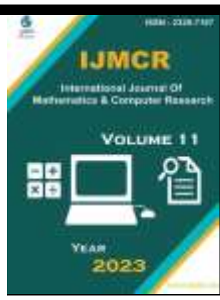
International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 11 Issue 02 February 2023, Page no. – 3236-3242

Index Copernicus ICV: 57.55, Impact Factor: 8.187

DOI: 10.47191/ijmcr/v11i2.02



Some Applications of Sectional Curvature in Differential Geometry

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ARTICLE INFO

Published Online:
14 February 2023

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ABSTRACT

The applications of the differential geometry of sectional curvature plays a great role in the field of physics, mathematics and engineering because it paves to knowledge of curves, surfaces, curvature, radius of curvature and sectional curvature . The study aims to explain some applications of sectional curvature. We followed the analytical induction mathematical method. We found the following some result: The sectional curvature indicate to know the behavior of some the functions and also we found that the sectional curvature $k(p,T_pM)=k(p)$ is the Gaussian curvature.

KEYWORDS: Curves, Surfaces, Sectional Curvature.

1. INTRODUCTION

Differential geometry is a discipline of mathematics that uses the techniques of calculus and linear algebra to study problem in geometry. The theory of plane, curves, surfaces and sectional curvature in three-dimensional Euclidean space formed the basis for development of differential geometry during the 18th and the 19th . Since the late 19th century, differential geometry has grown into a field concerned more generally with the geometric structure on differential manifolds. Differential geometry of curves is branch of geometry that deals with smooth curves in the plane and in Euclidean space by applying the concept of differential and integral calculus . The curves are represented in parametrized form and then their geometric properties and various quantities associated with them ,we turn our attention from curves to surfaces. The graphs of functions of two variables are familiar examples of surfaces from multivariable calculus. Whereas a curve locally looks like its tangent line $\mathbb{R}^1$, a surface locally looks like its tangent plane, . $\mathbb{R}^2$.Thus,we aremoving from intrinsically one-dimensional to intrinsically two-dimensional objects. such as curvature and sectional curvature . The differential geometry of sectional curvature is the curvature of two-dimensional sections of manifold, [5], pp(1-2)

2. DIFFERENTIAL GEOMETRY

The most important notion for the contents of this study (which is alsothe source of the name “differential geometry”) is that of derivativeor differentiation of real-valued functions which are defined on someopen set U $\subset R^n$ or, more generally, of maps defined on open setsU $\subset R^n$ to R^m. To say that a function is differentiable is to saythat it can be linearized up to terms of second order. More preciselya map F: U $\rightarrow R^m$ is said to be differentiable at a point x ∈ U,if there is a linear map A_x :$R^n \rightarrow R^m$ such that in a neighborhood$\cup_\epsilon(x)$(x) one has

$$F(x+\xi) = F(x) + A_x(\xi) + o(||\xi||). \quad (1)$$

Here, the symbol o(||ξ||) means that the terms indicated by it tendto zero as ξ → 0, even after previous division by ||ξ||. Then Ax is thelinear map described by the Jacobi matrix or the Jacobianof f .

Differential geometry uses calculus to study curved shapes, surfaces and curvature

i. **Curves:**

Definition (2.1):[1], pp2. If $f(x,y)=c$,where f is a function of x and y and c is a constant. From this point of view a curve is a set of points, namely

$$C=\{(x,y)\in\mathbb{R}^2|f(x,y)=c\}. \quad (1)$$

Its example for curves in the plane $\mathbb{R}^2$, but we can also consider curves in $\mathbb{R}^3$.For example, the x-axis in$\mathbb{R}^3$is straight line given by $y = 0,\ z = 0,$ and more generally a curve in $\mathbb{R}^3$ might be defined by a pair of equations

$$f_1(x, y, z) = c_1, \quad f_2(x, y, z) = c_2.$$

Curves of this kind are called level curves, the idea being that the curve in Equation(1),for example, is the set of points (x, y)in the plane at which the quantity $f(x, y)$reaches the 'level' c. But there another way to think about curves which turns out to be more useful in many situation. For this, a curve is viewed as the path traced out by a moving point. Thus, if $\gamma(t)$is the position of the point at time t, the curve is described by a function γ of a scalar paramerter t with vector values (in $\mathbb{R}^2$for a plane curve, in $\mathbb{R}^3$for a curve in space).[1]

Definition(2.2):[13],pp4.

For a given function F of two real variables x, y the equation F(x, y) = 0 describes a "curve" whenever the gradient of F does not vanish, that is to say if $\frac{\delta F}{\delta x} \neq 0$ or$\frac{\delta F}{\delta y}$ = 0 at every point satisfying F(x, y) = 0. If this assumption is satisfied, then this curve can always be parametrized locally as a regular parametrized curve in the sense of Definition (2.2) below.For a given function F of three variables x, y, z the equationF(x, y, z) = 0 describes a "surface" whenever the gradient of Fdoes not vanish, i.e., if $\frac{\delta F}{\delta x}$ = 0 or $\frac{\delta F}{\delta y}$ = 0 or $\frac{\delta F}{\delta z}$ = 0. If this assumption on thegradient is satisfied, then this surface can alwaysbe parametrized locally as a parametrized surface element in thesense of Definition 3.1 below.

Definition(2.3)[8], pp9. Let σ: I → R^n be a parametrized curve of class (atleast)C^1. The vector σ'(t) is the tangent vector to the curve at the point σ(t).If $t_0 \in I$ is such that σ'(t_0) ≠ O, then the line through σ(t_0) and parallelto σ'(t_0) is the affine tangent line to the curve at the point σ(t_0). Finally,if σ'(t) ≠ O for all t∈ I we shall say that σ is regular. **Remark (2.4): [8], pp9**. The notion of a tangent vector depends on the parametriza-tionwe have chosen, while the affine tangent line (if any) and the fact of beingregu-lar are properties of the curve. Indeed, let σ: I → R^n and ˜σ: I˜ → R^n betwo equivalent parametrized curves of class C1, and h: I˜ → I the parametertwo equivalent parametrized curves of class C1, and h: I˜ → I the parameterchange. Then, by computing ˜σ = σ ∘ h, we find:

$$\tilde{\sigma}'(t) = h'(t)\sigma'\big(h(t)\big). \quad (2)$$

Since h' is never zero, we see that the length of the tangent vector depends onour particular parametrization, but its direction does not; so the affine tangent line in $\tilde{\sigma}(t) = \sigma\,(h(t))\sigma$ determined by ˜σ is the same as that determined by σ. Moreover, $\tilde{\sigma}'$ is never zero if and only if $\tilde{\sigma}$ is never zero; so, being regularis a property of the curve, rather than of a particular representative.

Definition(2.5):[7],pp2 . A parametrized differentiable curve is a differentiablemap α: I → R^3 of an open interval I = (a, b) of the real line R into R^3.The word differentiable in this definition means that α is a correspondence which maps each t ∈ I into a point *α(t) = (x(t), y(t), z(t))* ∈R^3 in such away that the functions x(t), y(t), z(t) are differentiable. The variable t is called the parameter of the curve. The word interval is taken in a generalized sense,so that we do not exclude the cases a = −∞, b = +∞. If we denote by *x '(t)* the first derivative of x at the point t and use similarnotations for the functions *y* and *z*, the vector *(x '(t), y '(t), z '(t)) = α'(t)* ∈$R^3$is called the tangent vector (or velocity vector) of the curve α at t. The image set *α(I)* ⊂R^3 is called the trace of α.

Example (2.6):[7],pp2. The parametrized differentiable curve given by:

$$\alpha(t) = (a\cos t, a\sin t, bt)\,, t \in R, \quad (3)$$

has as its trace in R^3 a helix of pitch 2πb on the cylinder $x^2 + y^2 = a^2$. The parameter t here measures the angle which the x axis makes with the line joining the origin 0 to the projection of the point α(t) over the $x\,y$ plane

Definition(2.7):[9],pp52. Let **r** : $[a, b] \to R^i$, *i*= 2 or 3, denote a regular parametrization of a curve C with $P_t$corresponding to the parameter value t, i.e., **r**(t) = $\overrightarrow{OP_t}$. Fix a point $P_{t_0} \in C$. Any line through P_{t_0} and another point$P_t$is called a *secant line* throughP_{t_0}.

What happens to those secant lines when *t* approaches t_0, and thus *Pt* approaches P_{t_0}?We should show that, for a curve with a *regular* parametrization, there is a limit position: the *tangent line* through P_{t_0}.

The secant line through P_{t_0} and$P_t$has $\overrightarrow{P_{t_0}P_t}$as a parallel vector. The following definition tells us what we mean by a limit position:

Definition(2.8):[9],pp53. The positive, resp. negative unit semi-tangent vectors to the curve C at Pt0 are given by:

$$t+(t_0) = \lim_{t\to t_0+} \frac{\overrightarrow{P_{t_0}P_t}}{|\overrightarrow{P_{t_0}P_0}|} resp.\, t_-(t_0) = \lim_{t\to t_0-} \frac{\overrightarrow{P_{t_0}P_t}}{|\overrightarrow{P_{t_0}P_0}|}. \quad (4)$$

If $t_-(t_0)$ = -$t_+(t_0)$, then C has a tangent line with t(t_0) = $t_+(t_0)$ as a parallel vector.

Remark (2.9):[9],pp5.

I . The limits of the vector functions above may be taken coordinate wise.

ii .In general, the limits above need not exist. But the following result shows, thatwe do not need to worry for curves with *regular* parametrizations:

Definition(2.10):[9],pp59. again C be a curve with a regular parametrization $[a, b] \to R^i$,*i*= 2 or *i*= 3. What is the Let length l of the piece of curve between the starting point *Pa* and a point *Pt*, *t* ∈ [*a*, *b*]? What is the length of the entire curve from P_a *to* P_b?

Let the length of the piece of curve from Pa to Pt be denoted by $s(t)$. Of course, s(a) = 0, and s is an increasing (ordinary) function on $[a, b]$. Moreover, we found the speed of the parametrization as the function $v : [a, b] \to \mathbf{R}$, $v(t) = |\mathbf{r}'(t)|$. The speed is the differential increment of the length:

$$v(t) = \lim_{h\to 0}\frac{s(t+h)-s(t)}{h} = s'(t), \text{and hence} \quad (5)$$

Definition(2.11):[9], pp59. The arc length function s : [a, b] → **R** corresponding to the curve C parametrized by the vector function **r** above is defined as

$$s(t_0) = \int_a^{t_0} v(t)dt = \int_a^{t_0}|r'(t)|dt. \quad (6)$$

More explicitly, if $\mathbf{r}(t) = [x(t), y(t), z(t)]$, then

$$s(t_0) = \int_a^{t_0}\sqrt{(x')^2(t)+(y')^2(t)+(z')(t)}\,dt. \quad (7)$$

The length of the entire curve C is:

$$I = s(b) = \int_a^b |r'(t)|\,dt = \int_a^b\sqrt{(x')^2(t)+(y')^2(t)+(z')(t)}dt \quad (8)$$

The Fundamental Theorem of Calculus allows to calculate the derivative of the arc length function $s(t)$: In fact, $s' = v$, the speed, as it should! Furthermore, one can show – using integration by substitution – that the definition of the arc length function above (2.10) is independent of the chosen parametrization r.

ii. **Surfaces:**

By passing from curves to surfaces we in principle just replace the parameter of the curve by two independent parameters, which then describe a two-dimensional object, which is what is called a parametrized surface. For a proper development of the theory we require that the surface is not just given by a differentiable map in two vari- able, but that moreover it admits a geometric linearization in the sense that at every point there is a linear surface (i.e., a plane) which touches the surface at least to order one at that point. Hence it is quite natural to demand that the derivative of the param- etrization every point has maximal rank. A map satisfying this condition is called an immersion. A surface is a subset of $\mathbb{R}^3$ that looks like a piece of $\mathbb{R}^2$ in the vicinity or any given point, just as the surface of the Earth, although actually nearly spherical, appears to be a flat plane to an observer on the surface who sees only to the horizon. To make the phrases 'looks like' and 'in the vicinity' precise, we must first introduce some preliminary material. We describe this for $\mathbb{R}^n$ for any $n \geq 1$, although we shall need it only for $n = 1$, 2, or 3.

First, a subset U of $\mathbb{R}^n$ is called openif, whenever **a** is a point in U, there is a positive number such that every point **u** $\in\mathbb{R}^n$ within a distance ϵ of **a** is also in U:

$$a \in U \text{and } \|u-a\| < \epsilon \Longrightarrow \quad u \in U.$$

For example, the whole of $\mathbb{R}^n$ is an open set, as is

$$D_r(a) = \{u \in \mathbb{R}^n | \|u-a\| < r\},$$

the open ball with centre a and radius $r > 0$. (If $n = 1$, anopen ball is called anopen interval; if $n = 2$ it is called an open disc.) However

$$\bar{D}_r(a) = \{u \in \mathbb{R}^n | \|u-a\| \leq r\}$$

is not open because however small the positive number ϵ is , there is a point within a distance $\epsilon$of the point $(a_1 + r, a_2, \dots, a_n) \in \bar{D}_r(a)$ (say)that is not in $\bar{D}_r(a)$(forexample, the point $(a_1 + r + \frac{\epsilon}{2}, a_2, \dots, a_n)$.

We are now in a position to make our first attempt at defining the notion of a surface in $\mathbb{R}^3$. [1]

Definition(2.12)[13],pp56 Let U $\subset R^2$ be an open set. . A parametrized surface element is an immersion$f : U \to R^3$,f is also called a parametrization**,** the elements of U are called parameters, and their images under f are called points. The cartesian coordinates in Uare then mapped by f onto coordinate lines in the surface element , for such a grid of coordinate lines . A (non-parametrized) surface elementis an equivalence class of parametrized surface elements, where two parametrizations $f: U \to R^3$ and $\tilde{f}: \tilde{U} \to R^3$ are viewed as being equivalent if there is a diffeomorphism$\varphi: \tilde{U} \to U \text{ such that } \tilde{f} = f o\varphi$.

Sometimes one also speaks of *regular* surface elements if the rank of the map f is maximal, i.e., if f is an immersion. If there turn out to be points, however, where the rank is not maximal, one speaks of singular points or singularities. Similarly, one defines a hypersurface elementin IRn+1 by means of an immersion of an open subset U of R^n in R^{n+1}, even more generally a k-dimensional surface element in R^n. **Definition (2.13):[8], pp118**. An immersed (or parametrized) surfacein space is a map φ: $U \to R^3$of class C^∞, where $U \subseteq R^2$is an open set, such that the differential $d_{\varphi_x} : R^2 \to R^3$is injective (that is, has rank 2) in every point $x \in U$. The image $\varphi(U)$ of$\varphi$is the supportof the immersed surface.**Corollary (2.14):[8],pp118.** Let ϕ: U → R3 be an immersed surface. Then everyx_0 ∈U has a neighborhood $U_1 \subseteq U$ such that $\varphi|_{U_1}: U_1 \to R^3$is a homeomorphism with its image.

Proof. Let $G: \Omega \to W$be the diffeomorphism , $\pi: R^3 \to R^2$the projection on first coordinates, and set $U_1 = \pi(\Omega \cap (U \times \{0\}))$. Then $\varphi((x) = G(x,0)$ for all $x \in U_1$ and so $\varphi|_{U_1}$is a homeomorphism with its image, as required.

Definition(2.15):[8], pp121. A connected subset $S \subseteq R^3$is a (regular or embedded) surface in space if for all p ∈ S there exists a map φ: U → R^3 of class C^∞, whereU $\subseteq R^2$is an open subset, such that:

i. $\varphi(U) \subseteq S$ is an open neighborhood of p in S (or, equivalently, there exists an open neighborhood $W \subseteq R^3$ of p in R^3 such that $\varphi(U) = W \cap S$);

ii. φ is a homeomorphism with its image;

iii. the differential $d_{\varphi_x}: R^2 \to R^3$ is injective (that is, it has maximum rank,

i.e., 2) for all $x \in U$. Any map φ satisfying (a)–(c) is a local (or regular) parametrization in p; if $O \in U$ and $\varphi(O) = p$, we say that the local parametrization is centered in p. The inverse map $\varphi^{-1}: \varphi(U) \to U$ is called local chart in p; the neighborhood $\varphi(U)$ of p in S is called a coordinate neighborhood, the coordinates$\left(x_1(p), x_2(p)\right) = \varphi^{-1}(p))$ are called local coordinates of$p$; and, for j = 1, 2, the curve $t \to \varphi\left(x_0, + t_{e_j}\right)$ is the j-th coordinate curve (or line) through $\varphi(x_0)$.

Example(2.16):[6], pp(128-129). We will show that the cylinder{ $C = (x, y, z) \in R^3 \,\middle|\, x^2 + y^2 = 1\}$is a regular surface. Every point $p = (x, y, z) \in C$ with $x \neq -1$, lies in the image of the smooth function $\sigma: \underbrace{(-\pi, \pi) \,\vdots\!\times R}_{U} \to$

$C\ defined\ as\ \sigma(u, v) = \underbrace{(\cos(u))}_{x(u,v)}, \underbrace{\sin(u)}_{y(u,v)}, \underbrace{v}_{z(u,v)}\ ;\ see\ Figure\ No.1$

The image $V = \sigma(U)$is open in C, because it is the intersection with C of the open set $\{(x, y, z) \in R^3 | x \neq -1\}$. It is straightforward to verify thatσ is injective. To complete the verification that σ is a diffeomorphism (and is thus a valid surf- ace patch), it remains to show that the inverse $\sigma^{-1}: V \to U$ is smooth. At points of $V_+ = \{(x, y, z) \in V | x > 0\}, themap\ \sigma^{-1}$ is given by the formula$(x, y, z) \to \left(tan^{-1}\left({}^{y}/_{x}\right), z\right)$, which is smooth becauseit extends to the smooth function with the same formula on the open set $\{(x, y, z) \in R^3 \,|\, x > 0\}$. }. The smoothness of σ^{-1}at other points of V can verified in a similar manner. Thus, σ is a surface patch. The points of C at which $x = -1$can be covered by the second function$\mu: (-\pi, \pi) \times R \to C$ defined as$\mu(u, v) = (\cos(u + \pi), \sin(u + \pi), v)$,),
which isa valid surface patch by similar arguments. Notice that μ is the composition of σ with the rigid motion of R^3that rotates 180◦ about the $z - axis$.This completes the verification that C is a regular surface. For laterreference, we will nowindepend-ently verify the rank-2 condition guaranteed for the surface patch σ.The Jacobian matrix at$q = (u, v) \in is$

$$d_{\sigma_q} = \begin{pmatrix} \frac{\partial x}{\partial u}(q) & \frac{\partial x}{\partial v}(q) \\ \frac{\partial y}{\partial u}(q) & \frac{\partial y}{\partial v}(q) \\ \frac{\partial z}{\partial u}(q) & \frac{\partial z}{\partial v}(q) \end{pmatrix} = \begin{pmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 0 & 1 \end{pmatrix}. \qquad (9)$$

It has rank 2 for all choices of $q \in U$, because the two columns $\sigma_u(q) = (-\sin(u), \cos(u), 0)$, $\sigma_v(q) = (0,0,1)$, are linearly independent. Notice that $\sigma_u(q)$ points "around," while $\sigma_v(q)$ points "up".We knowthat the upper hemisphere (z > 0)ofS^2 is diffeomorphic to an open disk in R^2. The other five hemispheres(z <0, x >0, x <0, y >0, and y <0) are also diffeomorphic to disks,by similar arguments. For example, the hemisphere y <0 is covered by the surface patch $\sigma(x, y) = (x, -\sqrt{1 - x^2 - z^2}, z)$,whose domain is an open disk in the $xy - plane$,$\{(x, y) \in R^2 | x^2 + z^2 < 1\}$. Thus ,$S^2$ is a regular surface covered by atlas of six surface patches.

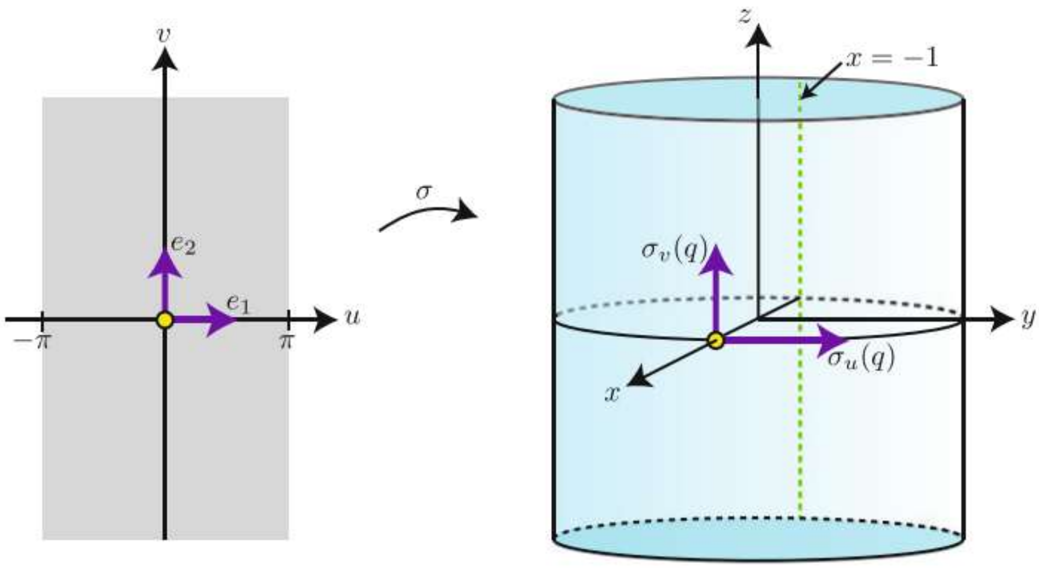

Figure: 2.1.The surface patch σ covers all but the green line

3. SECTIONAL CURVATURE

For our investigation of sectional curvature ,we need some symmetry properties of the curvature tensor , which are not so immediate .

Lemma(3.1):[13],pp242. For arbitrary vector fields X,Y,Z ,V the following hold

i. $R(X,Y)Z = -R((Y,X)Z;$ (10)

ii. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0;$ 1st (Bianchi identity) (11)

iii. $(\nabla \times R)(Y,Z)V + (\nabla YR)(Z,X)V + (\nabla ZR)(X,Y)V = 0;$ $(2nd\ Bianchi\ identity\ )$(12)

iv. $\langle R(X,Y)Z,V\rangle = -\langle R(X,Y)V,Z\rangle;$ (13)

v. $\langle R(X,Y)Z,V\rangle = \langle R(Z,V)X,Y\rangle$(14)

Definition(3.2):[13],pp246. With respect to a given Riemannian metric $\langle -,-\rangle$, the standard curvature tensor R_1 is defined by the relation $R_1(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y$. We then set $k_1(X,Y) = \langle R_1(X,Y)Y,X\rangle = \langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2$, $k(X,Y) = \langle R(X,Y)Y,X\rangle$. Let $\sigma \subset T_pM$ be two-dimensional subspace , spanned by X,Y. Then the quantity $K_\sigma = \frac{k(X,Y)}{k_1(X,Y}$, is called the sectional curvature of the Riemannian manifold with respect to the plane σ. If X,Y are orthonormal , then one has simply $K_\sigma = \langle R(X,Y)Y,X\rangle$. For case $n = 2$ we recognize the theorema fgregium with $K_\sigma = K$ (Gaussian curvature).

Definition(3.3):[2],pp(409-410). The sectional curvature is the curvature of two-dimensional sections of manifold Given any two vectors $u,v \in T_pM$,recally by Cauchy-Schwarz that $\langle u,v\rangle_p^2 \leq \langle u,v\rangle_p\langle v,v\rangle_p$, with equality if u,v are linearly dependent . Consequently, if $u\ and\ v$ are linearly independent, we have $\langle u,v\rangle_p\langle v,v\rangle_p - \langle u,v\rangle^2 \neq 0$. In this case, we claim that the ratio

$$K_p(u,v) = \frac{R_p(u,v,u,v)}{\langle u,v\rangle_p\langle v,v\rangle_p - \langle v,v\rangle_p^2} = \frac{\langle R_p(u,v)u,v\rangle}{\langle u,v\rangle_p\langle v,v\rangle_p - \langle u,v\rangle_p^2} \quad (15)$$

Is independent of the plane II spanned by $u\ and\ v. If\ (x,y)$is another basis of II, then $x = au + bv$, $y = cu + dv$, we get $\langle x,x\rangle_p\langle y,y\rangle_p - \langle x,y\rangle_p^2 = (ad-bc)^2(\langle u,u\rangle_p\langle v,v\rangle_p - \langle u,v\rangle^2)$, and similarly, $R_p(x,y,x,y) = \langle R_p(x,y)x,y\rangle = (ad-bc)^2 R_p(u,v,u,v)$. (16)

Definition(3.4):[2].pp410. Let (M,$\langle -,-\rangle$ be any Riemannian manifold equipped with the Lev-Civita connection. For every $p \in T_pM$,for every 2-plane II$\subset T_pM$,the sectio-nal curvature

$$K_p(II) = K_p(x,y) = \frac{R_p(x,y,x,y)}{\langle x,y\rangle_p\langle y,y\rangle_p - \langle x,y\rangle_p^2}, \quad (16)$$

for any basis(x,y) of II

The Differential Geometry of Sectional Curvature:

i. Constant Sectional Curvature:

Definition (4.1):[3], pp281. Let $k \in \boldsymbol{R}$ and $m \geq 2$ be an integer. An m-manifold $M \subset R^2$is said to have constant sectional curvature k iff $k(p,e) = k$, for every p$\in M$ and every 2-dimensional linear subspace $E \subset T_pM$.

Proposition(4.2):[13],pp(248-249). (F. Schur 1886). When the sectional curvature K_σ of a connected manifold of dimension $n \geq 3$ does not depend on the plane σ,but only on the point p at which it is calculated, then it is constant, i.e., does not depend on the point.

Proof:

First of all we have by $R = KR_1$ the relation $R(Y,Z)V = K.R_1(Y,Z)V$ with a differentiable function $K: M \to \boldsymbol{R}$. By taking derivatives we get

$$(\nabla_x R)(Y,Z)V = K.(\nabla_X R_1)(Y,Z)V + X(K).R_1(Y,Z)V = X(K).R_1(Y,Z)V, \quad (17)$$

Because $\nabla \times R_1 = 0$, we now wish to show that $X(K) = O$, for all X. By cyclically permuting the arguments, we get

$$(\nabla_x R)(Y,Z)V = X(K)(\langle Z,V\rangle Y - \langle Y,V\rangle Z),$$
$$(\nabla\_Y\ R)(Z,X)V = Y(K)(\langle X,V\rangle Z - \langle Z,V\rangle X),$$
$$(\nabla_Z R)(X,Y)V = Z(K)(\langle Y,V\rangle X - \langle X,V\rangle Y.$$

Now when we take the sum of these equations, the left-hand side vanishes because of the third equation in Lemma (3.1) and hence we have

$$0 = (Z(K)\langle Y,V\rangle - Y(K)\langle Z,V\rangle X + (X(K)\langle Z,V\rangle - Z(K)\langle X,V\rangle)Y + (Y(K)\langle X.V\rangle - X(K)\langle Y,V\rangle)Z, \quad (18)$$

For all X,Y,Z,V. By our assumption on the dimension there are three orthogonal vectors X,Y,Z. We first set $V = X$, yielding

$$0 = -Z(K)Y + Y(K)Z,$$

And consequently $Y(K) = Z(K) = 0$. Now we choose similarly $V = Y$, yielding $0 = Z(K)X - X(K)Z,$

And then also $Y(K) = 0$. Since at least one of the three vectors may be chosen arbitrarily , it follows that $X(K) = 0$, for every X. Thus K is locally constant, and by connectedness of M it is globally constant.

Definition(4.3):[13],pp249. If on a Riemannian K_σ is a constant or, equivalently, if $R = K.R_1$ with $k \in \boldsymbol{R}$, the manifold is called a pace of constant curvature.

Theorem(4.4):[3],pp(281-282). Let $M \subset R^n$ be an m-manifoldand fix an element $p \in M$and a real number k. then the following are equivalent

i. $K(p,E) = K$ for every 2-dimensional linear subspace $E \subset T_pM$

ii. The Riemann curvature tensor of$M$ at p given by

$$\langle R_p(v_1,v_2)v_3,v_4\rangle = K(\langle v_1,v_4\rangle\langle v_2,v_3\rangle - \langle v_1,v_3\rangle\langle v_2,v_4\rangle \quad (19)$$
$$for\ all\ v_1,v_2,v_3,v_4 \in T_pM$$

Proof: That (ii) implies (i) follows directly from the definition of sectional curvature in (3.4) y taking $v_1 = v_4 = u$ and $v_2 = v_3 = v$ in $K(II,E) = \frac{1}{4}|[\xi,\eta]|^2$, conversely ,assume (i) and defined the multi-linear map$Q: T_p \to R$ by$Q(v_1, v_2, v_3, v_4) = \langle R_p(v_1, v_2)v_3, v_4\rangle - K(\langle v_1, v_4\rangle\langle v_2, v_3\rangle - \langle v_1, v_3`\rangle\langle v_2, v_4\rangle$. (20)

Then for all $u, u, v_1, v_2, v_3, v_4 \in T_pM$.The Q satisfies the equation

$$Q(v_1, v_2, v_3, v_4) + (v_2, v_1, v_3, v_4) = 0, \quad (21)$$

$$Q(v_1, v_2, v_3, v_4) + Q(v_2, v_3, v_1, v_4) + (v_3, v_1, v_2, v_4) = 0, \quad (22)$$

$$\mathrm{Q}(v_1, v_2, v_3, v_4) - Q(v_3, v_4, v_1, v_2) = 0, \quad (23)$$

$$\mathrm{Q}(u, v, u, v) = 0.$$

Here the firstthree equations from Lemma(3.1)and the last follows from the definition of Qand the hypothesis that the sectional curvature is $K(p,E) = K$ for every 2-dimensional linear subspace $E \subset T_pM$.We mustprove that Q vanishes using equations (23)and (24) we find

$$0 = Q(u, v_1 + v_2, u, v_1 + v_2) = Q(u, v_1, u, v_2) + Q(u, v_2, u, v_1) = 2Q(u, v_1, u, v_2)$$

For all $u, v_1, v_2 \in T_pM$. This implies

$0 = Q(u_1 + u_2, v_1, u_1 + u_2, v_2) = Q(u_1, v_1, u_2, v_2) + Q(u_2, v_1, u_1, v_2)$ for all $u_1, u_2, v_1, v_2 \in T_pM$. Hence

$$Q(v_1, v_2, v_3, v_4) = -Q(v_3, v_2, v_1, v_4) = Q(v_2, v_3, v_1, v_4) = -Q(v_3, v_1, v_2, v_4) - Q(v_1, v_2, v_3, v_4).$$

Here the equation(21)and the equation (22). Thus

$$Q(v_1, v_2, v_3, v_4) = -\frac{1}{2}Q(v_3, v_1, v_2, v_4) - Q(v_1, v_2, v_3, v_4)$$

for all$v_1, v_2, v_3, v_4 \epsilon T_pM$ and repeating this argument,

$$Q(v_1, v_2, v_3, v_4) = \frac{1}{4}Q(v_1, v_2, v_3, v_4). \quad (25)$$

Hence $Q \equiv 0$ as claimed .This proves Theorem(4.4).

Corollary(4.5):[3],pp284. Any two connected , simply connected complete Riemannian manifold with the same constant sectional curvature and the same dimension are isometric.

Proof: Theorem (4.4)

Example(4.6)[3],pp285. Any flat Riemannian manifold has constant sectional curvature $K = 0$.

ii. Nonpositive Sectional Curvature:

Definition(4.7):[3].pp293. A Riemannian manifold $M$is said to have nonpositive sectional curvature iff $K(p,E) \le 0$ for every $p \in M$and evry2-dimensional linear subspace $E \subset T_pM$ or , equivalently$\langle R_p(u,v)v, u\rangle \le 0$ for all $p \in M$ and all $u, v \in T_pM$. A nonempty, connected, simply connected, complete Riemannian manifold with nonpositive sectional curvature is called aHadamard manifold.

iii. **Positive Sectional Curvature:**

The definition of positive sectional curvature with standard definition can easily be discussed by using the following corollary

Corollary(4.7):[3], pp312. Let$M \subset \boldsymbol{R}^2$be a complete, connected manifold of dimensional $m \ge 2$ and suppose that there three exists a $\delta > 0$ such that $K(p,E) \ge \delta$ for every $p \in M$ and every 2-dimensional linear subspace $E \subset T_pM$. $d(p,q) \le \frac{\pi}{\sqrt{\delta}}$, for all $p, q \in M$ and hence M is complete. (24)

5. APPLICATIONS OF THE DIFFERENTIAL GEOMETRY OF SECTIONAL CURVATURE

Example (5.1)[3], pp290. If $M \subset R^3$is a 2-manifold, then the sectional curvature $K(p, T_pM) = K(p)$ is the Gaussian curvature of M at p. More generally, for any 2-m- anifold$M \subset R^n$(where or not it has codimension one) we define the Gaussian curvat- ure of M at p by

$$K(p) = K(p, T_pM) \quad (26)$$

Example(5.2):[3], pp280. If $M \subset R^{m+1}$ is a submanifold of codimension one and $v: M \to S^m$ is a Gaussian map, then the sectional curvature of a 2-dimensional subspace $E \subset T_pM$ spanned by two linearly independent tangent vectors $u, v \in T_pM$ is given by

$$K(p,E) = \frac{\langle u, dv(p)u\rangle\langle v, dv(p)v\rangle - \langle u, dv(p)v\rangle^2}{|u|^2|v|^2 - \langle u, v\rangle^2}, \quad (27)$$

Which holds in all dimensions. In particular, when $M = S^m$, we have $v(p) = p$ and hence $K(p,) = 1$ for all p *and* E. For a sphere of radius r we have $v(p) = \frac{p}{r}$ and hence $K(p,E) = \frac{1}{r^2}$.

Example (5.3)[9], pp(147-148). Let $f: R^2 \to R$ denote the function $f(x,y) = \cos x + \cos y - 2$,with parametrization $r(u,v) = [u, v, \cos u + \cos v - 2]$. Since$f_u(u,v) = -\sin u$, $f_v(u,v) = -\sin v$, $f_{uu}(u,v) = -\cos u$, $f_{uv}(u,v) = 0$, $f_{vv}(u,v) = -v$, we obtain: $E(u,v) = 1 + (\sin u)^2$, $F(u,v) = \sin u \sin v$, $G(u,v) = 1 + (\sin v)^2$,

$$e(u,v) = \frac{-\cos u}{\sqrt{1 + (\sin u)^2 + (\sin v)^2}} f(u,v) = 0;$$

$$g(u,v) = \frac{-\cos v}{\sqrt{1 + (\sin u)^2 + (\sin v)^2}};$$

$$K(u,v)=\frac{\cos u\,cos v}{(1+(\sin u)^2+(\sin v)^2};$$

$$H(u,v)=\frac{\cos u(1+(\sin u)^2)-\cos v(1+(\sin v)^{2)}}{2(1+(\sin u)^2+(\sin v)^{2)^{\frac{3}{2}}}}.$$

The point $p_0:(0,0)$ is critical with respect to the function $f, i.e.$, both partial derivative vanish at p_0. The other critical points have coordinates $(K\pi, l\pi)$with integer k and l. At p_0 the graph of f has the $xy-planez=0$ as its tangent plane ; at the other critical points the tangent plane is one of the horizontal planes $z=0$ or $z=-2$ or $z=-4$. The Gaussian curvature at p_0 is $k(0,0)=1$, the mean curvatureis $H(0,0)=-1$. The principal curvature at this point are $k_1(p_0)=k_2(p_o)=-1$. This means, that alldirections are principal directions.The approximating paraboloid at p_0 is a sphere of radius 1. Since the Gaussian curvature is positive, the function has a local extremum(in fact , a local maximum) at p_0.

At $(k\pi, l\pi)$, the Gaussian curvature is positive , if k and l are either both even or both odd. The function f has a local maximum at each a point $(f(k\pi, l\pi)=0)$ if k and l are both even and a local minimum $(f(k\pi, l\pi)=-4)$, if k and l are both odd. If k is even and $l$is odd -or vice versa -, then $(k\pi, l\pi)$with horizontal tangent plane $z=-2$.

RESULTS

The differential geometry of sectional curvature indicates to know behavior of some of the functions and we showed that the calculation of the parametrized differentiable curves, surfaces and sectional curvature . And also we explained that the sectional curvature is the Gaussian curvature.

CONCLUSION

Finally we can say that any sectional curvature is Gaussian curvature**.**

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