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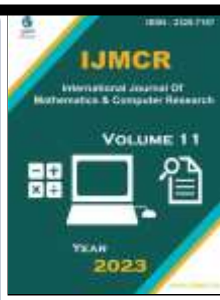

Reduced 2-Dimensional Birkhoff Surfaces of Section of the Desymmetrized *PSL*(2,Z) Group: The Anosov Characterization

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ABSTRACT

New 2-dimensional Birkhoff surfaces of sections are defined for the desymmetrized *PSL*(2,Z) group. The new object is demonstrated to be apt to study the geodesics flow solution of the Hamiltonian problem. The new definitions of the return maps of the new 2-dimensional Birkhoff surface of section are provided with; The demonstration relays on the self-adjointed-ness of the operators on which the conjugacy subclasses needed in the application to reduced surds act

KEYWORDS: Surfaces of Section; Return maps.

I. INTRODUCTION

The Artin systems [1] describe the geodesics flow associated with the Hamiltonian problem of a point particle moving of geodesic motion on the *PSL*(2,Z) domain the Upper Poincar´e Half Plane. The system is strongly chaotic, ergodic, and strong mixing: it s one realization of the Anosov flow.

Within the Artin formulation, the system is represented as a billiard, in which the point-particle moving on the geodesics undergo (hyperbolic) reflections on the sides of the group domains: the composition of the hyperbolic reflection lays the basis of the symbolic-dynamic analysis of the chaotic (billiard) systems.

In [2], analogues of the Artin factorization formulas are worked for the factorization formula of the Selberg zeta function. Very importantly, the determinant of a(n automoprhic) scattering matrix of arbitrary subgroups (of finite index) of a Fuchsian group of the first kind is used.

The Artin formalism of the desymmetrized *PSL*(2,Z) group is applied in [3].

Particular cases of the Artin formalism provided with the Selberg zeta function are exposed in [4].

In [3], the transfer operators of the return map of the geodesics belonging to the geodesics flow of the Hamiltonian problem on the Upper Poincar´e Half Plane on the surface of section individuated after the choice of the $u = 0$, $v \geq 1$ are newly derived. As an important result, also the singular orbits are taken into account: after this result, the analysis of the cusp forms is pobble even after the geometric tools followed in [3].

In [5], a dynamical zeta function is described, which is apt to relate the formulation of the Hamiltonian problem according to the Artin description of the fully-chaotic Hamiltonian flow (in billiard-like descriptions).

Generalized transfer operators are introduced after a new dynamical proof of the factorization formula. The geodesics flow remains this way related with the Hamiltonian problem; as a new outline, the problem is lead to a decomposition of the eigenfunctions of the Hamiltonian problem into two different sets obeying different boundary conditions.

The dynamical zeta function of the symbolic dynamics of Artin billiards is further characterized in [5]. In particular, The dynamical zeta function is here explained to relate the formulation of the Hamiltonian problem according to the Artin description and the fully-chaotic Hamiltonian flow (in a billiard-like description). The particularities of generalised transfer operators are further outlined after a new dynamical proof of the factorization formula. The geodesics flow related with the Hamiltonian problem leads to a decomposition of the eigenfunction into two sets obeying different boundary conditions.

The problem of defining Transfer operators of the Anosov flows is formulated in [6].

The definition of the transfer operator form Banach spaces of special pre-quantum operators of the normalized oriented geodesics flow is studied in [7].

The relation between Anosov flows and abstract Birkhoff section are summarized in [8]; in [9], oriented surfaces are taken into account for the analysis.

The consideration of an arbitrary Poincar´e surface of section of a chaotic system is presented in [10]: as from Formula (19) and from Formula (20) in [10], the composition of operators which will be here demonstrated to act 'on the left' of the symbolic-dynamics representation of the conjugacy subclass corresponding to a chosen orbit produces the effect to further factorize the Fredholm determinant formula.

The study of the definition of Birkhoff sections for Anosov flows was addressed in [11]; as a result, any transitive pseudo-Anosov flow is shown to have a Birkhoff section with two boundary components after the method of triangulations.

The aim of the present investigation is the study of new arbitrary reduced 2-dimensional oriented Birkhoff surfaces of section of the complete phase space of the geodesics flow solution of a Hamiltonian problem on the upper Poincar´e Half Plane. To this purpose, arbitrary oriented 3-dimensional Birkhoff sections are considered, and the opportune hypotheses for the reduction of the degrees of freedom of the problem are formulated. The definition of the Hamiltonian problem allows one to impose conditions on the phase space: under certain conditions, the number of degrees of freedom of the phase space is reduced, and so are the dimensions of the Birkhoff surface of section of the phase space. The arbitrariness of the choice of the new Birkhoff section is the choice of one non overlapping one of the (in the analysed literature, i.e. such as [3]) sides (or a preferred one, such as the ($u = 0$, $v \geq 1$) side) of the *PSL*(2,Z) group.

The study of [3] is specialized with the symmetrized *PSL*(2,Z) group domain, as for the direct application of the billiard map Formula 15.

The definition of reduced surds [12] will be applied to the considered system.

The need of oriented surfaces within the framework of the present analysis is dictated after the consideration of the (normalized) geodesics flow, whose geodesics are oriented ones (where the orientation of the geodesics can be defined after the specificities of the phase space after the choice of the orientation of the endpoints in the coordinate space).

The new definition of reduced oriented Birkhoff sections is this way obtained. More in detail, the orientation for the reduced Birkhoff sections is chosen by the orientation of the velocity-normalized geodesics whose flow is associated with the quantum (Hamiltonian) problem.

The present study is motivated after the analysis of [20], which is based on the self-adjointed-ness of the needed operators, which, in the present case, is the operator T_s from [13] in [3], whose features are studied with respect to the needed conjugacy subclasses of the groups studied in the present paper. The classification of all the periodic orbits according to the maps permits one to reconstruct the complete spectrum of the operator.

As a new item of investigation, the results are consistent with the considered conjugacy subclasses enumerated as those needed for the definition of reduced surds, and those needed for the reconstruction of the geodesics flow for arbitrary (and oriented) Birkhoff surfaces of sections.

The study allows one to recover tori from the Upper Poincar´e Half Plane.

The further study of perturbations of the periodic orbits, differently, is newly found to provide one with the complete tessellation of the Upper Poincar´e Half Plane after the study of the related topological entropy of the geodesics-flow analysis, where the latter is performed with respect ti the new definition of reduced oriented Birkhoff sections. The paper is organized as follows.

In Section II, the main objects to be investigated and newly-analysed are summarised.

In Section III, the descriptions of generic non-oriented 3-dimensional Birkhoff section after Anosov flows are reviewed; factorization formulas are introduced according to the different proofs methods.

In Section IV, the methods of the Artin factorization formula are explained.

In Section V, some specific properties of quantum maps and transfer operators, which are needed for the new investigations of the objects defined in Section III, are briefly recalled.

In VI, the symbolic dynamics of the solutions of the free Hamiltonian problem occurring on the desymmetrized *PSL*(2,Z) group domain is birefly recalled.

In Section VII, the definition of arbitrary Poincar´e surfaces of section is proposed.

In Section VIII, novel 2-dimensional oriented Birkhoff surfaces of sections are defined.

In Section X, the return map of the 2-dimensional oriented Birkhoff surfaces of section are newly defined.

Outlook and perspectives follow in Section XI.

In the Appendix A, complements of the geodesics-flow analyses are provided with.

II. INTRDUCTORY MATERIAL

It is possible to investigate the properties of the Anosov flow X associated with the geodesics flow (i.e. solution of the Hamiltonian problem) after the definition of a generic (non-oriented) 3-dimensional Birkhoff surfaces of section.

The results of the analyses of pseudo-Anosov maps are recapitulated in [**?**] from [14].

As from [15], any transitive Anosov flow on an oriented 3-manifold is proven to admit a Birkhoff section Σ; the first return map of the geodesics of the flow on the Birkhoff section induces a pseudo-Anosov homeomorphism on a closed surface Σ obtained from Σ after 'collapsing' every component of the frontier˜ FΣ of the Birkhoff section Σ.

III. 3-DIMENSIONAL GENERIC BIRKHOFF SURFACES OF SECTIONS

It is here convenient to study a 3-dimensional Birkhoff section [9]. The 3-dimensional Birkhoff section is defined after an Anosov flow A and a 3-manifold M. The Birkhoff section Σ is a compact surface; the interior of Σ is embedded in the 3-manifold M in a transverse manner to the Anosov flow X, and its boundary FΣ is made of finitely many periodic orbits of the Anosov flow. Such orbits are hypothesized to match the Birkhoff section Σ within a finite time. As investigated in [15], the successive return maps can be defined after the techniques developped in [16] and [17].

As from [17], for any disjoint set of simple closed geodesics, the geodesics flow is Anosov.

The Thurston description [18] allows one to include sngular trajectories within the geodesics flow, i.e. it allows one to include the cusp point within the frontier of the group domains.

The complete Anosov flow is thus reconstructed after the techniques exposed in [19] after the analysis of the Markov decomposition in the considered 3-dimensional case.

The multiplicity of the orbits can be used, within other tools, for the reconstruction of the Anosov flow X associated with any component on the frontier FΣ, which is apt to describe the topological position of the surface of section Σ. These notions will be used in Section IX for the reconstruction of the return maps on an arbitrary surface of section.

From Theorem A in [15], an Anosov flow on the 3-manifold M can admit a positive Birkhoff section.

IV. THE ARTIN FACTORIZATION OF THE SELBERG ZETA FUNCTION

The determinant of the automorphic scattering matrix can be calculated in different manners [2].

From [20], the Artin-Takagi formula is obtained as a consequence of a more general spectral structure for the resolvent of a self-adjoint operator in the space of automorphic functions, in terms of which functions the Selberg zeta function is expressed. The case of non-compact-domains groups is taken into account; it can be demonstrated that for each of these groups there exists a normal subgroup of finite index, for which the Roelke [21] conjecture holds.

From [2], the spectral-theory properties of automorphic functions are exploited for the factorization of the formula. The present analysis is motivated after and follows form the analysis of [20], as the latter is based on the properties of self-adjointed-ness of the used operators.

V. THE TRANSFER OPERATORS AND THE QUANTUM MAPS

The existence of transfer operators associates with quantum maps is derived in [3] after the existence of opportune zeta functions defined within the framework of a continuous time dynamics of the solution(s) of the Hamiltonian problem: more in detail, in [3] pag. 25, the demonstration is based on the existence and uniqueness of the Banach spaces in which the various operators have spectral properties.

Quasi-hyperbolic geometry is obtained from the Banach spaces. The quasi-hyperbolic geometry implies quasihyperbolic lengths, which define quasi-hyperbolic distances, with the properties of uniform convexity, uniform smoothness and locally-uniform round spaces.

Given Σ_2 the Poincar´e surface of section, a transfer operator T_E is defined, on which the Poincar´e return map is defined.

It is possible to determine the classical trajectories in the coordinate representation connecting two points q and q' (from the phase space) on Σ_2 by means of a(n oriented, opportunely-normalized) geodesics (of a point-particles moving on the defined geodesics with) velocity $\dot{q}$ at both q and q' with the same orientation on Σ_2.

Correspondingly, the value of the action $S_E(q',q)$ calculated along the classical trajectories at a fixed (with respect to the phase space) energy E (E being an eigenvalue of the classical Hamiltonian problem), as

$$S_E(q',q) = \int_q^{q'} p(x,E) \quad (1)$$

with p the momentum; the number of degrees of freedom minus one corresponds to the dimension N of the Poincar´e surface of section (from which N is in the present purposes calculated as 2 of Σ_2), with the Maslov index of the trajectory ν.

The semiclassical Poincar´e map upgrade $T_E\psi(q')$ of the function $\psi(q)$ from the point q to the point q' is therefore calculated as

$$T_E(q') = \int_{\Sigma 2} T_E(q',q)\psi(q)d^Nq. \quad (2)$$

The quantum limit can be implemented. The Hamiltonian problem is here thus obtained after the semiclassical upgrade with quantized energy levels determined after the condition that the spectrum contain the eigenvalue 1. At the value E of the energy (level) there corresponds an eigenvalue E_n of the Hamiltonian H.

The operator T_E is this way endowed with an invariant function $\tilde{}(\psi)(q)$ which relates to the map as

$$\tilde{\psi}(q') = \int_{\Sigma 2} T_E(q',q)\tilde{\psi}(q)d^Nq. \quad (3)$$

The compatibility of the construction is demonstrated in [3] after the references therein from the vanishing of the Fredholm determinant as

$$det[1 - T_E] = 0 \quad (4)$$

under the suitable conditions (specified in [22] and [23]).

Under the specified suitable conditions, the determinant in Eq. 4 coincides with the dynamical ζ function constructed as an infinite product over all periodic orbits.

The monodromy matrix M_p (in the momentum p representation) consists of the linear term of the expansion of the Poincar´e maps in powers of the coordinates perpendiculars of the periodic-orbit system; in the case of the Hamiltonian problem, the specified monodromy matrix $\hat{M}_p$ has the property

$$\hat{M}_p = 1. \tag{5}$$

The study in the above can be specified with respect to the 2-dimensional case.

At each hyperbolic orbit, the monodromy matrix admits two (different possibilities of sets of) eigenvalues parameterized after the Ljapunov exponent on the periodic orbit λ_p (as spelled out in [3]).

Hamiltonian systems can therefore be realised as billiard systems with constant negative curvature.

The specification of the analyses allows one to define the Markov partitions. Let F be a piece-wise analytic map of an interval I, i.e.

$$F : I \to I. \tag{6}$$

The piece-wise analytic map F induces a Markov partition of the interval I such that

$$I = \bigcup_i I_i \tag{7}$$

with an incidence matrix $S(i \leftarrow j)$.

Let G be the set of local inverse of F, i.e.

$$G_i : I_i \to I, \tag{8}$$

which therefore corresponds to the branches of the inverses of the piece-wise analytic map F.

Accordingly, the Poincar´e surface of section Σ_2 associated with the study of [3] is therefore chosen as the side a of the considered group domain.

VI. SYMBOLIC DYNAMICS OF THE DESYMMETRIZED *PSL*(2,Z) GROUP AND REDUCED-SURDS CONJUGACY SUBCLASSES

The dynamics of the geodesics flow of the free Hamiltonian problem taking place on the desymmetrized *PSL*(2,Z) domain is here spelled out after the application of the definition of reduced surds [12]. In [12], the conjugacy classes of periodic orbits are classified according to the request that both the oriented endpoints are defined as reduced. The generators of the desymmetrised *PSL*(2,Z) group are defined as

(9a) $$T_1 : T_1 z \to \bar{z} + 1$$

(9b) $$T_2 : T_2 z \to -\bar{z},$$

(9c) $$T_3 : T_3 z \to \frac{1}{\bar{z}},$$

which are the (hyperbolic) reflections on the sides b, a, and c of the group domain, which also corresponds to the operators B, A, and C.

Oriented geodesics are defined according to the oriented endpoints u^+ and u^-, of radius $r = (u^+ - u^-)/2$ and centre $u_0 = (u^+ + u^-)/2$.

An example of the system on the Upper Poincar´e Half plane is drawn in Fig. 1.

Reduced surds are defined in [12] as the velocity-normalised oriented geodesics whose endpoints are classified as

$$-2 \le u^- \le 1, \qquad (10a)$$

$$u^+ \ge 1 \qquad (10b)$$

an corresponds to the periodic orbits whose cutting trajectory is the first *ba* trajectory of the system.

The conjugacy subclasses needed to compare the trajectory of the system to the definition of reduced surds are worked out of [**?**]. In [**?**], the conjugacy subclasses are encoded in Formula J15; Formula J15 of [**?**] is spelled out as

$$z' = z = \prod_i (T_1 T_2)^{F(n_i)} T_3 z \tag{11}$$

for the coordinates $z = u + iv$ of the Upper Poincar´e Halp Plane.

VII. ARBITRARY POINCARE SURFACES OF SECTION´

It is possible to define an arbitrary 1-dimensional Poincar´e surface of section of a graoup domain on which the dynamics of the solutions of the free Hamiltonian system occurs. In particular, the arbitrary Poincar´e surface of section is chosen as not corresponding with one (preferred) side of the groups domain.

More in detail, the following definition is worked out.

Definition 1 the arbitrary 1-dimensional Poincar´e surface of section of the desymmetrized *PSL*(2,Z) group domain is here chosen as a (degenerate) geodesics $u = u^* = const$, with

$$-\frac{1}{2} < u^* < 0 \tag{12}$$

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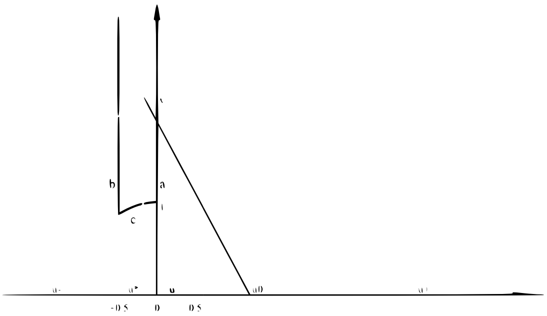

Figure 1: The desymmetrized domain of the *PSL*(2,Z) group domain on th Upper Poincar´e Half Plane, endowed with a Poincar´e surface of section (grey degenerate geodesics), on which the return map of the solutions of the Hamiltonian problem (an example is the green solid segment of geodesic) are defined. The parameterization of the return map is term of the radius of the geodesics solutions of the Hamiltonian problem is evidentiated.

VIII. NEW REDUCED 2-DIMENSIONAL ORIENTED BIRKHOFF SECTIONS

It is the purpose of the present Section to newly investigate the description of 2-dimensional oriented Birkhoff surfaces of section according to the number of degrees of freedom calculated after [12] after the specifications of Section VII.

The specifications of the Markov partitions in Section V from [3] allow one to analyse the Anosov-related problem. The definition of the Birkhoff section of the first return map of the (normalized) geodesics (whose flow is demonstrated in [15] as a pseudo-Anosov homeomorphism) can be considered, within the present framework, after the definition of the complete Artin system (billiard) [1].

The finitely many periodic orbits defined in [9] can be classified according to the definition of reduced surds in [12].

The definition of the Birkhoff surfaces of section results as redundant after the study of the definition of reduced surds. Indeed, the definition of reduced surds encodes the conjugacy subclasses of the desymmetrized *PSL*(2,Z) group which are needed to reconstruct the geodesics flow as with the oriented cutting trajectory as one defined in [12], i.e. as one defined as connecting the side b with the side a, whose oriented endpoints are in the opportune, with unit velocity.

The definition of reduced surds allows one therefore to eliminate one degree of freedom of the Hamiltonian system, i. e. the definition allows one to consider 2-dimensional surfaces of sections of the complete phase space.

In [3], the surfaces of sections are chosen as two-dimensional surfaces of the phase-space $\Sigma_a(u)$, i. e. as coinciding with the side a: $u = 0$, $v \geq 1$ of the billiard system independently of the intervals of the u direction in which the oriented endpoints are defined. In the present case, the definition of (oriented) reduced surds thus allows one to consider the lower-dimensional surface of section a: $u = 0$, $v \geq 1$.

The definition of an arbitrary surface of section Σ^* of reduced surds is constructed after encoding the needed conjugacy subclass ’on the left’ of the definition of the reduced-surd of the surface of section Σ_a.

The definition of the Markov decomposition of the Anosov flow of a 3-manifold is here specified for arbitrary surfaces of section, defined after the values of u and θ.

The return map of the periodic orbits on an arbitrary surface of section does not therefore modify the the τ_E operator after the insertion of all the items of transformations needed after considering the surface $\Sigma_{\cos\theta\, u.}$. Therefore, the return maps can be considered.

According to the developped analyses, the following definitions holds.

Definition 2: a 2-dimensional oriented Birkhoff surface of section is a 2 dimensional surface embedded in the reduced 3-dimansional phase space of the Hamiltonian system, of which one side corresponds to the arbitrarily-chosen one-dimensional Poincar´e surface of section, and the other side is such that the orientation of the Birkhoff surface is defined positive in the outward direction of the *ba* outgoing velocity-normalised oriented geodesics.

Definition 3: The 2-dimensional reduced Birkhoff section is therefore specified as a 2-dimensional surface whose frontiers are defined after the chosen u^* degenerated geodesics and the side of the reduced phase space specified after the angle θ comprehended within the axis of abscissae and the radius r of the geodesics connecting the center of the geodesics u_0 with the intersection point between the generalized Poincar´e surface of section u^* and the same geodesics, such that $\cos\theta = (u_0 - u^*)/r$. ✷.

A Poincar´e surface of section is depicted in Fig. 1 according to Definition 3.

From Definition 2 and from Definition 3, the following theorems hold.

Theorem 1: From Theorem A in [15], a positive Birkhoff section is therefore admitted after the specifications imposed on the considered system.

Proof: to prove the Theorem 1, it is sufficient to apply the consideration that the application of the composition of operators corresponding to the composition of hyperbolic reflections of which the conjugacy subclasses implied after the definition of reduced surds do not modify the T_E operator. ✷.

Theorem 2 The multiplicity of the finitely-many periodic orbit is investigated after the composition of operators corresponding to the composition of hyperbolic reflections which defined the reduced surds. **Proof**: by construction. ✷

The definition of return maps depends on the number of operators of hyperbolic reflections which constitute the each conjugacy subclass which correspond to the description of the periodic orbits within the framework of the symbolic dynamics.

IX. 2-DIMENSIONAL REDUCED BIRKHOFF SURFACE OF SECTION AND THE RETURN MAPS

The return maps of the Poincar´e surface of section on the *a* side of the desymmetrized *PSL*(2,Z) domain are constructed after the symbolic dynamics.

Theorem 3: the first return map is defined after the operator *C* which precedes the first occurrence of the *A* operator in the conjugacy subclass corresponding to the periodic orbit of a reduced surd.

Proof: by construction. ✷.

Corollary 3.1: The first return map therefore corresponds to the cutting trajectory of a reduced periodic surd.

Theorem 4: the successive *n*-th return map is defined by means of the occurrence of the operators which precede the *n*-th occurrence of the *A* operator corresponding to conjugacy subclass of the periodic orbit corresponding to a reduced surd.

Proof: by construction. ✷.

The features of the reduced surds allow one to chose an orientation of the 2-dimensional reduced Birkhoff surface of section.

Definition 4: The orientation of the 2-dimensional reduced Birkhoff surface of section is chosen as positive as the outgoing directionof the trajectory defining the return maps.

XI. OUTLOOK AND PERSPECTIVES

The unfolding of the perturbations of periodic orbits lead to non-periodic configurations, which allows one to complete tessellate the Poincar´e Upper Half Plane rather than obtaining tori. To study this construction, it is useful to consider the topological entropy associated with geodesics flows and that associated with Anosov flows [24]. The comparison of the topological entropy of the diffeomorphisms and that of continuous maps has been investigated in [25] as far as the eigenvalues of the continuous maps the topological entropy induces in real homology.

Sectional-Anosov flows are demonstrated to define incompressible transverse tori after certain types of periodic configurations [26].

The definition of reduced Birkhoff surfaces allows therefore one to relate the Artin symbolic-dynamics description with the Anosov problem, where the chosen maps have been illustrated.

Due to the properties of the Artin system(s), the perturbations of the periodic orbits provide with an unfolding able to reconstruct the complete tesselation of the Upper Poincar´e Hal Plane.

The study of the topological entropy of the corresponding systems allows one to chose the perturbation to be applied on a chosen periodic configuration.

The relation between the dynamics of pseudo-Anosov mapping classes and that of H´enon mappings is still proposed as an open question [27].

The application of the H´enon map is useful in the study of quadratic surds.

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APPENDIX A: COMPLEMENTS OF GEODESICS FLOWS

The study of a compact Riemann manifold of negative constant curvature allows one to define homology classes for geodesics [28]. The number of prime closed geodesics on the considered Riemann surface is asymptotically calculated as an exponential function of the topological entropy of the geodesics flow and a suitable positive constant [29].

In [30], the deomnstration is provided with that, with an Anosov flow leaving the smooth volume element of a compact Riemann manifold *m* as invariant, the union of periodic orbits is dense in *m*.

In [31], after the study of asymptotic cycles and the features of the space of invariant measures for hyperbolic flows of a volume-preserving Anosov flow on a compact manifold, the closed orbits are proven to span the first homology (with real coefficients).

In [32], for transitive Anosov flows, an asymptotic formula for the number of definite closed orbits in a fixed homology class if found.

From [33], the basic notion of mapping class groups are introduced. Further properties of Anosov flows are collected in [34].

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