boundary conditions have to be discussed for the sides of the domain with the generators (identifications); differently, the Hamiltonian problem associated with the Laplace-Beltrami operator is imposed boundary conditions on the geodesics contained in the domain, which undergo the actions of the generators.

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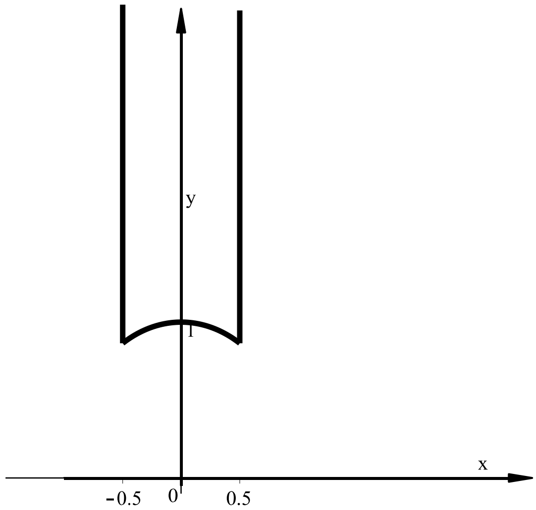

Figure 1: The domain of the group *PSL*(2,Z) on the Upper Poincar´e Half Plane (black solid line).

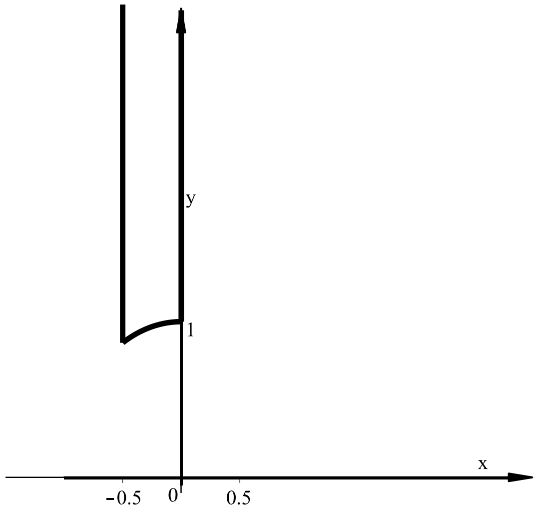

Figure 2: The domain of the desymmetrized group *PSL*(2,Z) on the Upper Poincar´e Half Plane (black solid line).

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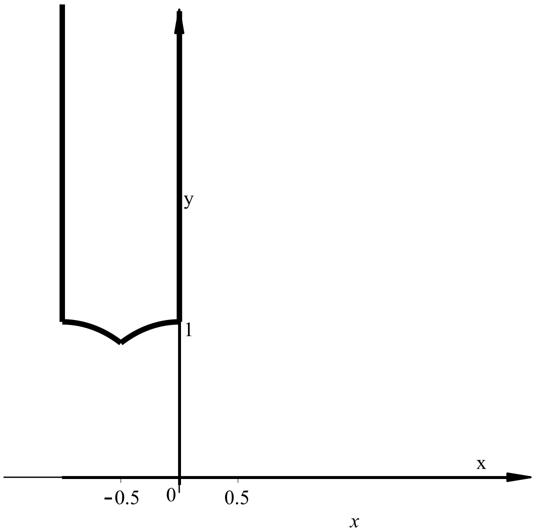

Figure 3: The domain of the congruence subgroup Γ0 of the desymmetrized *PSL*(2,Z) group on the Upper Poincar´e Half Plane (black solid line).

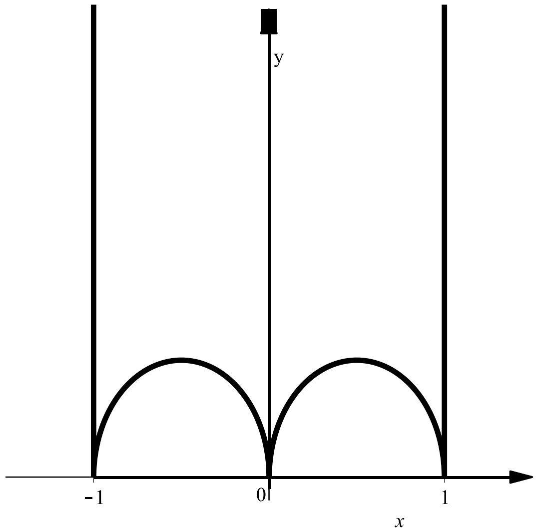

Figure 4: The domain of a leaky torus after [6] as a congruence subgroup of the symmetric *PSL*(2,Z) group on the Upper Poincar´e Half Plane (black solid line).

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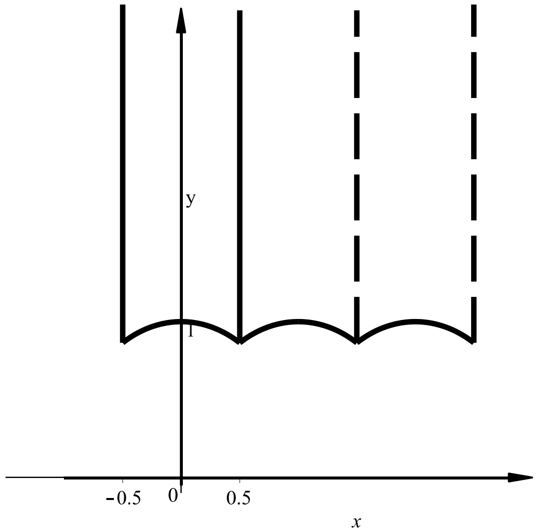

Figure 5: The domain of a leaky torus after [6] as a congruence subgroup of the symmetric *PSL*(2,Z) group from the representation of the congruence subgroup in [1] pag. 181 on the Upper Poincar´e Half Plane (black solid line); the segments of degenerate geodesics (’vertical’, dashed lines) are the symmetry lines with respect to which the generators of the hyperbolic reflections *T* from Eq.’s (11) act.

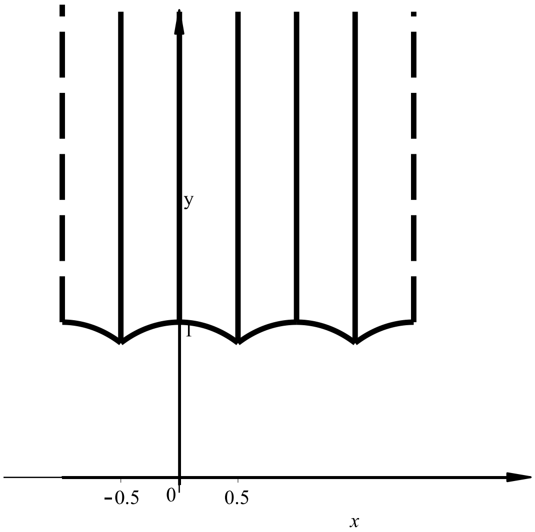

Figure 6: The domain of the leaky torus after as a congruence subgroup of the desymmetrized *PSL*(2,Z) group from the representation Eq.’s (16) on the Upper Poincar´e Half Plane (black solid line); the segments of degenerate geodesics (’vertical’, dashed lines) are the symmetry lines with respect to which the generators of the hyperbolic reflections *T* from Eq.’s (16) act.

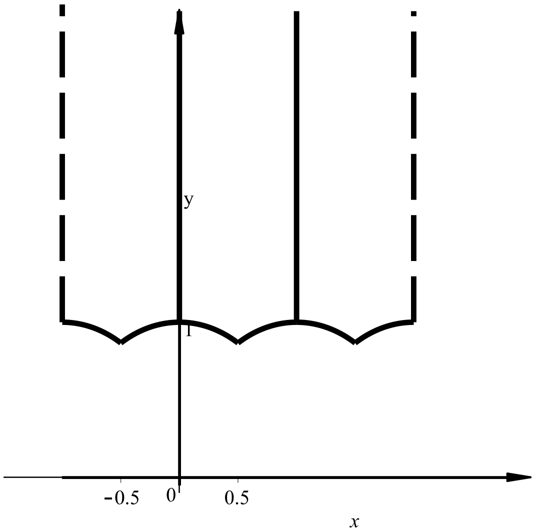

Figure 7: The domain of the leaky torus after as a congruence subgroup of the Γ0 congruence subgroup of the desymmetrized *PSL*(2,Z) group from the representation Eq.’s (16) on the Upper Poincar´e Half Plane (black solid line); the segments of degenerate geodesics (’vertical’, dashed lines) are the symmetry lines with respect to which the generators of the hyperbolic reflections *T* from Eq.’s (16) act.