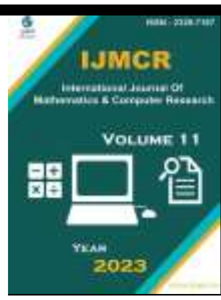
International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 11 Issue 05 May 2023, Page no. – 3410-3415

Index Copernicus ICV: 57.55, Impact Factor: 7.362

DOI: 10.47191/ijmcr/v11i5.03



η-Ricci Solitons on Lorentzian β-Kenmotsu Manifold

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ARTICLE INFO	ABSTRACT
Published Online: 16 May 2023 Corresponding Author: **R. N. Singh**	η-Ricci solitons on Lorentzian β- Kenmotsu manifold are considered an manifolds satisfying certain curvature Conditions, R(ξ,X).S=0, S(ξ,X).R=0, $W_2(\xi,X).S=0$, $S(\xi,X).W_2=0$ We proved that in Lorentzian β-Kenmotsu manifold (M,ϕ,ξ,η,g). Then the existence of an η-Ricci solitons implies that M is Einstein manifold and if the Ricci curvature tensor satisfies, S(ξ,X).R=0, then Ricci solitons M is steady. If the condition μ=0, then λ=0, which shows that λ is steady.

KEYWORDS: β-Kenmotsu manifolds, η-Ricci solitons, W_2- curvature tensor etc.

2010 Mathematics Subject Classification: 53C25, 53C15.

INTRODUCTION

The concept of Ricci solitons was introduced by Hamilton [20, 21] in the Year 1982. In differential geometry, a Ricci solitons is a special type of Riemannian metric such metric evalve under Ricci flow only be symmetries of the flow and ofter arise as limits of dilations of Singularities in the Ricci flow [21, 14, 33]. They can be viewed as generalizations of Einstein metrics. They can viewed as fixed point of the Ricci flow, as a dynamical system on the space of Riemmanian metrics moduls diffeomorphisms ans scallings. Ricci solitons have been studied in many contexts: on Kahler manifolds [18], on contact and Lorentzian manifolds [12,14, 37, 83, 10, 14, 23],on Sasakian [6], α-Sasakian manifolds [4, 37] and K-contact manifolds [34], on Kenmotsu [1, 2, 26, 29, 35], and f-Kenmotsu manifolds [13], etc. In Para contact geometry, Ricci solitons firstly appeared in the papers of G. Calvaruso and D. Perrone [15], Recently, C. L. Bejan and M. Crasmareanu studied Ricci solitons on 3-dimensional normal Para contact manifolds [8, 34]. As a generalizations of Ricci solitons, the notion of η-Ricci solitons introduced by J. T. Cho and M. Kimura [17], which was treated by C. Calin and Crasmareanu on Hopf Hypersurfaces in complex space forms [16, 14]. The concept is named after Gegorio Ricci-Curbastro. In 2015, A.M. Blaga have obtained some results on η-Ricci solitons satisfying certain curvature conditions. Ricci solitons were introduced by R. S. Hamilton as natural generalization of Einstein metric [20, 21], and have studied in many contexts on α-Sasakian manifold [4, 37], trans-sasakian etc. Ricci soliton firstly appeared in the paper of G. Calvaruso and D. Perrone [15]. Recently C.L. Bejan and M. Crasmareanu [13, 14], defined with Ricci soliton an α-Sasakian manifold [4]. In [17], Cho and Kimura studied Ricci solitons of real hyper surfaces in non-flat complex space from and they defined η-Ricci soliton, which satisfies the equation

$$\mathcal{L}_V g+2S+2\lambda g+2\mu\eta\otimes\eta=0, \quad (1.1)$$

where λ and μ are real constants. The η-Ricci solitons are studied on Hopt hyper surfaces in the paper [14]. Also it may be noted that a generalized of η-Ricci Einstein geometry is provided by Ricci soliton for this frame work are studied. In 1970, Pokhariyal and Mishra [32], have introduced new curvature tensor called W_2-curvature tensor in a Riemannian manifold and studied some properties. Further, Pokhariyal [32], has studied some properties of this curvature tensor in a Sasakian manifold. Matsumoto, Ianus and Mihai [22, 23], Ahmet Yildiz and U. C. De [1, 2] and Venkatesha, C. S. Bagewadi, and K. T. Pradeep Kumar have studied W_2-curvature in P-Sasakian ,Kenmotsu and Lorentzian para-Sasakian manifolds respectively.

In [36], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of a plane sections containing is a constant, say C. He showed that they can be divided into three classes:(1), Homogeneous normal contact Riemannian manifolds with c> 0: (2), Global Riemannian Products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if c=0 and: (3), A wared product space $\mathbb{R}\times_f C$ if $c>0$. It is know that the manifolds of class (1) are characterized by admitting a Sasakian structure. In the Gray-Hervella classification of almost Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [8, 12, 18, 19]. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [5, 24, 25], if the product manifold M× ℝ belongs to the class W_2.The class $C_6 \otimes C_5$ [9].coincides with the class of the trans-Sasakian structures of type (α,β).in fact, in [24]. Local nature of the two subclasses, namely, C_5 and C_6, structures of trans-Sasakian structures are characterized completely. We note that trans-Sasakian structure of type (0,0),(0,β) and (α, 0) are cosymlectic, β -kenmotsu [3], and α -Sasakian [4], Respectively. In [24], it is proved that trans-Sasakian structures are generalized quasi-Sasakian Thus, trans-Sasakian structures also provide a lorge class of generalized quasi-Sasakian structures. An almost contact metric structure (ϕ,ξ,η,g) on M is called a trans-Sasakian structure [24, 25], if (M× $\mathbb{R}, J, G$) belongs to the class W_4 [24], where J is the almost complex structure on M × ℝ , defined by

$$J(X, f\frac{d}{dt})=(\phi X\text{-}f,\ \eta(X)\frac{d}{dt}), \qquad (1.2)$$

for all vector fields X on M and smooth functions f on M× ℝ. and G is the product metric on M× ℝ. this may be expressed by the condition.

$$(\nabla_X\phi)Y=\alpha(g(X,Y)\xi\text{-}\eta(Y)X) + \beta(g(\phi X,Y)\text{-}\eta(Y)\phi X), \qquad (1.3)$$

for some smooth functions α and β on M, and we say that the trans-Sasakian structure is of type (α,β). A trans-Sasakian structure of type (α,β) is α-Sasakian if β=0, and α a non zero constant [5]. If α =1, then α-Sasakian manifold is a Sasakian manifold. Motivated by the above studied the object of the present paper is to study η-Ricci solitons on α-Sasakian manifold which satisfy certain curvature properties R(ξ,X).S=0, S(ξ,X).R=0, respectively. Remark that in [26], H.G. Nagaraja and Premalatha have obtained some results on η-Ricci soliton satisfying conditions of the following type $W_2(\xi,X).S$=0, and $S(\xi,X).W_2$=0, we also proved that α-Sasakian manifold of constant curvature supporting an η-Ricci soliton is locally isometric to sphere. Then ξ is (i) concurrent and (ii) Killing vector field. During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci sol itons to solve the long standing Poincare conjecture posed in 1904. There after Ricci solitons in contact metric manifolds have been studied by various authors such as Bejan and Crasmareanu [7], Blaga [10, 11], Nagaraja and Premalatta [26], I. T. Cho and M. Kirmura [17], U. C. De, and many others.

Preliminaries

A differentiable manifold M of dimension n is called Lorentzian β- Kenmotsu manifold . If it admits a (1.1)-tensor field ϕ contravarient vector field ξ, a covariant vector field η and Lorentzian metric g which g satisfy endowed with an almost contact structure (ϕ,ξ,η,g) satisfies

$$\eta(\xi)=1, \phi.\xi=0,\ \eta(\phi X)=0, \qquad (2.1)$$

$$\phi^2X= X + \eta(X)\xi ,\ \eta(X)= g(X,\xi), \qquad 2.2)$$

$$g(\phi X,\phi Y)= g(X,Y)\text{-}\eta(X)\eta(Y), \qquad (2.3)$$

for any vector field X,YϵTM,

Also if Lorentzian β- Kenmotsu manifold M satisfies

$$\nabla_X\xi=\beta[X\text{-}\eta(X)\xi], \qquad (2.4)$$

$$(\nabla_X\phi)(Y)=\beta[g(\phi X,Y)\text{-}\eta(Y)\phi X], \qquad (2.5)$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian Metric g. Then M is called Lorentzian β-Kenmotsu manifold. Further on an Lorentzian manifold M the following relation hold

$$\eta(R(X,Y)Z\ )=\ \beta^2[g(X,Z)\eta(Y)\ -\ g(Y,Z)\eta(X)], \quad (2.6),$$

$$R(\ \xi,\ X)\ Y=\beta^2[\eta(Y)X-g(X,Y)\xi]\ , \quad (2.7) \qquad R(X,\ Y)\xi=\beta^2[\eta(X)Y-\eta(Y)X], \qquad (2.8)$$

$$S(X,\xi)\text{=-}(n\text{-}1)\beta^2\ \eta(X), \qquad (2.9)$$

$$Q\xi=-(n-1)\beta^2\xi, \qquad (2.10)$$

$$S(\xi,\xi)\text{=-}(n\text{-}1)\beta^2, \qquad (2.11)$$

In Pokhariyal and Mishra have defined the curvature tensor W_2 given by

$$W_2(X,Y,U,V)=R(X,Y,U,V) + \frac{1}{n-1}[g(X,U)S(Y,V)\text{-}\ g(Y,U)S(X,V)], (2.12)$$

where S is a Ricci tensor of type (0,2).

Consider an Lorentzian β-Kenmotsu manifold satisfying W_2 =0, In view of the equation(2.12), then we have

$$R(X,Y,U,V)=\frac{1}{n-1}[g(X,U)S(Y,V) \text{-}\ g(Y,U)S(X,V)], \qquad (2.13)$$

Putting Y=V= ξ, in above equation (2.13), then using equations (2.9) and (2.11), we obtain

$$S(X,U)\text{=-}\beta^2[g(X,Y)\text{+}\eta(U)\eta(X)], \qquad (2.14)$$

Thus M is an Einstein manifold.

Theorem (2.1): If η-Ricci solitons on Lorentzian β-Kenmotsu manifol M, the condition W_2=0, holds, then M is an Einstein manifold.

Definition: An η-Ricci solitons on Lorentzian β-Kenmotsu manifold is called W_2-semi-symmetric if it satisfies

$$R(X,Y).\ W_2=0, \qquad (2.15)$$

where R(X,Y) is to be considered as a derivation of tensor algebra at each point of the manifold for tangent vector X and Y. In an β-Kenmotsu manifold the W_2-curvature tensor satisfies the condition

$$\eta(W_2(X,Y)Z=0, \qquad (2.16)$$

3. η-Ricci solitons on Lorentzian β-Kenmo0, anifold

Let (M,ϕ,ξ,η,g) be a η-Ricci solitons on Lorentzian β-Kenmotsu manifold. Consider the equation

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta\otimes\eta=0, \qquad (3.1)$$

where L_ξ is the Lie derivative operator along the vector field ξ, S is Ricci tensor of the metric g, λ and μ are real constants. Now, from above equations (3.1) and (2.4), we get

$2S(X,Y)=- g(\nabla_X\xi, Y)- g(X,\nabla_Y\xi)-2\lambda g(X,Y)$
$-2\mu\eta(X)\eta(Y)$, (3.2)

Now, from above equations (3.2) and (2.4), we get

$S(X,Y) = (\lambda+\beta)g(X,Y)$
$+(\beta-\mu)\eta(X)\eta(Y)$, (3.3)

for any $X,Y \epsilon \chi(M)$, or the data (g,ξ,λ,μ) which satisfy the equation (3.1) is said to be an η-Ricci soliton on M, in Particular, if μ=0, (g,ξ,λ) is a Ricci solitons [] and it is called shrinking, steady or expanding according as λ is negative, zero or positive respectively.

Proposition (3.1): η-Ricci solitons on Lorentzian β-Kenmotsu manifold (M,ϕ,ξ,η,g), the following relations holds.

$\nabla_X\xi = \beta[X-\eta(X)\xi]$, (3.4)
$\eta(\nabla_X\xi)=0$, (3.5)
$R(X,Y)\xi = \beta^2[\eta(X)Y-\eta(Y)X]$, (3.6)
$\eta(R(X,Y)Z) =$
$\beta^2[g(X,Z)\eta(Y)-g(Y,Z)\eta(X)]$, (3.7)
$(\nabla_X\eta)(Y) = \beta[g(X,Y)-\eta(X)\eta(Y)]$,
$(\nabla_X\eta)=0$, (3.9)
$L_\xi g(X,Y) = 2\beta[g(X,Y)-\eta(X)\eta(Y)]$ (3.10)

where R is the Riemannian curvature tensor field and ∇ denotes is the Levi-Civita connection Associated to g.

$(L_\xi\phi)(X)=\nabla_\xi\phi X-\phi(\nabla_\xi X)$, (3.11)

By virtue of the above equation (3.11), in above equation, X=ξ, we get

$(L_\xi\phi)(\xi)=\nabla_\xi\phi\xi-\phi(\nabla_\xi\xi)$, (3.12)

Now, taking from above equations (3.12) and (2.1), in above equation, we get

$(L_\xi\phi)=0$,

In view of the using from the above equation (3.1), we get

$L_\xi g(X,Y)+2S(X,Y)+2\lambda g(X,Y)$
$+2\mu\eta(X)\eta(Y)=0$, (3.13)

which on using in above equation (3.13), putting Y=ξ, we get

$L_\xi\eta(X)+2S(X,\xi)+(2\lambda+2\mu)\eta(X)$
$=0$, (3.14)

In virtue of the using from the above equations (3.14) and (2.9), we get

$L_\xi\eta(X)+[(2\lambda+2\mu-(n-1)\beta^2)]$ =0, (3.15)

Now, Taking from the above equation (3.15) in X=ξ, we get

$2(\lambda+\mu)=(n-1)\beta^2$, (3.16)

Again, we have

$(L_\xi\eta)(X)=-\eta(\nabla_\xi X)$, (3.17)

In view of the above equation (3.17), in above equation, X=ξ, we get

$(L_\xi\eta)=0$,

Again, we have

$(L_\xi\eta)(X)=-\eta(\nabla_\xi X)-2\beta[\eta(X)]$, (3.18)

Now, taking from the above equation (3.17), as X=ξ,we get

B=0, (3.19)

Again we have

$(L_\xi g)(X,Y)=-(\nabla_\xi g)(X,Y)$
$+2\beta[g(X,Y)-\eta(X)\eta(Y)]$, (3.20)
where $(\nabla_\xi g)(X,Y)=0$,
$(L_\xi g)(X,Y)$
$=2\beta[g(X,Y)-\eta(X)\eta(Y)]=0$, (3.21)

for this proposition we get the (0,2), tensor field

$\Omega(X,Y)= g(X,\phi X)$,

Is symmetric and satisfies

$\Omega(\phi X,Y)=\Omega(X,\phi Y)$, $\Omega(\phi X,\phi Y)=\Omega(X,Y)$
$(\nabla_X\Omega)(Y,Z)=\eta(Y)g(X,Z)+\eta(Z)g(X,Y)+2\eta(X)\eta(Y)\eta(Z)$, for any X, Y,Z $\epsilon\chi(M)$.

Remark: η- Ricci Soliton on Lorentzian β-Kenmotsu manifold (M,ϕ,ξ,η,g) we deduce that

Proposition (3.2): If η-Ricci Soliton on Lorentzian β-Kenmotsu Manifold (M,ϕ,ξ,η,g) the para-contact from η is closed and the Nijenhuis tensor field of structural endomorphism ϕ vanishes identically.

(1) The 1-form η is closed indeed, from the above equation (2.5), we get

$(d\eta)(X,Y)= g(Y,\nabla_X\xi)- g(X,\nabla_Y\xi)$, (3.17)

Taking from above equations (3.17) and (2.7), in above equation, we get

$(d\eta)(X,Y)=0$,

(ii) The Nijenhuis tensor field associated to ϕ.

$N_\phi(X,Y)=0$,

In [3] and [10] the authors proved that on a η-Ricci soliton on Lorentzian β-Kenmotsu manifold (M,ϕ,ξ,η,g) tensor field satisfies

Now, from the above equations (3.3) and (2.9), in above equation, Y=ξ, we get

$S(X,\xi)=(-\lambda+2\beta+\mu)(X)$. (3.18)

which on using from above equations (3.18) and (2.14), we get

$(n-1)\beta^2=(\lambda+2\beta-\mu)$, (3.19)

In virtue of the the above equations (3.19) and (3.18), we get

$\lambda=\mu$. (3.20)

If μ=0, then λ=0,which shows that λ is steady. Thus we can states as follows-

Theorem (3.2): If η-Ricci Soliton on β-Kenmotsu structure on the manifold M. Let (ϕ,ξ,η,g) and (g,ξ,λ,μ) then

(i) If the manifold M and Let (M,) has cyclic tensor
$\nabla_X S(Y,Z)+\nabla_Y S(Z,X)+\nabla_Z S(X,Y)=0$.

(ii) If the manifold (M , g) has cyclic η-recurrent Ricci tensor

$\nabla_X S(Y,Z)+\nabla_Y S(Z,X)+\nabla_Z S(X,Y)$
$=-(\beta-\mu)\beta[g(X,Y)\eta(Z)$
$+g(X,Z)\eta(Y)+ g(Y,Z)\eta(X)$
$-3\eta(X)\eta(Y)\eta(Z)]$,

Proof: Replacing the expansion of S form

$\nabla_X S(Y,Z)=(\beta-\mu)\nabla_X g(Y,Z)$
$+(\beta-\mu)[\eta(Z)\nabla_X\eta(Y)+\eta(Y)(\nabla_X\eta)(Y)]$. (3.21)

Now, taking from the above equations (3.21) and (2.5), we get

$\nabla_X S(Y, Z) = (\beta - \mu)\beta[g(X,Y)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z) + g(X,Z)\eta(Y)]$. (3.22)

By cyclic permutation X, Y and Z, we get

$\nabla_Y S(Z, X) = (\beta - \mu)\beta[g(Y,Z)\eta(X) - 2\eta(X)\eta(Y)\eta(Z) + g(Y,X)\eta(Z)]$, (3.23)

and

$\nabla_Z S(X, Y) = (\beta - \mu)\beta[g(Z,X)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z) + g(Z,Y)\eta(X)]$. (3.24)

Adding from the above equations (3.20), (3.21) and (3.22), we get

$\nabla_X S(Y, Z) + \nabla_Y S(Z,X) + \nabla_Z S(X,Y)$
$= (\beta - \mu)\beta[g(X,Y)\eta(Z) + g(X,Z)\eta(Y)$
$+ g(Y,Z)\eta(X) - 3\eta(X)\eta(Y)\eta(Z)]$, (3.23)

Corollary (3.3): If η-Ricci soliton Lorentzian β-Kenmotsu manifold. (M, ϕ, ξ, η, g) having cyclic Ricci tensor or cyclic η-recurrent Ricci tensor, then is no Ricci solitons with potential vector field ξ.

Proposition (3.4): Let (ϕ, ξ, η, g) be an η-Ricci soliton Lorentzian β-Kenmotsu structure on the manifold M and Let (g, ξ, λ, μ) be an η-Ricci soliton on M. If the Manifold (M, g) is Ricci symmetric ∇S=0, then λ=0, λ is steady.

Proof: If ∇s=0, taking from the above equation (3.3), we get

$(\nabla_X S)(Y, Z) = (\beta - \mu)[\eta(Z)(\nabla_X \eta)(Y)$
$+ \eta(Y)(\nabla_X \eta)(Z)]$, (3.24)

Now, from above equations (3.24) and (2.5), we get

$(\nabla_X S)(Y, Z) = (\beta - \mu)\beta[g(X,Y)\eta(Z)$
$- 2\eta(X)\eta(Y)\eta(Z) + g(X,Z)\eta(Y)]$, (3.25)

which on using from the above equation (3.25), Z=ξ,we get

$(\nabla_X S)(Y, \xi) = (\beta - \mu)\beta\ [\ g\ (X,\ \ Y)\ - \eta(X)\eta(Y)]$, (3.26)

where $(\nabla_X S)(Y, \xi)$=0, β=0, or $\beta = \mu$

If β=0, or μ=0.

Now, from above equation (3.20), we get

λ=0.

which shows that λ is steady. Thus we can states as follows-

Corollary (3.5): If a η-Ricci soliton Lorentzian β-Kenmotsu manifold (M, ϕ, ξ, η, g) is Ricci symmetric or has η-recurrent Ricci tensor, then M is no Ricci solitons with the potential vector field ξ.

In what follows we shall consider η-Ricci soliton requiring for the curvature to satisfy R(ξ, X).S=0 and S.(ξ, X).R=0, respectively.

Theorem (3.6): If (ϕ, ξ, η, g) to η-Ricci soliton on Lorentzian β-Kenmotsu structure on the manifold (M, g, ξ, λ, μ) is an η-Ricci soliton on M and R(ξ, X).S=0, then β=0.

Proof: Letus suppose that R(ξ, X).S=0, Then we have

S(R(ξ, X)Y,Z)+S(Y,R(ξ, X)Z)=0, (3.27)

Replacing the expression of S from above equation (2.7) and from the symmetric of R, we get

$\beta^2[\eta(Y)S(X, Z) - g(X,Y)S(\xi, Z) + \eta(Z)S(X, Y)$
$- g(X, Z)S(Y, \xi)$=0. (3.28)

Now, from above the equations (3.28) and (2.9), we get

$\beta^2[\eta(Y)S(X, Z) + \eta(Z)S(X, Y)$
+(n-1) $\beta^2\{ g(X,Y)\ \eta\ (\ Z\ ) + g(X, Z)\eta(Y)\}]$ =0. (3.29)

which on using from the above equation (3.29), X=Y=ξ,we get

$\beta^2[S(\xi, Z) + \eta(Z)S(\xi, \xi) + (n-1)\beta^2\{g(\xi,\xi)\eta(Z)$
$+ g(\xi, Z)\}]$=0, (3.30)

In view of the above equation (3.30), in above equation, Z=ξ,we get

β=0,

Theorem (3.7): If (ϕ, ξ, η, g) is η-Ricci soliton on Lorentzian β-Kenmotsu structure on the manifold (M, g,ξ, λ, μ) is an η-Ricci soliton on M and R(ξ, X).S=0 , β=0, then λ=0, where is no Ricci soliton with potential vector field ξ. Ricci and η-Ricci soliton on Lorentzian β-Kenmotsu manifold satisfying S(ξ, X).R=0. If λ=0, then β=0, Letus suppose that S(ξ, X).R=0, then we have

S(X,R(Y,Z)W)ξ-S($\xi, R(Y, Z)W$)X
+S(X,Y)R(ξ, Z)W-S(ξ, Y)R(X,Z)W
+S(X,Z)R(Y,ξ)W-S(ξ, Z)R(Y,X)W
+S(X,W)R(Y,Z)ξ-S(ξ, W)R(Y,Z)X=0, (3.31)

Now, taking inner product of above equation (3.31), with ξ, we get

$S(X, R(Y,Z)W) - \eta(X)S(\xi, R(Y, Z)W)$
$+ S(X, Y)\eta(R(\xi, Z)W) - S(\xi, Y)\eta(R(X, Z)W)$
$+ S(X, Z)\eta(R(Y, \xi)W) - S(\xi, Z)\eta(R(Y, X)W)$
$+ S(X, W)\eta(R(Y, Z)\xi) + S(\xi, W)\eta(R(Y, Z)X)$=0. (3.32)

In virtue from the above equations (3.32) and (2.6), we get

$S(X,R(Y,Z)W) - \eta(X)S(\xi, R(Y, Z)W)$
$= \beta^2[S(X, Y)\{g(Y, Z) - \eta(Y)\eta(Z)\}$
$+ S(\xi, Y)\{g(X, W)\eta(Z) - g(Z, W)\eta(X)\}$
$- S(X, Z)\{g(Y, W) - \eta(Y)\eta(W)\}$
$+ S(\xi, Z)\{g(Y, W)\eta(X) - g(X, W)\eta(Y)\}$
$- S(\xi, W)\{g(Y, X)\eta(Z) - g(Z, X)\eta(Y)\}$=0. (3.33)

In view of the above equations (3.33) and (2.14), (2.9), we get

$(n-2)\eta(R(Y, Z)W) - g(X, R(Y, Z)W)$
$= \beta^2[g(X, U)g(Y, Z) - g(X, Z)\eta(Y)\eta(Z)$
$+ g(Y, Z)\eta(X)\eta(U) - 2\eta(X)\eta(Y)\eta(Z)$
$+ g(X, Z)g(Y, W) - g(X, Z)\eta(Y)\eta(W)$
$+ g(Y, W)\eta(X)\eta(Z)]$
$+ (n-1)[g(Y, W)\eta(X)\eta(Z)$
$- g(X, W)\eta(Y)\eta(Z) + g(Y, X)\eta(W)\eta(Z)$
$- g(Z, X)\eta(Y)\eta(W) - g(X, W)\eta(Y)\eta(Z)$
$+ g(Z, W)\eta(X)\eta(Y)]$, (3.34)

which on using from the above equation (3.34), as X=ξ,we get

$(n-3)\eta(R(Y, Z)W) = -\beta^2[g(Y, Z)\eta(U)$
$+2\ g\ (Y,W)\ \eta(Z) - \eta(U)\eta(Y)\eta(Z)$
$- \eta(Y)\eta(Z)\eta(W) - 2\eta(Y)\eta(Z)]$
$+ (n-1)[g(Y, W)\eta(Z)$
$- 2\eta(Y)\eta(Z)\eta(W) + g(Z, W)\eta(Y)]$. (3.35)

Now, from above equations (3.35) and (2.6), Putting Y=W=ξ,we get

$\beta = 0$, Since $[1 - \eta(U)]\eta(Z) \neq 0$.

If λ=0, which shows that λ is steady.Thus we can states as follows-

Example (3.8): we consider 3-dimensional η-Ricci solitons on Lorentzian β-Kenmotsu manifold with the Schouten-van Kanpen connection we consider the 3-dimensional manifold M= $\{(X,Y,Z) \epsilon R^3, U \neq 0\}$, where (X,Y,Z) are the Standard coordinates in R^3. Let (e_1, e_2, e_3) are linearly independent at each point of M. Let g be the Riemannian metric g defined by

$$e_1 = U^2 \frac{d}{dX}, \quad e_2 = U^2 \frac{d}{dY}, \quad e_3 = \frac{d}{dU},$$

$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$, $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$.

Let η be the 1-form defined by $\eta(U) = g(U, e_3)$ for any $U \epsilon \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3)=0$, then using linearity of ϕ and g, we have

$$\eta(e_3) = 1, \phi^2 U = -U + \eta(U)e_3,$$

$g(\phi U, \phi W) = g(U,W) - \eta(U)\eta(W)$,

for any U,W $\epsilon \chi(M)$. Thus for $e_3 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on M. $[e_1, e_2]=0$,

$$[e_2, e_3] = -\frac{2}{U} e_2, [e_1, e_3] = -\frac{2}{Z} e_1.$$

The Riemannian connection of the metric tensor g is given by the Koszul's formula which is

$2g(\nabla_X Y, U) = Xg(Y,U) + Yg(U,X) - 2g(X,Y) - g(X,[Y,U]) - g(Y,[X,U]) + g(U,[X,Y])$, (3.36)

Using from above equation (3.36), we get

$2g(\nabla_{e_1} e_3, e_1) = 2g(-\frac{2}{Z} e_1$, $2g(\nabla_{e_1} e_3, e_2)=0$, and $2g(\nabla e_1, e_3, e_3)=0$,

Hence $\nabla_{e_1} e_3 = \frac{2}{U} e_1$,

Similary $\nabla_{e_2} e_3 = -\frac{2}{U} e_2$, $\nabla_{e_3} e_3=0$,

further yields

$\nabla_{e_1} e_2 = 0$, $\nabla_{e_1} e_1 = \frac{2}{U} e_3$, $\nabla_{e_2} e_2 = \frac{2}{U} e_3, \nabla_{e_2} e_1 = 0$, $\nabla_{e_3} e_2=0$, $\nabla_{e_3} e_1=0$, (3.37)

from (3.37), we see that the manifold satisfies

$\nabla_X \xi = \beta[X - \eta(X)\xi]$, for $\xi = e_3$ where β=-$\frac{2}{U}$,

Hence we conclude that M is an η-Ricci solitons on Lorentzian β-Kenmotsu manifold. Also then β=0, known that $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$,

$R(e_1, e_2)e_3=0$, $R(e_2, e_3)e_3 = -\frac{\epsilon}{\beta^2} e_2$, $R(e_1, e_2)e_3 = -\frac{\epsilon}{\beta^2} e_1, R(e_1, e_2)e_2 = -\frac{4}{\beta^2} e_1$,

$R(e_3, e_2)e_2 = -\frac{\epsilon}{\beta^2} e_3$, $R(e_1, e_3)e_2=0$, $R(e_1, e_2)e_1 = \frac{4}{\beta^2} e_2, R(e_2, e_3)e_1 = 0$,

$R(e_1, e_3)e_1 = \frac{6}{\beta^2} e_3$.

The Schouten-Van Kampen connection on M is given by

$\nabla_{e_1} e_3 = \left(-\frac{2}{U} - \beta\right) e_1, \nabla_{e_2} e_3 = \left(-\frac{2}{U} - \beta\right) e_2, \nabla_{e_3} e_3 = -\beta(e_3 - \xi), \nabla_{e_1} e_2=0$,

$\overline{\nabla}_{e_2} e_2 = \frac{2}{\beta}(e_3 - \xi), \nabla_{e_3} e_2=0$, $\nabla_{e_1} e_1 = \frac{2}{U}(e_3 - \xi), \nabla_{e_2} e_1=0$, $\nabla_{e_3} e_1=0$, $\nabla_{e_i} e_j=0$,

From (3.38), we can see that $\nabla_{e_i} e_j=0$, $(1 \leq i, j \leq 3)$ for $\xi = e_3$ and β=-$\frac{2}{U}$.

Hence M is a 3-dimensional η-Ricci solitons on Lorentzian β-Kenmotsu manifold with the Schouten-Van Kampen connection. Also using (3.38),it can be seen that R=0.Thus the manifold M is a flat manifold with respect to the Schouten-Van Kampen connection. Since a flat is a Ricci-flat manifold with respect to the Schouten-Van Kampen connection, the manifold M is both a Projectively flat and a conharmonically flat 3-dimensional η-Ricci solitons on Lorentzian β-Kenmotsu manifold with respect to the Schouten-Van Kampen connection. So, from theorem 1 and theorem 2 is η-Einstein manifold with respect to the Levi-Civita connection.

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