



Common Fixed Point Theorem for Weakly Compatible Mappings in Digital Metric Space

A. S. Saluja¹, Jyoti Jhade²

^{1,2}Department of Mathematics, Institute for Excellence in Higher Education, Bhopal – 462016, Madhya Pradesh, India

ARTICLE INFO	ABSTRACT
Published Online: 29 May 2023	In this paper, a common fixed point theorem for a pair of weakly compatible mappings in the setting of digital metric space is proved with a supportive example. Our result extends and improves the result of Ozgur Ege et al. [2].
Corresponding Author: A. S. Saluja	
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1. INTRODUCTION

In the fixed point theory, there exist many generalities of metric space and one of them is the digital metric space introduced by Ozgur Ege and Ismet Karaca [8]. The concept of digital metric space is related to digital topology in which we study the topological and geometrical digital properties of an image. An Image is used as an object in computer graphic design and many other computer-related business works. In this type of work, a digital image is taken as a set of arranged points called pixels or voxels. In digital topology, we study these points and the adjacency relation between them. Rosenfeld [10] was the first to use digital topology as an apparatus and studied the properties of almost fixed points of a digital image. Later, Boxer [6, 7] gives the topological concept in the digital form. Based on this concept Ozgur Ege and Ismet Karaca [8] established digital metric space in 2015 and proved the “Banach Contraction Principle” and several other fixed-point results in this space. In the whole article, *DMS* illustrates digital metric space.

The study of common fixed points for different types of maps has always been a very interesting area in the theory of fixed points. Jungck [3] was the first who introduced commutative mappings to complete metric space in 1976 and by using the properties of these mapping he proved some common fixed point results. After that, many authors generalize and extend many results for commutative mapping with different contractive conditions in several ways. In 1982, Sessa [13] define weakly commutative mappings. These mappings are more general than commutative mappings that

every commutative mapping is weakly commutative, but the converse may not be true. Again, G. Jungck [4, 5] defines compatible and weakly compatible mappings that are more general in nature and give fixed point results using their variants. Recently, Asha Rani et al. [1] introduced weakly commutative and commutative mappings to digital metric space, and Sunjay Kumar et al. [2, 12], Sumitra Dalal [11], and Rashmi Rani [9] present some results for weakly compatible, compatible and commutative maps in *DMS*. With the motivation in this paper, a common fixed point theorem for a pair of weakly compatible mappings which satisfy a contractive condition in *DMS* is presented. This result generalizes and extends the result of Ozgur Ege et al. [2]. Before we prove our main result, the following definitions are needed.

2. PRELIMINARIES

Let $F \subseteq \mathbb{Z}^n$, $n \in \mathbb{N}$ where \mathbb{Z}^n is a lattice point set in the Euclidean n -dimensional space and (F, Y) represent a digital image, with Y -adjacency relation between the members of F and (F, Φ, Y) represent a *DMS*, where (F, Φ) is a metric space.

Definition 2.1. [7] “Let l, n be two positive integers, where $1 \leq l \leq n$ and g, h are two distinct points,

$$g = (g_1, g_2, \dots, g_n), h = (h_1, h_2, \dots, h_n) \in \mathbb{Z}^n.$$

Then the points g and h are said to be Y_1 -adjacent if there are at most l indices i such that $|g_i - h_i| = 1$ and for all other indices j , $|g_j - h_j| \neq 1, g_j = h_j$.”

Definition 2.2. [7] Let $\kappa \in \mathbb{Z}^n$, then the set –

$$N_Y(\kappa) = \{ \sigma \mid \sigma \text{ is } Y\text{-adjacent to } \kappa \}$$

Represent the Y – neighbourhood of κ for $n \in \{1, 2, 3\}$.

Where $Y \in \{2, 4, 6, 8, 18, 26\}$.

Definition 2.3. [7] Let $\delta, \sigma \in \mathbb{Z}$ where $\delta < \sigma$, then the digital interval is –

$$[\delta, \sigma]_\alpha = \{ \alpha \in \mathbb{Z} \mid \delta \leq \alpha \leq \sigma \}.$$

Definition 2.4. [8] “The digital image $(F, Y) \subseteq \mathbb{Z}^n$ is called Y – connected if and only if for every pair of different points $g, h \in F$, there is a set $\{g_0, g_1, \dots, g_s\}$ of points of digital image (F, Y) , such that $g = g_0, h = g_s$, and g_e and g_{e+1} are Y – neighbours where $e = 0, 1, 2, \dots, s-1$.”

Definition 2.5. [8] Let $K: F \rightarrow K$ is a function and $(F, Y_0) \subseteq \mathbb{Z}^n, (K, Y_1) \subseteq \mathbb{Z}^n$ are two digital images. Then –

- (i). K is (Y_0, Y_1) – continuous if there exists Y_0 – connected subset σ of F , for every $K(\sigma), Y_1$ – connected subset of K .
- (ii). K is (Y_0, Y_1) – continuous if for every Y_0 – adjacent point $\{\sigma_0, \sigma_1\}$ of F , either $K(\sigma_0) = K(\sigma_1)$ or $K(\sigma_0)$ and $K(\sigma_1)$ are Y_1 – adjacent in K .
- (iii). K is said to be (Y_0, Y_1) – isomorphism, if K is (Y_0, Y_1) – continuous bijective and K^{-1} is (Y_0, Y_1) – continuous, also it is denoted by $F \cong K_{(Y_0, Y_1)}$.

Definition 2.6. [8] Let a $(2, Y)$ continuous function $K: [0, \sigma]_{\mathbb{Z}} \rightarrow F$ s.t. $K(0) = \alpha$ and $K(\sigma) = \beta$. Then in the digital image (F, Y) , it is called a digital Y – path from α to β .

Definition 2.7. [10] Let $K: (F, Y) \rightarrow (F, Y)$ be a (Y, Y) – continuous function on a digital image (F, Y) , then we said that the property of fixed point satisfied by the digital image (F, Y) if for every (Y, Y) – continuous function $K: F \rightarrow F$ there exists $\alpha \in F$ such that $K(\alpha) = \alpha$.

Definition 2.8. [8] “Let $\{u_n\}$ is a sequence in digital metric space (F, Φ, Y) , then the sequence $\{u_n\}$ is called–

- (i). Cauchy sequence if and only if there exists $q \in \mathbb{N}$ such that, $\Phi(u_n, u_m) < \epsilon, \forall n, m > q$.
- (ii). Converge to a limit point $\ell \in F$ if for every $\epsilon > 0$, there exists $q \in \mathbb{N}$ such that for all $n > q, \Phi(u_n, \ell) < \epsilon$.”

Theorem 2.9. [8] “A digital metric space (F, Φ, Y) is complete.”

Definition 2.10. [8] Let $K: (F, \Phi, Y) \rightarrow (F, \Phi, Y)$ be a self-map. Then K is called a digital contraction if, for all $u, \sigma \in F$ there exist $\tau \in [0, 1)$ such that,

$$\Phi(K(u), K(\sigma)) \leq \tau \Phi(u, \sigma).$$

Proposition 2.11. [8] “Every digital contraction map $K: (F, \Phi, Y) \rightarrow (F, \Phi, Y)$ is digitally Y – continuous.”

Definition 2.12. [11] Let $J, K: F \rightarrow F$ are two self-mappings on (F, Φ, Y) . Then the point $\sigma \in F$ is said to be a coincidence point of J and K if $J(\sigma) = K(\sigma)$. Furthermore, if $J(\sigma) = K(\sigma) = \eta$ then η is said to be a point of coincidence for mappings J and K .

Definition 2.13. [1] Let $J, K: (F, \Phi, Y) \rightarrow (F, \Phi, Y)$ are two mappings defined on the digital metric space (F, Φ, Y) . Then these mappings are called commutative mappings if $J(K(\sigma)) = K(J(\sigma)), \forall \sigma \in F$.

Definition 2.14. [11] Let $J, K: (F, \Phi, Y) \rightarrow (F, \Phi, Y)$ are two mappings defined on the digital metric space (F, Φ, Y) . If mappings J and K commute at coincidence points, then they are called weakly compatible mappings that is if $J(\sigma) = K(\sigma), \forall \sigma \in F$ then $J(K(\sigma)) = K(J(\sigma)), \forall \sigma \in F$.

Proposition 2.15. Let $J, K: F \rightarrow F$ are two weakly compatible maps on F and if a point η is a unique point of coincidence of mappings J and K i.e., $J(\sigma) = K(\sigma) = \eta$ then η is the unique common fixed point of the mappings J and K .

Proof. Since $J, K: F \rightarrow F$ are two weakly compatible mappings and $J(\sigma) = K(\sigma) = \eta$. Then we have $J(\sigma) = J(K(\sigma)) = K(J(\sigma)) = K(\sigma)$ i.e., $J(\sigma) = K(\sigma)$ be a point of coincidence of J and K . But we have the only point of coincidence of J and K is η . Hence $J(\eta) = K(\eta) = \eta$. Let α be another point of coincidence of J and K i.e., $J(\alpha) = K(\alpha) = \alpha$. Then by uniqueness, we get, $\eta = \alpha$. Therefore, η is the unique common fixed point of J and K . □

Remark 2.16. The mappings which are commutative are evidently weakly compatible, but the converse may not be true.

Example 2.17. Let $F = [1, \infty)$ and Φ is a usual metric on F . Let $J, K: F \rightarrow F$ are two mappings on F defined by $J(u) = 2u - 1$ and $K(u) = u^2, \forall u \in F$. Then we can see that J and K are weakly compatible mappings. Since they commute at the coincidence point 1 that is, for $J(1) = K(1)$, we have, $J(K(1)) = K(J(1))$. But not commutative because $J(K(\sigma)) \neq K(J(\sigma)), \forall \sigma \in F$.

3. MAIN RESULT

Theorem 3.1. Let (F, Φ, Y) represent a complete DMS, where Y is an adjacency and Φ is a usual Euclidean metric on \mathbb{Z}^n . let $J, K: F \rightarrow F$ are two self-mappings such that $J(F) \subseteq K(F)$ satisfying the following,

$$\Phi(Ju, Jq) \leq \xi [\Phi(Ju, Kq) + \Phi(Jq, Ku) + \Phi(Ju, Ku) + \Phi(Jq, Kq)], \forall u, q \in F \text{ \& } 0 < \xi < \frac{1}{4} \quad (3.1)$$

If $K(F) \subseteq F$ is complete and $\{J, K\}$ is a pair of weakly compatible mappings then there exists a unique common fixed point in F for mappings J and K .

Proof. Let $u_0 \in F$ be an arbitrary point. Since $J(F) \subseteq K(F)$, let $u_1 \in F$ be chosen such that $Ju_0 = Ku_1$. Continuing this procedure having chosen $u_n \in F$, we chose $u_{n+1} \in F$ such that,

$$Ju_n = Ku_{n+1}, \quad \forall n = 0, 1, 2, \dots$$

Now, by inequality (3.1), we have –

$$\Phi(Ju_{n+1}, Ju_n) = \Phi(Ku_n, Ku_{n-1})$$

$$\begin{aligned} &\leq \xi[\Phi(Ju_n, Ku_{n-1}) + \Phi(Ju_{n-1}, Ku_n) + \Phi(Ju_n, Ku_n) + \Phi(Ju_{n-1}, \\ &\quad Ku_{n-1})] \\ &\leq 2\xi[\Phi(Ku_{n+1}, Ku_n) + \Phi(Ku_n, Ku_{n-1})] \end{aligned}$$

So, we have

$$\Phi(Ku_{n+1}, Ku_n) \leq \mu\Phi(Ku_n, Ku_{n-1}), \quad \text{where, } \mu = \frac{2\xi}{1-2\xi}$$

Now, for $n > m$ by, the triangle inequality property, we have-

$$\begin{aligned} &\Phi(Ku_n, Ku_m) \leq \Phi(Ku_n, Ku_{n-1}) + \Phi(Ku_{n-1}, Ku_{n-2}) \\ &+ \dots + \Phi(Ku_{m+1}, Ku_m) \\ &\leq [\mu^{n-1} + \mu^{n-2} + \dots + \mu^m] \Phi(Ku_1, Ku_0) \\ &\leq \frac{\mu^m}{1-\mu} \Phi(Ku_1, Ku_0) \\ &\Phi(Ku_n, Ku_m) \leq \frac{\mu^m}{1-\mu} \Phi(Ku_1, Ku_0) \end{aligned}$$

Since $0 < \mu < 1$, $\frac{\mu^m}{1-\mu} \Phi(Ku_1, Ku_0) \rightarrow 0$ as $n, m \rightarrow \infty$. This leads to the conclusion that $\{Ku_n\}$ is a Cauchy sequence. Also, we have $K(F) \subseteq F$ is complete. Then there must be a point σ in $K(F)$ such that $Ku_n \rightarrow \sigma$ as $n \rightarrow \infty$. Subsequently, we can find $\eta \in F$ such that $K(\eta) = \sigma$. Further, from inequality (3.1), we have

$$\begin{aligned} &\Phi(Ku_n, J\eta) = \Phi(Ju_{n-1}, J\eta) \\ &\leq \xi[\Phi(Ju_{n-1}, K\eta) + \Phi(J\eta, Ku_{n-1}) + \Phi(Ju_{n-1}, Ku_{n-1}) + \Phi(J\eta, \\ &\quad K\eta)] \\ &\leq \xi\Phi(Ku_{n-1}, K\eta) \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we obtain,

$$\Phi(K\eta, J\eta) \leq \xi\Phi(K\eta, K\eta)$$

Implies that, $\Phi(K\eta, J\eta) = 0$.

Therefore, we get $K\eta = J\eta$. Hence, $K\eta = J\eta = \sigma$ that is σ is the point of coincidence of mappings J and K . We will now show the uniqueness of the point of coincidence σ . For this, let $\sigma_1 \in F$ is another point of coincidence of mappings J and K such that, $K\eta_1 = J\eta_1 = \sigma_1$. Now,

$$\begin{aligned} &\Phi(K\eta_1, K\eta) = \Phi(J\eta_1, J\eta) \quad (\because J\eta_1 = K\eta_1 \text{ and } J\eta = \\ &\quad K\eta) \\ &\leq \xi[\Phi(J\eta_1, K\eta) + \Phi(J\eta, K\eta_1) + \Phi(J\eta_1, K\eta_1) + \Phi(J\eta, K\eta)] \\ &\leq \xi\Phi(K\eta_1, K\eta) \end{aligned}$$

As $0 < \xi < 1$, we get $\Phi(K\eta_1, K\eta) = 0$ i.e., $K\eta_1 = K\eta$. This implies,

$$K\eta_1 = K\eta = J\eta = J\eta_1 = \sigma = \sigma_1.$$

Hence, by Proposition 2.15, it is clear that mappings J and K have a unique common fixed point. \square

We now give an example to illustrate the above Theorem (3.1).

Example 3.2. Let (F, Φ, Y) be a complete DMS with digital metric $\Phi(u, q) = |u - q|$, where $F = [0, 1]$ and let J and K are weakly compatible mappings on F defined by,

$$J(u) = \frac{u}{1+u} \quad \text{and} \quad K(u) = u, \quad \forall u \in F.$$

Then, it is easy to see that all the requirements and conditions which are given in Theorem (3.1) hold, and a unique common fixed point exists for both mappings at $\sigma = 0$ such that, $J(0) = K(0) = 0$.

4. CONCLUSION

This paper is aimed at introducing the perception of weakly compatible mappings in digital metric space and by using these mappings and their variants, establish a digital common fixed point theorem in this space. Our results broaden and extend the result of Ozgur Ege et al. [2]. This result is an application in fixed point theory. Which can be used to compress digital images and can be beneficial in processing and redefining image storage.

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