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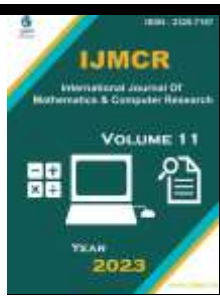

Optimization of a Fuzzy Inventory Model with Pentagonal Fuzzy Numbers

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ABSTRACT

This paper explores an optimal replenishment strategy for a two-echelon inventory model which has been considered and analyzed in a fuzzy environment. In fuzzy environment, carrying cost, ordering cost and the replenishment processing cost are assumed to be pentagonal fuzzy numbers. The purpose of this model is to minimize the total inventory cost in fuzzy scenario. There are two inventory models proposed in this paper. Crisp models are developed with fuzzy total inventory cost but crisp optimal order quantity. Fuzzy model is also formulated with fuzzy total inventory cost and fuzzy optimal order quantity. Graded mean integration formula is employed to defuzzify the total inventory cost and the Kuhn–tucker condition is used to determine the optimal order quantity. Finally, we develop an algorithm to obtain the optimal order quantity. A comparison of fuzzy model with classical inventory model is being made. Numerical results highlighting the sensitivity of various parameters are also elucidated. The result illustrates that this fuzzy model can be quite useful in determining the optimal order quantity and minimum total inventory cost.

KEYWORDS: Fuzzy inventory model, pentagonal fuzzy numbers, graded mean integration, kuhn-tucker condition, defuzzification.

1. INTRODUCTION

Supply chain management has the power to boost customer service, reduce operating costs and improve the financial standing of a company. Organizations increasingly find that they must rely on effective supply chains, or networks, to compete in the global market and networked economy. Multi-echelon inventory system has formed by an item in which it moves through more than one stage before reaching the final customer. Clark and Scarf (1960) were the first to study the two-echelon inventory model. Integrated inventory model for the buyer-vendor coordination was proposed by Goyal and Gupta (1989).

Inventory models considering lead time as a determinate variable have been developed by several researchers recently. Annadurai and Uthayakumar (2010a) developed a combination inventory model with backorders and lost sales in which the order quantity, reorder point, lead time and setup cost are ruling variables. It is assumed that an appearance order lot may contain some defective products and the number of defective products is a random variable. There are two inventory models scheduled in this paper, one with normally distributed demand and the another with distribution-free demand. Again Annadurai and Uthayakumar (2010b) proposed (T, R, L) inventory model to analyze the effects of increasing two different kinds of investments to lessen the lost-sales rate, in which the review period, lead time and lost-sales rate are preserved as decision variables.

EOQ – based inventory models minimize the sum of mainly two costs, which are the holding and the ordering costs. These models assume that the input parameters and the decision variables are described as crisp values or having crisp statistical distributions where their total inventory cost functions are minimized without ambiguity in the results. Further, using these models require inventory managers to have some flexibility when deciding on the sizes of the order quantities to reduce the cost of uncertainty. Hence, using fuzzy set theory to solve inventory problems, instead of the traditional probability theory, produces more accurate results can be clearly perceived in the fuzzy inventory models proposed by Guiffrida.

In inventory modeling, the closest possible approach in reality is the fuzzy set theory. Taha (1997) gave the kuhn-tucker condition used to solve uncertainty problems as mentioned in operations research. Fuzzy arithmetic theory and applications was introduced by Kaufmann and Gupta (1991). Chen (1985) discussed arithmetic operations on fuzzy numbers with function principle, which might be used as the fuzzy arithmetic operations with generalized fuzzy numbers. Chen and Hsieh (1999) elaborated a graded mean integration representation of the generalized fuzzy number. Sumana Saha and Tripti Chakrabarti (2012) proposed fuzzy economic order quantity model for time dependent deteriorating items and time dependent demand with shortage. Optimization of fuzzy production inventory model with repairable defective products under crisp or fuzzy production quantity was proposed by Chen et al. (2005). Fuzzy inventory model for deteriotating items with fluctuations demand and using inventory parameters as pentagonal fuzzy numbers was proposed by Harish Nagar and Priyanka Surana (2015). Dutta and Pavan Kumar (2013) considered an optimal policy for an inventory model without shortages by making fuzziness in demand, holding cost and ordering cost.

Syed and Aziz (2007) applied singed distance method to fuzzy inventory model without shortages. Dutta and Pavan Kumar (2012) proposed fuzzy inventory model without shortage using trapezoidal fuzzy number. Rajalakshmi and Michael Rosario (2017) proposed a fuzzy inventory model with allowable shortage using different fuzzy numbers. Jaggi et al. (2012) applied fuzziness in inventory model for deteriorating items with time-varying demand and shortages. Lee and Yao (1999) determined an economic order quantity in a fuzzy environment for an inventory model without backorder. Apurva Rawat (2011) proposed a fuzzy inventory model without shortages using triangular fuzzy number.

In this article, we address a fuzzy environment for an optimal inventory model. Initially, we start with the fuzzy inventory model for crisp-order quantity. Then fuzzy inventory model is proposed for fuzzy order quantity to find the total inventory cost in the fuzzy sense. Then the corresponding optimal order quantity has determined. The rest of the paper is organized as follows. In section 2 introduces the notations and assumptions required to state the problem. In section 3, we mathematically formulate the problem. In section 4, the methodology of Graded mean integration method, pentagonal fuzzy number and the Kuhn-Tucker condition is discussed. In section 5 describes the fuzzy inventory models. The numerical example along with the graphical representation is done to illustrate crisp and fuzzy sense in section 6. Finally, certain important managerial insights are also surmise from the sensitivity analysis in section 7. We conclude with final remarks in section 8.

2. NOTATIONS AND ASSUMPTIONS

In this paper, the proposed model is developed on the basis of the following notations and assumptions.

2.1 Notations:

TIC : Total inventory cost.

h : Inventory holding cost per unit time.

Q : Order quantity per cycle.

K : Ordering cost per order.

D : Demand quantity.

R : Replenishment processing cost.

T : Length of the plan.

Q^* : Optimal order quantity.

$T\tilde{I}C$: Fuzzy total inventory cost.

$\tilde{h}$: Fuzzy holding cost per item per unit time.

$\tilde{K}$: Fuzzy Ordering cost per order.

$\tilde{R}$: Fuzzy Replenishment processing cost.

$\tilde{Q}^*$: Fuzzy optimal order quantity.

2.2 Assumptions:

- Total demand is considered as a constant.
- Shortages are not allowed.
- Time plan is constant.
- Ordering cost, holding cost, and replenishment processing cost all are pentagonal fuzzy numbers.

3. MATHEMATICAL MODEL

For a crisp inventory model, the total inventory cost is the sum of the following elements

Ordering cost = $\dfrac{K*D}{Q}$,

Inventory holding cost = $\dfrac{h*Q}{2}$ and

Replenishment processing cost = $\dfrac{R*D}{Q}$.

That is, $$TIC = \frac{KD}{Q} + \frac{hQ}{2} + \frac{RD}{Q}. \qquad (1)$$

In order to find the minimization of TIC, the derivative of TIC with respect to Q is $\dfrac{\partial TIC}{\partial Q} = 0$, it becomes,

$$-\frac{KD}{Q^2} + \frac{h}{2} - \frac{RD}{Q^2} = 0. \qquad (2)$$

Since, $\dfrac{\partial^2 TIC}{\partial Q^2} = \dfrac{2D}{Q^3}(K+R) > 0.$

Hence, we find the optimal order quantity Q^*.

Solving Eq. (2) for Q, we obtain $Q^* = Q = \sqrt{\dfrac{2D(K+R)}{h}}$. $\qquad (3)$

4. METHODOLOGY

In this paper, Graded mean integration method and function principle are used to find the optimal order quantity with a fuzzy inventory model. When the quantities are fuzzy numbers Kuhn-Tucker method is used to solve the model.

4.1 Graded mean integration representation method

Chen and Hsieh (1999) introduced Graded mean integration representation method based on the integral value of graded mean h – level of generalized fuzzy number to defuzzify the same. Here, generalized fuzzy number is described as follows:

Any fuzzy subset of the real line R, whose membership function satisfies the following conditions, is a generalized fuzzy number.

i. $\mu_{\tilde{A}}(x)$ is a continuous mapping from R to the closed interval $[0,1]$,

ii. $\mu_{\tilde{A}}(x) = 0, -\infty \le x \le a_1$,

iii. $\mu_{\tilde{A}}(x) = L(\mathrm{x})$ is strictly increasing on $[a_1, a_2]$,

iv. $\mu_{\tilde{A}}(x) = w_A,\ a_2 \le x \le a_3$,

v. $\mu_{\tilde{A}}(x) = R(\mathrm{x})$ is strictly decreasing on $[a_3, a_4]$,

vi. $\mu_{\tilde{A}}(x) = 0,\ a_4 \le x \le \infty$, where a_1, a_2, a_3 and a_4 are real numbers and $0 < w_A \le 1$.

This type of generalized fuzzy number is also denoted as $\tilde{A} = (a_1, a_2, a_3, a_4, w_A)_{LR}$. When $w_A = 1$, it can be simplified as $\tilde{A} = (a_1, a_2, a_3, a_4)_{LR}$.

Defuzzification of $\tilde{F}$ can be found by graded mean integration representation. If $\tilde{F}$ is a pentagonal fuzzy number (a,b,c,d,e) then the graded mean integration representation formula is given by

$$G(\tilde{F},\tilde{0}) = \int_0^1 d\left(\left[A_L(\alpha),\ A_R(\alpha)\right],\ \tilde{0}\right) d\alpha = \frac{1}{12}(a+3b+4c+3d+e). \qquad (4)$$

4.1.1 Pentagonal fuzzy number

A pentagonal fuzzy number $\tilde{A} = (a,b,c,d,e)$ is represented with membership function $\mu_{\tilde{A}}$ as:

$$\mu_{\tilde{A}} = \begin{cases} L_1(x) = \dfrac{x-a}{b-a}, a \le x \le b, \\ L_2(x) = \dfrac{x-b}{c-b}, \mathrm{b} \le x \le c, \\ 1, \quad x = c, \\ R_1(x) = \dfrac{d-x}{d-c}, \mathrm{c} \le x \le d, \\ R_2(x) = \dfrac{e-x}{e-d}, \mathrm{d} \le x \le e, \\ 0, \quad Otherwise. \end{cases}$$

The α – cut of a pentagonal fuzzy number $\tilde{A} = (a,b,c,d,e)$, $0 \le \alpha \le 1$ is $A(\alpha) = [A_L(\alpha), A_R(\alpha)]$, where $A_L(\alpha) = a + (b-a)\alpha$ and $A_R(\alpha) = d + (d-c)\alpha$ are left and right end points of A_α. Here, we have

$A_{(L_1)}(\alpha) = a + (b-a)\alpha = L_1^{-1}(\alpha)$, $A_{(L_2)}(\alpha) = b + (c-b)\alpha = L_2^{-1}(\alpha)$,

$A_{R_1}(\alpha) = d - (d-c)\alpha = R_1^{-1}(\alpha)$ and $A_{R_2}(\alpha) = e - (e-d)\alpha = R_2^{-1}(\alpha)$.

So $L^{-1}(\alpha) = (L_1^{-1}(\alpha) + \mathrm{L}_2^{-1}(\alpha))/2 = (a + (b-a)\alpha + b + (c-b)\alpha)/2 = (a + b + (c-a)\alpha)/2$,

$R^{-1}(\alpha) = (R_1^{-1}(\alpha) + R_2^{-1}(\alpha))/2 = (d - (d-c)\alpha + e - (e-d)\alpha/2 = (d + e - (e-c)\alpha/2$.

The relationship between α – level set $A(\alpha) = [A_L(\alpha), A_R(\alpha)]$ of a pentagonal fuzzy number and α – level fuzzy interval $[L_\alpha, R_\alpha]$ can be expressed as follows, $A_L(\alpha) = L^{-1}(\alpha) = L_\alpha$, and $A_R(\alpha) = R^{-1}(\alpha) = R_\alpha$. For the fuzzy set $\tilde{A}$, $0 \le \alpha \le 1$ according to the decomposition theorem in the fuzzy theory, the following expression can be obtained as $\tilde{A} = \bigcup_{0 \le \alpha \le 1} \tilde{A}_\alpha = \bigcup_{0 \le \alpha \le 1} [L_\alpha, R_\alpha]$.

The graphical representation of the α – cut of pentagonal fuzzy number is shown in the Figure.1.

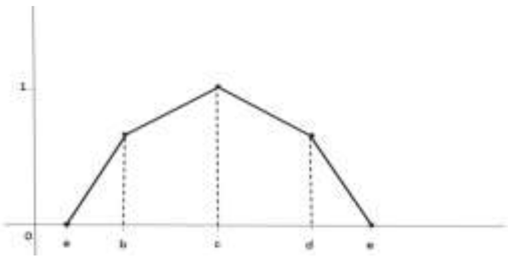

Figure 1: α – cut of pentagonal fuzzy number

4.1.2 Arithmetic operations on pentagonal fuzzy numbers under function principle

The function principle was introduced by Chen (1985) to treat arithmetical operations on fuzzy numbers. This principle is used for addition, subtraction, multiplication, and division of fuzzy numbers. We define some fuzzy arithmetical operations for the pentagonal fuzzy numbers. Let $\tilde{A} = (a_1, a_2, a_3, a_4, a_5)$ and $\tilde{B} = (b_1, b_2, b_3, b_4, b_5)$ be two pentagonal fuzzy numbers, then

1. The addition of $\tilde{A} = (a_1, a_2, a_3, a_4, a_5)$ and $\tilde{B} = (b_1, b_2, b_3, b_4, b_5)$ is given by

$\tilde{A} \oplus \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5)$,

where a_1, a_2, a_3, a_4, a_5 and b_1, b_2, b_3, b_4, b_5 are any real numbers.

2. The multiplication of $\tilde{A} = (a_1, a_2, a_3, a_4, a_5)$ and $\tilde{B} = (b_1, b_2, b_3, b_4, b_5)$ is given by

$\tilde{A} \otimes \tilde{B} = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4, a_5 b_5)$, where a_1, a_2, a_3, a_4, a_5 and b_1, b_2, b_3, b_4, b_5 are any non-zero positive real numbers.

3. $-\tilde{B} = (-b_5, -b_4, -b_3, -b_2, -b_1)$ then the subtraction of $\tilde{B}$ from $\tilde{A}$ is

$$\tilde{A} ! \ \tilde{B} = (a_1 - b_5, a_2 - b_4, a_3 - b_3, a_4 - b_2, a_5 - b_1),$$

where a_1, a_2, a_3, a_4, a_5 and b_1, b_2, b_3, b_4, b_5 are any real numbers.

4. $\dfrac{1}{\tilde{B}} = \tilde{B}^{-1} = \left(\dfrac{1}{b_5}, \dfrac{1}{b_4}, \dfrac{1}{b_3}, \dfrac{1}{b_2}, \dfrac{1}{b_1} \right)$, where b_1, b_2, b_3, b_4, b_5 are all non-zero positive real numbers, then division of $\tilde{A}$ and $\tilde{B}$ is

$$\tilde{A} \% \tilde{B} = \left(\frac{a_1}{b_5}, \frac{a_2}{b_4}, \frac{a_3}{b_3}, \frac{a_4}{b_2}, \frac{a_5}{b_1} \right).$$

5. For any real number k,

$$k \otimes \tilde{A} = \begin{cases} (ka_1, ka_2, ka_3, ka_4, ka_5), k \geq 0 \\ (ka_5, ka_4, ka_3, ka_2, ka_1), k < 0. \end{cases}$$

4.1.3 Kuhn-Tucker Condition

Taha (1997) discussed to solve the optimum solution of non- linear programming problem subject to inequality constraints by using Kuhn-Tucker conditions. The development of the Kuhn-Tucker condition is based on the Lagrangian method.

Suppose that the problem is given by Minimize $z = f(x)$, subject to $h_i(x) \geq 0,\ i = 1, 2, ..., m.$ The nonnegative constraints $x \geq 0$, if any, are included in the m constraints. The inequality constraints may be converted into equations by using nonnegative surplus variables. Let S_i^2 be the surplus quantity added to the ith constraint $h_i(x) \geq 0.$ Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m),$ $h(x) = (h_1(x), h_2(x), ..., h_m(x))$ and $S^2 = (S_1^2, S_2^2, ..., S_m^2).$ Then the Lagrangian functions are given by $\mathrm{L}(x, \mathrm{S}, \lambda) = \mathrm{f}(x) - \lambda(h(x) - S^2).$

Given the constraints $h_i(x) \geq 0.$

Taking the partial derivatives of L with respect to x, S and λ we obtain $\dfrac{\partial L}{\partial x} = \nabla f(x) - \lambda \nabla h(x) = 0,$ $\dfrac{\partial L}{\partial S_i} = 2\lambda_i S_i = 0,\ i = 1, 2, ..., m,$ and $\dfrac{\partial L}{\partial \lambda_i} = -h_i(x) + S_i^2 = 0,\ i = 1, 2, ..., m.$ From the second and third sets of equations it shows that $\lambda_i h_i(x) = 0,\ i = 1, 2, ..., m.$

The Kuhn-Tucker conditions need x and λ to be a stationary point of the minimization problem which can be summarized as following:

$$\lambda \leq 0,$$
$$\nabla f(\mathrm{x}) - \lambda \nabla h(\mathrm{x}) = 0,$$
$$\lambda_i h_i(\mathrm{x}) = 0, i = 1, 2, ..., m,$$
$$h_i(\mathrm{x}) \geq 0.$$

5. FUZZY INVENTORY MODELS

5.1 Fuzzy inventory model for crisp order quantity

Throughout this paper, following parameters are used in order to simplify the treatment of the fuzzy inventory models. Let $\tilde{K}$, $\tilde{R}$ and $\tilde{h}$ be fuzzy parameters. Now, fuzzy integrated inventory model is introduced with fuzzy parameters for crisp order quantity Q as follows. Suppose, $\tilde{K} = (K_1, K_2, K_3, K_4, K_5)$, $\tilde{R} = (R_1, R_2, R_3, R_4, R_5)$ and $\tilde{h} = (h_1, h_2, h_3, h_4, h_5)$ are non-negative pentagonal fuzzy numbers. Then we get the fuzzy total inventory cost as,

$$\tilde{TIC}=\left\{\left(\frac{K_1D}{Q}+\frac{h_1Q}{2}+\frac{R_1D}{Q}\right),\left(\frac{K_2D}{Q}+\frac{h_2Q}{2}+\frac{R_2D}{Q}\right),\left(\frac{K_3D}{Q}+\frac{h_3Q}{2}+\frac{R_3D}{Q}\right),\right.$$
$$\left.\left(\frac{K_4D}{Q}+\frac{h_4Q}{2}+\frac{R_4D}{Q}\right),\left(\frac{K_5D}{Q}+\frac{h_5Q}{2}+\frac{R_5D}{Q}\right)\right\}. \quad (5)$$

Now, we defuzzify the fuzzy total inventory cost Eq. (5) by using graded mean integration method. The representation of the fuzzy total inventory cost $\tilde{TIC}$ by graded mean integration method is given by

$$S(\tilde{TIC})=\frac{1}{12}\left\{\left(\frac{K_1D}{Q}+\frac{h_1Q}{2}+\frac{R_1D}{Q}\right)+3\left(\frac{K_2D}{Q}+\frac{h_2Q}{2}+\frac{R_2D}{Q}\right)+4\left(\frac{K_3D}{Q}+\frac{h_3Q}{2}+\frac{R_3D}{Q}\right)\right.$$
$$\left.+3\left(\frac{K_4D}{Q}+\frac{h_4Q}{2}+\frac{R_4D}{Q}\right)+\left(\frac{K_5D}{Q}+\frac{h_5Q}{2}+\frac{R_5D}{Q}\right)\right\}. \quad (6)$$

When defuzzified fuzzy total inventory cost $S(\tilde{TIC})$ is minimized, the fuzzy order quantity Q^* is obtained. The minimization of defuzzified fuzzy total inventory cost $S(\tilde{TIC})$ is derived from the solution of the first order partial derivative of the Eq. (6) with respect to Q. That is $\frac{\partial S(\tilde{TIC})}{\partial Q}=0$. Then the result is given by

$$\frac{1}{12}\left\{\frac{-D}{Q^2}(K_1+3K_2+4K_3+3K_4+K_5)+\frac{1}{2}(h_1+3h_2+4h_3+3h_4+h_5)\frac{-D}{Q^2}(R_1+3R_2+4R_3+3R_4+R_5)\right\}=0.$$

On simplification, the optimal order quantity Q^* is given by

$$Q^*=Q=\sqrt{\frac{2D(K_1+3K_2+4K_3+3K_4+K_5+R_1+3R_2+4R_3+3R_4+R_5)}{h_1+3h_2+4h_3+3h_4+h_5}}. \quad (7)$$

5.2 Fuzzy inventory model for fuzzy order quantity

In this section, an inventory model is introduced by changing the crisp order quantity into fuzzy order quantity. Suppose the fuzzy order quantity $\tilde{Q}$ be a pentagonal fuzzy number $\tilde{Q}=(Q_1,Q_2,Q_3,Q_4,Q_5)$ with $0\le Q_1\le Q_2\le Q_3\le Q_4\le Q_5$. The fuzzy total cost is given by

$$\tilde{TIC}=\left\{\left(\frac{K_1D}{Q_5}+\frac{h_1Q_1}{2}+\frac{R_1D}{Q_5}\right),\left(\frac{K_2D}{Q_4}+\frac{h_2Q_2}{2}+\frac{R_2D}{Q_4}\right),\left(\frac{K_3D}{Q_3}+\frac{h_3Q_3}{2}+\frac{R_3D}{Q_3}\right)\right.$$
$$\left.\left(\frac{K_4D}{Q_2}+\frac{h_4Q_4}{2}+\frac{R_4D}{Q_2}\right),\left(\frac{K_5D}{Q_1}+\frac{h_5Q_5}{2}+\frac{R_5D}{Q_1}\right)\right\}. \quad (8)$$

The graded mean integration representation of $\tilde{TIC}$ is given by

$$S(\tilde{TIC})=\frac{1}{12}\left\{\left(\frac{K_1D}{Q_5}+\frac{h_1Q_1}{2}+\frac{R_1D}{Q_5}\right)+3\left(\frac{K_2D}{Q_4}+\frac{h_2Q_2}{2}+\frac{R_2D}{Q_4}\right)+4\left(\frac{K_3D}{Q_3}+\frac{h_3Q_3}{2}+\frac{R_3D}{Q_3}\right)\right.$$
$$\left.+3\left(\frac{K_4D}{Q_2}+\frac{h_4Q_4}{2}+\frac{R_4D}{Q_2}\right)+\left(\frac{K_5D}{Q_1}+\frac{h_5Q_5}{2}+\frac{R_5D}{Q_1}\right)\right\}, \quad (9)$$

with $0\le Q_1\le Q_2\le Q_3\le Q_4\le Q_5$. The minimization of $S(\tilde{TIC})$ is obtained by differentiating the Eq. (9) with respect to Q_1,Q_2,Q_3,Q_4,Q_5. Thereby, letting all the partial derivatives equals to zero and solving for Q_1,Q_2,Q_3,Q_4,Q_5. We get,

$Q_1 = \sqrt{\frac{2(K_5 D + R_5 D)}{h_1}}$, $Q_2 = \sqrt{\frac{2(3K_4 D + 3R_4 D)}{3h_2}}$, $Q_3 = \sqrt{\frac{2(4K_3 D + 4R_3 D)}{4h_3}}$, $Q_3 = \sqrt{\frac{2(4K_3 D + 4R_3 D)}{4h_3}}$, and $Q_5 = \sqrt{\frac{2(K_1 D + R_1 D)}{h_5}}$.

Kuhn-Tucker condition is applied for the following linear programming problem to get the optimal order quantity $\tilde{Q}^*$.

Minimize $S(T\tilde{I}C)$

Subject to the constraints

$$Q_2 - Q_1 \geq 0$$
$$Q_3 - Q_2 \geq 0$$
$$Q_4 - Q_3 \geq 0$$
$$Q_5 - Q_4 \geq 0$$
$$Q_1 \geq 0$$

We get,

$$KT(Q_1, Q_2, Q_3, Q_4, Q_5, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = S((T\tilde{I}C)(Q, K, R, h)) - \lambda_1(Q_2 - Q_1) - \lambda_2(Q_3 - Q_2) - \lambda_3(Q_4 - Q_3) - \lambda_4(Q_5 - Q_4) - \lambda_5 Q_1 \quad (10)$$

The Kuhn-Tucker conditions are given as follows,

$\lambda \leq 0,$

$\tilde{\mathrm{N}} f(S(T\tilde{I}C)) - \lambda_i \tilde{\mathrm{N}} h(\tilde{\mathrm{Q}}) = 0,$

$\lambda_i\, h_i(\mathrm{Q}) = 0,\ i = 1, 2, \ldots, m,$

$h_i(\mathrm{Q}) \geq 0.$

These conditions simplify to the following,

$\lambda_1 \leq 0,\ \lambda_2 \leq 0,\ \lambda_3 \leq 0,\ \lambda_4 \leq 0,$ and $\lambda_5 \leq 0,$ (11.1)

$$\left[\frac{h_1}{2} - \frac{K_5 D}{Q_1^2} - \frac{R_5 D}{Q_1^2}\right] + \lambda_1 - \lambda_5 = 0, \quad (11.2)$$

$$\left[\frac{h_2}{2} - \frac{K_4 D}{Q_2^2} - \frac{R_4 D}{Q_2^2}\right] - \lambda_1 + \lambda_2 = 0, \quad (11.3)$$

$$\left[\frac{h_3}{2} - \frac{K_3 D}{Q_3^2} - \frac{R_3 D}{Q_3^2}\right] - \lambda_2 + \lambda_3 = 0, \quad (11.4)$$

$$\left[\frac{h_4}{2} - \frac{K_2 D}{Q_4^2} - \frac{R_2 D}{Q_4^2}\right] - \lambda_3 + \lambda_4 = 0, \quad (11.5)$$

$$\left[\frac{h_5}{2} - \frac{K_1 D}{Q_5^2} - \frac{R_1 D}{Q_5^2}\right] - \lambda_4 = 0, \quad (11.6)$$

$$\lambda_1(Q_2 - Q_1) = 0, \quad (11.7)$$

$$\lambda_2(Q_3 - Q_2) = 0, \quad (11.8)$$

$$\lambda_3(Q_4 - Q_3) = 0, \quad (11.9)$$

$$\lambda_4(Q_5 - Q_4) = 0, \quad (11.10)$$

$$\lambda_5(Q_1) = 0, \quad (11.11)$$

$Q_2 - Q_1 \geq 0$, $Q_3 - Q_2 \geq 0$, $Q_4 - Q_3 \geq 0$, $Q_5 - Q_4 \geq 0$, and $Q_1 \geq 0$. Since $Q_1 > 0$ and $\lambda_5(Q_1) = 0$, then $\lambda_5 = 0$. If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, then $Q_5 < Q_4 < Q_3 < Q_2 < Q_1$ and it does not satisfy the constraints $0 \leq Q_1 \leq Q_2 \leq Q_3 \leq Q_4 \leq Q_5$. Therefore $Q_2 = Q_1$, $Q_3 = Q_2$, $Q_4 = Q_3$, and $Q_5 = Q_4$. That is $Q_1 = Q_2 = Q_3 = Q_4 = Q_5 = \tilde{Q}^*$. Then the optimal fuzzy order quantity is defined as follows. $\tilde{Q}^* = (Q^*, Q^*, Q^*, Q^*, Q^*)$. Hence, we get the optimal order quantity $\tilde{Q}^*$ from the Eqs. (11. 2 - 11. 6).

That is $$\tilde{Q}^* = \sqrt{\frac{2D\left(K_1 + 3K_2 + 4K_3 + 3K_4 + K_5 + R_1 + 3R_2 + 4R_3 + 3R_4 + R_5\right)}{h_1 + 3h_2 + 4h_3 + 3h_4 + h_5}}. \tag{12}$$

It shows that the Eq. (12) becomes Eq. (3). That is, if $K_1 = K_2 = K_3 = K_4 = K_5 = K$, $R_1 = R_2 = R_3 = R_4 = R_5 = R$ and $h_1 = h_2 = h_3 = h_4 = h_5 = h$, then Eq. (12) can be revised as $Q = \sqrt{\frac{2D(K+R)}{h}}$.

Now we use the following algorithm to find the fuzzy total inventory cost and fuzzy optimal order quantity.

5.3 Algorithm

Step 1: Calculate total inventory cost for the crisp model for the given crisp value of K, R and h.

Step 2: Determine the fuzzy total inventory cost using fuzzy arithmetic operations on fuzzy ordering cost, fuzzy inventory holding cost and fuzzy replenishment processing cost which is taken as pentagonal fuzzy numbers.

Step 3: Use graded mean integration representation method to defuzzify the total inventory cost $\tilde{TIC}$ in order to find the fuzzy order quantity Q^*. It can be obtained by putting the first derivative of the defuzzified fuzzy total inventory cost $S(T\tilde{I}C)$ to zero.

Step 4: Use Kuhn-Tucker condition to find the fuzzy optimal order quantity $\tilde{Q}^*$.

Step 5: To check whether the fuzzy order quantity obtained by Kuhn-Tucker condition is same as the crisp order quantity and compare the total inventory cost obtained from both the crisp model and the fuzzy model along with their savings.

6. NUMERICAL EXAMPLE

To illustrate the solution procedure developed in the crisp model, let us consider the system with the data $D = 2300$, $K = 10$, $R = 20$ and $h = 2$. Using Eqs. (1) and (3), we get the minimum total cost $TIC = 525.36$ and optimal order quantity $Q^* = 262.68$. Here K, R and h are transferred into the fuzzy parameters. That is $\tilde{K} =$ (5, 7, 9, 14, 16), $\tilde{R} =$ (10, 12, 17, 30, 36) and $\tilde{h} = (1.5, 1.7, 2, 2.3, 2.5)$ is pentagonal fuzzy numbers. Using Eqs. (9) and (12) we get the defuzzified fuzzy total inventory cost $S(T\tilde{I}C) = 506.29$ and the optimal order quantity $\tilde{Q}^* = 262.28$.

7. SENSITIVITY ANALYSIS

We examine the effects of changes in the system parameters. Sensitivity analysis is ensured by changing the value of several input parameters and analyzes its effects on the optimal solution. The impact of fuzziness in the cost components is analyzed by assigning pentagonal fuzzy numbers to the parameters. The results of sensitivity analysis are presented in Table 1-3 and graphically shown in Fig. 2- 4. Based on the sensitivity analysis, we obtain the following managerial insights.

- Table 1 gives the optimal solution for several values of K. Increases in ordering cost results increases in order quantity and total cost. Therefore, an increase in ordering cost will lead to an increase in order quantity and total cost both in crisp and fuzzy model. But the total inventory cost is optimized in the fuzzy model than in the crisp model, which results in the system savings.

- Table 2 shows that, for both the model, increasing in the replenishment processing cost R results in increasing order quantity and the total cost. Also, it is perceived that the total inventory cost is effectively minimized in the fuzzy model when compare to the crisp model.
- Table 3 shows that when the holding cost h increases, the total cost increases. This is expected, because the holding cost is one of the significant components of the average total cost. In addition, order quantity decreases with increase in holding cost. Also, it is noticeable that the total inventory cost is effectively minimized in the fuzzy model than in the crisp model.

Table 1: Sensitivity analysis for $\tilde{K}$

Parameter K	Q^*	TIC	Parameter $\tilde{K}$	$\tilde{Q}^*$	$S(T\tilde{I}C)$	Savings %
6	244.54	489.08	(1,3,5,10,12)	244.54	466.97	4.5
8	253.77	507.54	(3, 5, 7, 12, 14)	253.77	487.10	4
10	262.68	525.36	(5, 7, 9, 14, 16)	262.68	506.29	3.6
12	271.29	542.59	(7, 9, 11, 16, 18)	271.29	524.68	3.3
14	279.64	559.29	(9, 11, 13, 18, 20)	279.64	542.36	3.1
16	287.42	575.50	(11, 13, 15, 20, 21)	287.42	558.97	2.9

Table 2: Sensitivity analysis for $\tilde{R}$

Parameter R	Q^*	TIC	Parameter $\tilde{R}$	$\tilde{Q}^*$	$S(T\tilde{I}C)$	Savings %
18	253.77	507.54	(8, 10, 15, 28,34)	253.772	487.10	4.0
19	258.26	516.53	(9, 11, 16, 29, 35)	258.263	496.80	3.8
20	262.68	525.36	(10, 12, 17, 30, 36)	262.679	506.29	3.6
21	267.02	534.04	(11, 13, 18, 31, 37)	267.021	515.58	3.5
22	271.29	542.59	(12, 14, 19, 32, 38)	271.293	524.68	3.3
23	275.50	551	(13, 15, 20, 33, 39)	275.500	533.60	3.2

Table 3: Sensitivity analysis for $\tilde{h}$

Parameter h	Q^*	TIC	Parameter $\tilde{h}$	$\tilde{Q}^*$	$S(T\tilde{I}C)$	Savings %
1	371.48	371.48	(0.5, 0.7, 1, 1.3, 1.5)	371.48	349.28	6.0
1.5	303.32	454.97	(1, 1.2, 1.5, 1.8, 2)	303.32	435.27	4.3
2	262.68	525.36	(1.5, 1.7, 2, 2.3, 2.5)	262.68	506.29	3.6
2.5	234.95	587.37	(2, 2.2, 2.5, 2.8, 3)	234.95	568.32	3.2
3	214.48	643.43	(2.5, 2.7, 3, 3.3, 3.5)	214.48	624.13	3.0
3.5	198.57	694.98	(3, 3.2, 3.5, 3.8, 4)	198.57	675.31	2.8

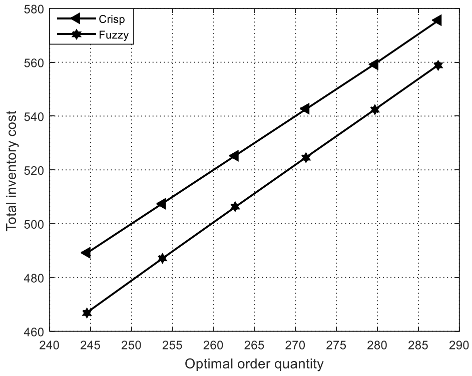

Figure 2. Sensitivity analysis of K

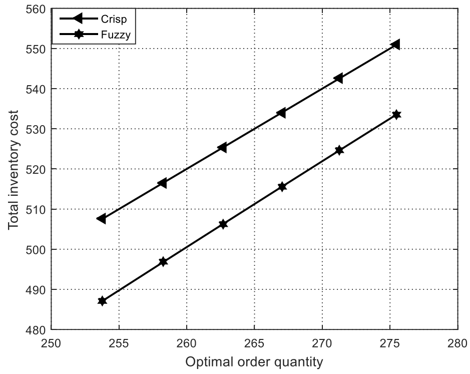

Figure 3. Sensitivity analysis of R

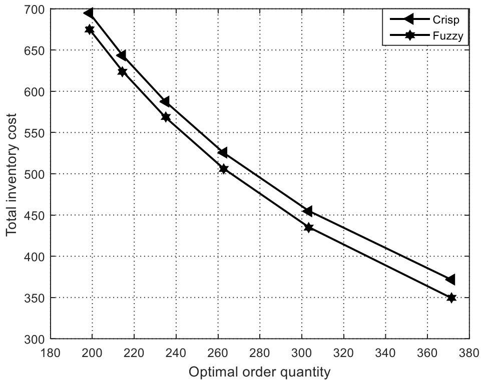

Figure 4. Sensitivity analysis of h

8. CONCLUSION

In this paper, a mathematical model for an optimal inventory model in fuzzy environment is developed in both crisp and fuzzy environments. In fuzzy environment, all related inventory parameters are assumed to be pentagonal fuzzy numbers. For defuzzification, graded mean integration method is employed to evaluate the minimum total cost. Kuhn - Tucker method is used to determine the optimal order quantity. A computational algorithm is framed to investigate the effects of fuzzy parameters on the optimal order quantity and minimum total cost of the proposed model. Graphical representation of numerical examples shows that by using the proposed fuzzy model, one can obtain a significant amount of savings. As a result of the proposed strategy, Pentagonal fuzzy numbers provide minimum total cost and system savings. After comparing both the conventional crisp and fuzzy models, it is observed that the fuzzy model is better than the conventional crisp model.

Future researches on this problem can deal with inventory constraints, ordering constraints, etc. Further, various type of multi-echelon supply chain models can be considered in crisp sense, fuzzy sense or both.

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