[Volume 4 issue 06 June 2016]
PageNo.1481-1492 ISSN : 2320-7167

INTERNATIONAL JOURNAL OF MATHEMATICS AND COMPUTER RESEARCH

Haar Wavelet Method For The Solution Of Elliptic Partial Differential Equations

Sumana R. Shesha[1], Tejuswini M.[2] and Achala L. Nargund[3]
[1,2,3]P. G. Department of Mathematics and Research Centre in Applied Mathematics,
M. E. S. College of Arts, Commerce and Science,15th cross, Malleswaram, Bangalore - 560003.
Email ID: [1]sumana.shesha@gmail.com, [2]9tejuswini3@gmail.com, [3]anargund1960@gmail.com

Abstract: Elliptic partial differential equations arise in the mathematical modelling of many physical phenomena arising in science and engineering. In this paper, we use Haar wavelet method for the numerical solution of Laplace and Poisson equation. The basic idea of Haar wavelet collocation method is to convert the partial differential equation into a system of algebraic equations that involves a finite number of variables. The numerical results are compared with the exact solution to prove the accuracy of the Haar wavelet method.

Keywords: Elliptic partial differential equations, Laplace equation, Poisson equation, Haar wavelets, collocation points.

Mathematics Subject Classification: 65T60

1 Introduction

Partial differential equations (PDEs) are used to model many physical phenomena arising in science and engineering. Elliptic PDEs have applications in the fields of electromagnetism, astronomy, fluid mechanics, electrostatics, mechanical engineering and theoretical physics. Laplace equation and Poisson equation are the simplest examples of elliptic PDEs. Laplace equation is also known as the steady state heat equation. Poisson equation is used to describe the potential energy field caused by a given charge or mass density distribution.

Many analytical, semi-analytical and numerical methods have been used to solve elliptic PDEs. Jones et. al. [1] obtained the numerical solution of elliptic PDEs using the method of lines. Evans and Okalie [2] applied spectral resolution method to obtain a solution of elliptic PDEs with periodic boundary conditions. Lewis and Rehm [3] determined the solution of a nonseparable elliptic partial differential equation by preconditioned conjugate gradients. Dehghan and Shirzadi [4] used meshless method of radial basis functions to solve stochastic elliptic PDEs. Hashemzadeh et. al. [5] solved linear elliptic PDEs in polygonal domains. Rangogni [6] obtained the solution of the generalized Laplace equation by coupling the boundary element method and the perturbation method. Tatari and Dehghan [7] solved Laplace equation in a disk using the Adomian decomposition method. Sohail and Din [8] used differential transform method to solve Laplace equation. Aminataei and Mazarei [9] obtained the solution of Poisson equation using radial basis function networks on the polar coordinate. Bennour and Said [10] solved Poisson equation with Dirichlet boundary conditions.

The Haar wavelet is the principal known wavelet and was proposed in 1909 by Alfred Haar. The Haar wavelet is likewise the least complex conceivable wavelet. Over the recent decades, wavelets by and large have picked up a respectable status because of their applications in different disciplines and in that capacity have numerous examples of overcoming adversity. Prominent effects of their studies are in the fields of signal and image processing, numerical analysis, differential and integral equations, tomography, and so on. A standout amongst the best utilizations of wavelets has been in image processing. The FBI has built up a wavelet based algorithm for fingerprint compression. Wavelets have the capability to designate functions at different levels of resolution, which permits building up a chain of approximate solutions of equations. Compactly supported wavelets are localized in space, wherein solutions can be refined in regions of sharp variations/transients without going for new grid generation, which is the basic methodology in established numerical schemes. Sumana and Achala [11] have given a brief report on Haar wavelets.

Chen and Hsiao [12] recommended to expand into the Haar series the highest order derivatives appearing in the differential equation. This idea has been very prolific and it is being abundantly applied for the solution of PDEs. The wavelet coefficients appearing in the Haar series are calculated either using Collocation method or Galerkin method. Lepik [13, 14, 15, 16] used Haar wavelet method to solve linear Fredholm integral equation, nonlinear Volterra integral equation, stiff differential equations, Duffing equation, diffusion equation, Burgers equation and Sine-Gordon equation. Bujurke et al. [17] have computed eigenvalues and solutions of regular Sturm-Liouville problems using Haar wavelets. More recently, Hariharan et al. [18] have solved Klein-Gordon and Sine-Gordon equations using Haar wavelet methods.

The paper is organized as follows. The Haar wavelet preliminaries and the function approximation are presented in Section 2 and Section 3 respectively. The method of solution of the two-dimensional Laplace and Poisson equations using Haar wavelets are proposed in Section 4. The numerical examples and discussions are presented in Section 5. The conclusions drawn are presented in Section 6.

2 Preliminaries of Haar Wavelets

The Haar wavelet family for $x \in [0,1]$ is defined [19] as follows

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_1, \xi_2) \\ -1 & \text{for } x \in [\xi_2, \xi_3) \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

where

$$\xi_1 = \frac{k}{m},\ \xi_2 = \frac{k+0.5}{m},\ \xi_3 = \frac{k+1}{m}. \tag{2}$$

Here $m = 2^n,\ n = 0,1,\cdots,J$ indicates the level of the wavelet; $k = 0,1,\cdots,m-1$ is the translation parameter. J is the maximum level of resolution. The index i in equation (1) is calculated by the formula $i = m + k + 1$. In the case of minimum values $m = 1,\ k = 0$ we have $i = 2$. The maximum value of i is $i = 2M = 2^{J+1}$. For $i = 1$, $h_1(x)$ is assumed to be the scaling function which is defined as follows.

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0,1) \\ 0 & \text{elsewhere} \end{cases} \tag{3}$$

We require the following integrals in order to solve second order partial differential equations.

$$p_i(x) = \int_0^x h_i(x)dx = \begin{cases} x - \xi_1 & \text{for } x \in [\xi_1, \xi_2) \\ \xi_3 - x & \text{for } x \in [\xi_2, \xi_3) \\ 0 & \text{elsewhere} \end{cases} \tag{4}$$

$$q_i(x) = \int_0^x p_i(x)dx = \begin{cases} \frac{1}{2}(x-\xi_1)^2 & \text{for } x \in [\xi_1, \xi_2) \\ \frac{1}{4m^2} - \frac{1}{2}(\xi_3 - x)^2 & \text{for } x \in [\xi_2, \xi_3) \\ \frac{1}{4m^2} & \text{for } x \in [\xi_3, 1] \\ 0 & \text{elsewhere} \end{cases} \tag{5}$$

3 Function Approximation

According to the two-dimensional multi-resolution analysis, any function $f(x, y)$ which is square integrable on $[0,1)\times[0,1)$ can be expressed in terms of two-dimensional Haar series as follows.

$$f(x,y)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a(i,j)\,h_i(x)\,h_j(y). \tag{6}$$

Here, the expansion of $f(x, y)$ is an infinite series. If $f(x, y)$ is approximated as piecewise constant in each sub-area, then it will be terminated at finite terms, that is,

$$f(x,y)=\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2}a(i,j)\,h_i(x)\,h_j(y), \tag{7}$$

where the wavelet coefficients $a(i, j),\ i=1,2,\cdots,2M_1,\ j=1,2,\cdots,2M_2$ are to be determined.

4 Method of Solution

4.1 Laplace Equation

Consider the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0\le x\le 1, 0\le y\le 1, \tag{8}$$

with boundary conditions

$$\left.\begin{aligned}u(x,0)&=f_1(x)\\ u(x,1)&=f_2(x)\end{aligned}\right\}0\le x\le 1, \tag{9}$$

$$\left.\begin{aligned}u(0,y)&=g_1(y)\\ u(1,y)&=g_2(y)\end{aligned}\right\}0\le y\le 1. \tag{10}$$

Let the Haar wavelet solution be in the form

$$u_{xxyy}(x,y)=\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2}a(i,j)h_i(x)h_j(y) \tag{11}$$

Integrating equation (11) twice w.r.t. y in the limits $[0, y]$ and using equation (9) gives

$$u_{xx}(x,y)=\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2}a(i,j)h_i(x)[q_j(y)-yq_j(1)]+yf_2''(x)+(1-y)f_1''(x) \tag{12}$$

Integrating equation (11) twice w.r.t. x in the limits $[0, x]$ and using equation (10) leads to

$$u_{yy}(x,y)=\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2}a(i,j)[q_i(x)-xq_i(1)]h_j(y)+xg_2''(y)+(1-x)g_1''(y) \tag{13}$$

Integrating equation (12) twice w.r.t. x in the limits $[0, x]$ and using equation (10), we arrive at

$$\begin{aligned}u(x,y)=&\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2}a(i,j)[q_i(x)-xq_i(1)][q_j(y)-yq_j(1)]+xg_2(y)+(1-x)g_1(y)\\&+yf_2(x)+(1-y)f_1(x)-xyf_2(1)-x(1-y)f_1(1)-(1-x)yf_2(0)\\&-(1-x)(1-y)f_1(0)\end{aligned} \tag{14}$$

The wavelet collocation points are defined as

$$x_l=\frac{l-0.5}{2M_1},\quad l=1,2,\cdots,2M_1 \tag{15}$$

$$y_n=\frac{n-0.5}{2M_2},\quad n=1,2,\cdots,2M_2 \tag{16}$$

Substituting equations (12) and (13) in equation (8), and taking $x \to x_l$ and $y \to y_n$ in the resultant equations and equation (14), we get

$$\sum_{i=1}^{2M_1} \sum_{j=1}^{2M_2} a(i, j) A(i, j, l, n) = \phi(x_l, y_n) \tag{17}$$

where

$$A(i, j, l, n) = h_i(x_l)[q_j(y_n) - y_n q_j(1)] + [q_i(x_l) - x_l q_i(1)] h_j(y_n) \tag{18}$$

$$\phi(x_l, y_n) = (y_n - 1) f_1''(x_l) - y_n f_2''(x_l) + (x_l - 1) g_1''(y_n) - x_l g_2''(y_n) \tag{19}$$

$$\begin{aligned} u(x_l, y_n) = & \sum_{i=1}^{2M_1} \sum_{j=1}^{2M_2} a(i, j)[q_i(x_l) - x_l q_i(1)][q_j(y_n) - y_n q_j(1)] + x_l g_2(y_n) \\ & + (1 - x_l) g_1(y_n) + y_n f_2(x_l) + (1 - y_n) f_1(x_l) - x_l y_n f_2(1) \\ & - x_l (1 - y_n) f_1(1) - (1 - x_l) y_n f_2(0) - (1 - x_l)(1 - y_n) f_1(0) \end{aligned} \tag{20}$$

The wavelet coefficients $a(i, j)$, $i = 1, 2, \cdots, 2M_1$, $j = 1, 2, \cdots, 2M_2$ can be calculated from equation (17). These coefficients are then substituted in equation (20) to obtain the Haar wavelet solution at the collocation points $x_l, l = 1, 2, \cdots, 2M_1$, $y_n, n = 1, 2, \cdots, 2M_2$.

4.2 Poisson Equation

Consider the two-dimensional Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x, y), \quad 0 \le x \le 1, 0 \le y \le 1 \tag{21}$$

with boundary conditions

$$\left. \begin{aligned} u(x, 0) &= f_1(x) \\ u(x, 1) &= f_2(x) \end{aligned} \right\} 0 \le x \le 1, \tag{22}$$

$$\left. \begin{aligned} u(0, y) &= g_1(y) \\ u(1, y) &= g_2(y) \end{aligned} \right\} 0 \le y \le 1. \tag{23}$$

Substituting equations (12) and (13) in equation (8), and taking $x \to x_l$ and $y \to y_n$ in the resultant equation, we obtain

$$\sum_{i=1}^{2M_1} \sum_{j=1}^{2M_2} a(i, j) A(i, j, l, n) = \phi(x_l, y_n) \tag{24}$$

where

$$A(i, j, l, n) = h_i(x_l)[q_j(y_n) - y_n q_j(1)] + [q_i(x_l) - x_l q_i(1)] h_j(y_n) \tag{25}$$

$$\phi(x_l, y_n) = (y_n - 1) f_1''(x_l) - y_n f_2''(x_l) + (x_l - 1) g_1''(y_n) - x_l g_2''(y_n) + F(x_l, y_n) \tag{26}$$

5 Numerical Examples and Discussion

In this section, examples are considered to check the efficiency and accuracy of the Haar wavelet collocation method (HWCM). Lagrange bivariate interpolation is used to find the solution at the specified points. The entire computational work has been done with the help of MATLAB software.

Example 1:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le 1, 0 \le y \le 1,$$

$$\left.\begin{matrix} u(x,0) = 0 \\ u(x,1) = \sin(\pi x) \end{matrix}\right\} 0 \le x \le 1, \tag{27}$$

$$\left.\begin{matrix} u(0,y) = 0 \\ u(1,y) = 0 \end{matrix}\right\} 0 \le y \le 1.$$

The exact solution is

$$u(x,y) = \frac{\sin(\pi x)\sinh(\pi y)}{\sinh(\pi)} \tag{28}$$

The HWCM solution of the example with $M_1 = 8$, $M_2 = 8$ in Table 1 and Figure 1. The results are compared with the exact solution. Figure 2 shows the physical behavior of the HWCM solution in contour and 3D. If $u_{ex}(x,y)$ is the exact solution (28), we define the error estimate as

$$\sigma = \frac{1}{2M_1 2M_2}\left\| u(x,y) - u_{ex}(x,y) \right\| \tag{29}$$

We have obtained the following error estimates for $M_1 = 8$, $M_2 = 8$.

1. $\sigma = 2.0486E-06$ in L_2 space.
2. $\sigma = 2.4982E-06$ in L_∞ space.

Example 2:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le 1, 0 \le y \le 1,$$

$$\left.\begin{matrix} u(x,0) = 2\log(x) \\ u(x,1) = \log(x^2 + 2x + 2) \end{matrix}\right\} 0 \le x \le 1, \tag{30}$$

$$\left.\begin{matrix} u(1,y) = \log(y^2 + 1) \\ u(2,y) = \log(y^2 + 4) \end{matrix}\right\} 0 \le y \le 1.$$

The exact solution is

$$u(x,y) = \log((x+1)^2 + y^2) \tag{31}$$

The HWCM solution of the example with $M_1 = 8$, $M_2 = 8$ in Table 2 and Figure 3. The results are compared with the exact solution. Figure 4 shows the physical behavior of the HWCM solution in contour and 3D. We have obtained the following error estimates for $M_1 = 8$, $M_2 = 8$.

1. $\sigma = 1.6334E-07$ in L_2 space.
2. $\sigma = 2.1668E-07$ in L_∞ space.

Example 3:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=(x^2+y^2)e^{xy}, \quad 0\le x\le 1, 0\le y\le 1,$$

$$\left.\begin{aligned} u(x,0)&=1 \\ u(x,1)&=e^x \end{aligned}\right\} 0\le x\le 1, \qquad (32)$$

$$\left.\begin{aligned} u(0,y)&=1 \\ u(1,y)&=e^y \end{aligned}\right\} 0\le y\le 1.$$

The exact solution is

$$u(x,y)=e^{xy} \qquad (33)$$

The HWCM solution of the example with $M_1=8$, $M_2=8$ in Table 3 and Figure 5. The results are compared with the exact solution. Figure 6 shows the physical behavior of the HWCM solution in contour and 3D. We have obtained the following error estimates for $M_1=8$, $M_2=8$.

1. $\sigma=5.3716E-07$ in L_2 space.
2. $\sigma=6.7485E-07$ in L_∞ space.

Example 4:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{x+1}{y+1}+\frac{y+1}{x+1}, \quad 0\le x\le 1, 0\le y\le 1,$$

$$\left.\begin{aligned} u(x,0)&=(x+1)\log(x+1) \\ u(x,1)&=(x+1)\log\left(4(x+1)^2\right) \end{aligned}\right\} 0\le x\le 1, \qquad (34)$$

$$\left.\begin{aligned} u(0,y)&=(y+1)\log(y+1) \\ u(1,y)&=(y+1)\log\left(4(y+1)^2\right) \end{aligned}\right\} 0\le y\le 1.$$

The exact solution is

$$u(x,y)=(x+1)(y+1)\ln\left((x+1)(y+1)\right) \qquad (35)$$

The HWCM solution of the example with $M_1=4$, $M_2=4$ in Table 4 and Figure 7. The results are compared with the exact solution. Figure 8 shows the physical behavior of the HWCM solution in contour and 3D. We have obtained the following error estimates for $M_1=4$, $M_2=4$.

1. $\sigma=4.4423E-17$ in L_2 space.
2. $\sigma=7.2858E-17$ in L_∞ space.

6 Conclusion

In this paper, an efficient numerical scheme based on uniform Haar wavelets is used to solve elliptic partial differential equations, namely, two-dimensional Laplace and Poisson equations. The numerical scheme is tested for four examples. The obtained numerical results are compared with the exact solutions. We observe that the error values are negligibly small which indicate that the HWCM solution is very close to the exact solution. Thus the Haar wavelet method guarantees the necessary accuracy with a small number of grid points and a wide class of PDEs can be solved using this approach. This method takes care of boundary conditions automatically and hence it is the most convenient method for solving boundary value problems. This method can also be used to solve nonlinear PDEs.

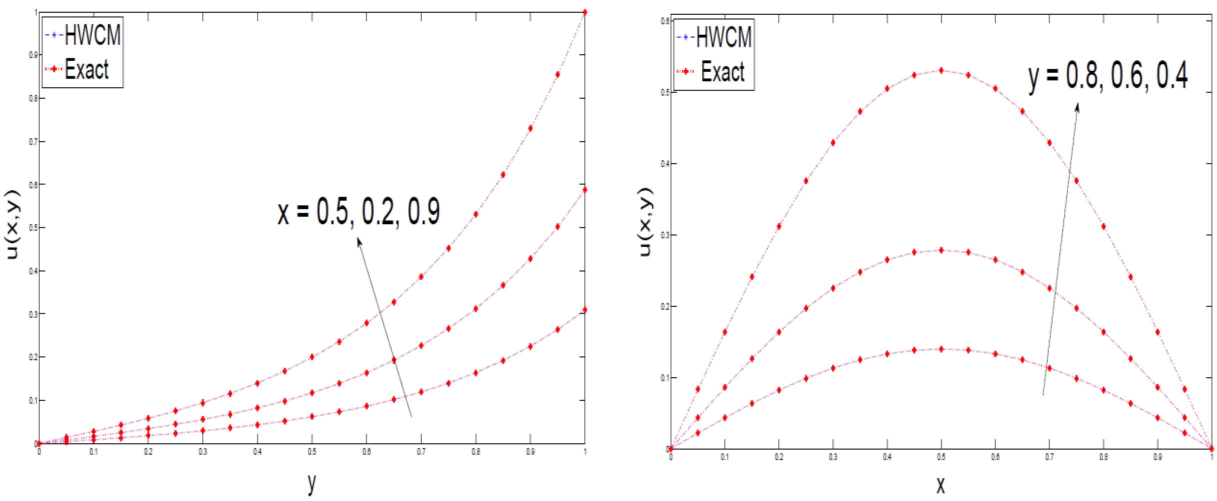

Figure 1: Comparison of the HWCM solution and exact solution of Example 1

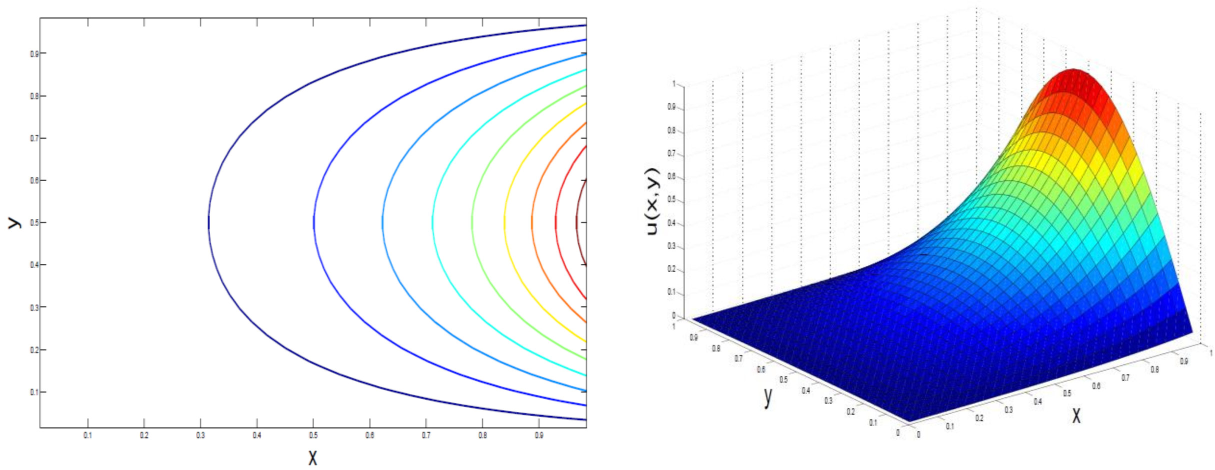

Figure 2: Physical behaviour of the HWCM solution of Example 1

Table 1: Comparison of HWCM solution and exact solution of Example 1

(x, y)	HWCM	Exact	(x, y)	HWCM	Exact
(0.1,0.2)	0.01795263	0.01794057	(0.5,0.6)	0.27855811	0.27856833
(0.1,0.4)	0.04321284	0.04319989	(0.5,0.8)	0.53089801	0.53097922
(0.1,0.6)	0.08607919	0.08608235	(0.7,0.2)	0.04700060	0.04696902
(0.1,0.8)	0.16405651	0.16408160	(0.7,0.4)	0.11313267	0.11309877
(0.3,0.2)	0.04700060	0.04696902	(0.7,0.6)	0.22535825	0.22536652
(0.3,0.4)	0.11313267	0.11309877	(0.7,0.8)	0.42950551	0.42957122
(0.3,0.6)	0.22535825	0.22536652	(0.9,0.2)	0.01795263	0.01794057
(0.3,0.8)	0.42950551	0.42957122	(0.9,0.4)	0.04321284	0.04319989
(0.5,0.2)	0.05809594	0.05805690	(0.9,0.6)	0.08607919	0.08608235
(0.5,0.4)	0.13983968	0.13979777	(0.9,0.8)	0.16405651	0.16408160

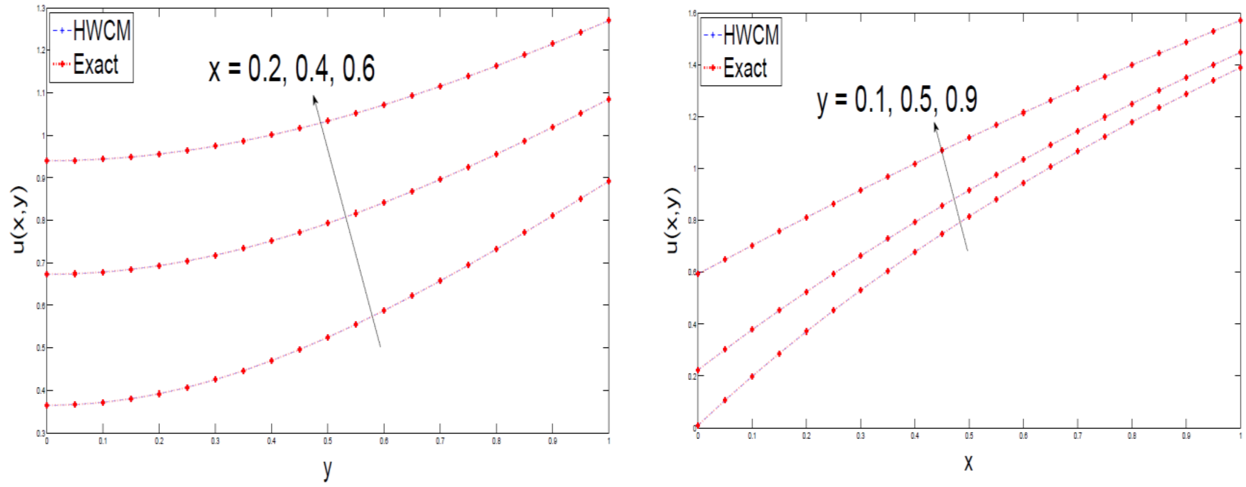

Figure 3: Comparison of the HWCM solution and exact solution of Example 2

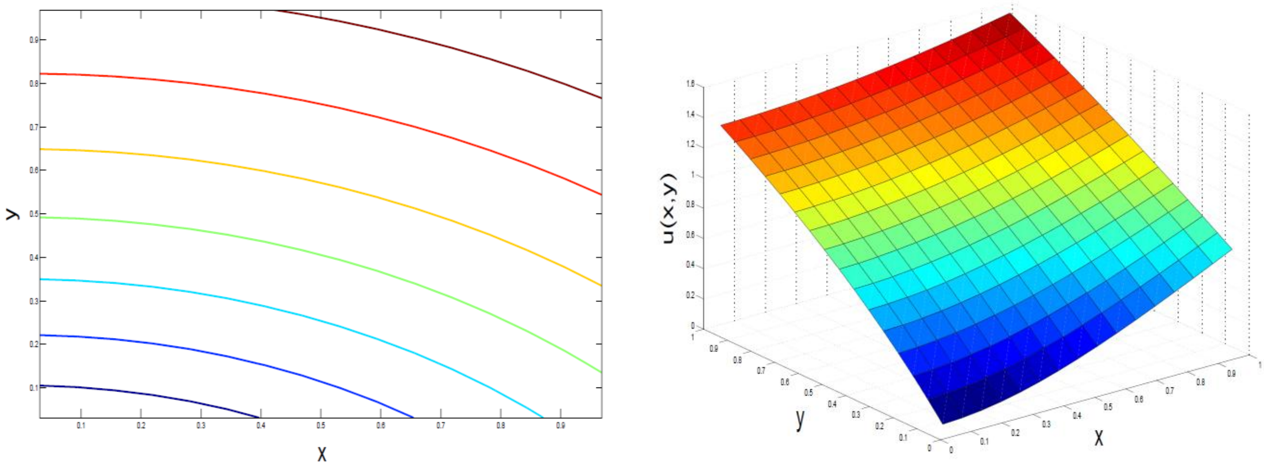

Figure 4: Physical behaviour of the HWCM solution of Example 2

Table 2: Comparison of HWCM solution and exact solution of Example 2

(x, y)	HWCM	Exact	(x, y)	HWCM	Exact
(0.1,0.2)	0.22313984	0.22314355	(0.5,0.6)	0.95935457	0.95935022
(0.1,0.4)	0.31480734	0.31481074	(0.5,0.8)	1.06126071	1.06125650
(0.1,0.6)	0.45107303	0.45107562	(0.7,0.2)	1.07499861	1.07500242
(0.1,0.8)	0.61518424	0.61518564	(0.7,0.4)	1.11514217	1.11514159
(0.3,0.2)	0.54811478	0.54812141	(0.7,0.6)	1.17865928	1.17865500
(0.3,0.4)	0.61518308	0.61518564	(0.7,0.8)	1.26130181	1.26129787
(0.3,0.6)	0.71784078	0.71783979	(0.9,0.2)	1.29472591	1.29472717
(0.3,0.8)	0.84586987	0.84586827	(0.9,0.4)	1.32707538	1.32707500
(0.5,0.2)	0.82854590	0.82855182	(0.9,0.6)	1.37876783	1.37876609
(0.5,0.4)	0.87962639	0.87962675	(0.9,0.8)	1.44692054	1.44691898

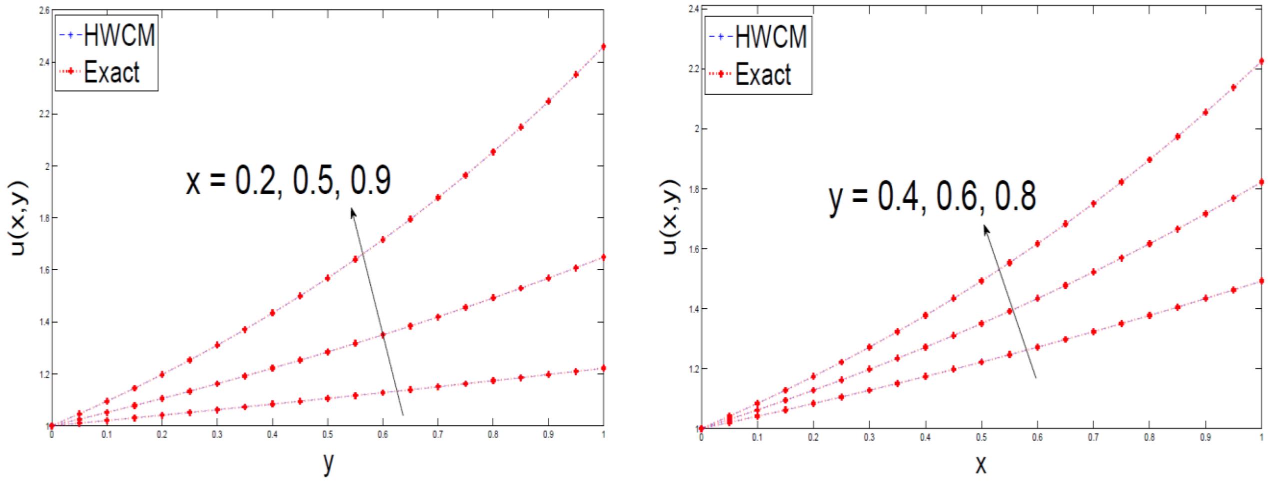

Figure 5: Comparison of the HWCM solution and exact solution of Example 3

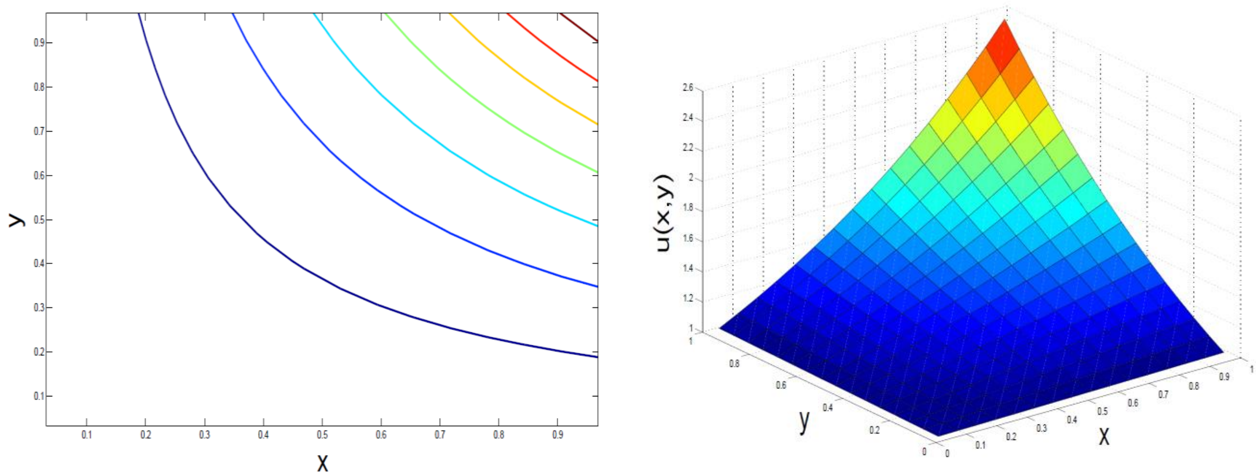

Figure 6: Physical behaviour of the HWCM solution of Example 3

Table 3: Comparison of HWCM solution and exact solution of Example 3

(x, y)	HWCM	Exact	(x, y)	HWCM	Exact
(0.1,0.2)	1.02020387	1.02020134	(0.5,0.6)	1.34987482	1.34985881
(0.1,0.4)	1.04081501	1.04081077	(0.5,0.8)	1.49183648	1.49182470
(0.1,0.6)	1.06184125	1.06183655	(0.7,0.2)	1.15028181	1.15027380
(0.1,0.8)	1.08329048	1.08328707	(0.7,0.4)	1.32314316	1.32312981
(0.3,0.2)	1.06184310	1.06183655	(0.7,0.6)	1.52197664	1.52196156
(0.3,0.4)	1.12750772	1.12749685	(0.7,0.8)	1.75068384	1.75067250
(0.3,0.6)	1.19722941	1.19721736	(0.9,0.2)	1.19722101	1.19721736
(0.3,0.8)	1.27125791	1.27124915	(0.9,0.4)	1.43333556	1.43332941
(0.5,0.2)	1.10517958	1.10517092	(0.9,0.6)	1.71601395	1.71600686
(0.5,0.4)	1.22141712	1.22140276	(0.9,0.8)	2.05443870	2.05443321

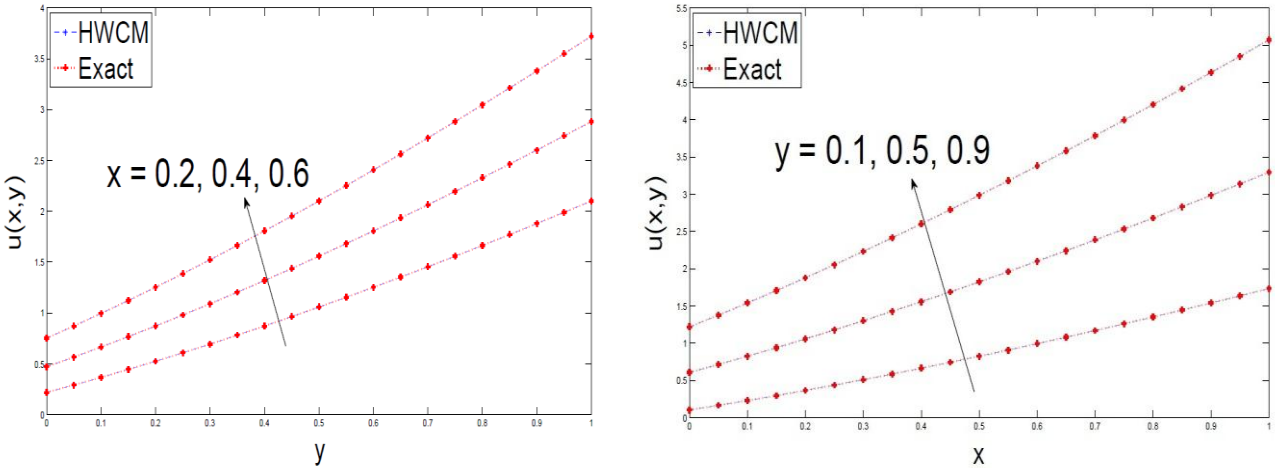

Figure 7: Comparison of the HWCM solution and exact solution of Example 4

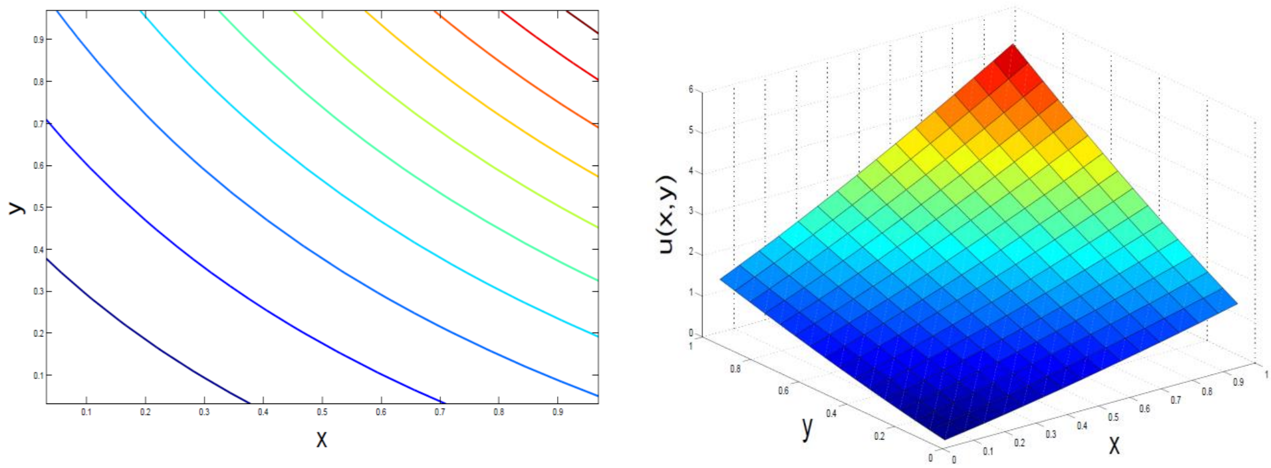

Figure 8: Physical behaviour of the HWCM solution of Example 4

Table 4: Comparison of HWCM solution and exact solution of Example 4

(x, y)	HWCM	Exact	(x, y)	HWCM	Exact
(0.1,0.2)	0.36647395	0.36647389	(0.5,0.6)	2.10112497	2.10112497
(0.1,0.4)	0.66494500	0.66494492	(0.5,0.8)	2.68177978	2.68177979
(0.1,0.6)	0.99495240	0.99495230	(0.7,0.2)	1.45441760	1.45441761
(0.1,0.8)	1.35253185	1.35253175	(0.7,0.4)	2.06369916	2.06369916
(0.3,0.2)	0.69370987	0.69370988	(0.7,0.6)	2.72171872	2.72171871
(0.3,0.4)	1.08988243	1.08988243	(0.7,0.8)	3.42234963	3.42234964
(0.3,0.6)	1.52332522	1.52332522	(0.9,0.2)	1.87912004	1.87912001
(0.3,0.8)	1.98935317	1.98935317	(0.9,0.4)	2.60234754	2.60234749
(0.5,0.2)	1.05801599	1.05801600	(0.9,0.6)	3.38004691	3.38004685
(0.5,0.4)	1.55806842	1.55806842	(0.9,0.8)	4.20537074	4.20537068

Acknowledgements

The authors are obliged to Prof. N. M. Bujurke, INSA Senior Scientist for introducing them to Haar wavelets and providing needful inputs during the preparation of this manuscript. The first author is thankful to the University Grants Commission (UGC) for their financial support through Award No.: MRP(S)-0385/13-14/KABA031/UGC-SWRO, Bangalore. The third author is thankful to Vision Group of

Science and Technology (VGST) for providing financial assistance through GRD 105, CISE.

References

[1] D.J Jones, J.C South Jr., E.B Klunker, *On the numerical solution of elliptic partial differential equations by the method of lines*, J. Comput. Phys. 9(3) (1972) 496-527.

[2] D.J Evans, S.O Okolie, *The numerical solution of an elliptic PDE with periodic boundary conditions in a rectangular region by the spectral resolution method*, J. Comput. Appl. Math. 8(4) (1982) 237-271.

[3] J.G. Lewis, R.G. Rehm, *The numerical solution of a nonseparable elliptic partial differential equation by preconditioned conjugate gradients*, J. Res. Nat. Bur. Stand. 88(5) (1980) 367-390.

[4] M. Dehghan, M. Shirzadi, *Numerical solution of stochastic elliptic partial differential equations using the meshless method of radial basis functions*, Eng. Anal. Bound. Elem. 50 (2015) 21-303.

[5] P. Hashemzadeh, A.S. Fokas, S.A. Smitheman, *A numerical technique for linear elliptic partial differential equations in polygonal domains*, Proc. R. Soc. 471 (2175) (2015).

[6] R. Rangogni, *Numerical solution of the generalized Laplace equation by coupling the boundary element method and the perturbation method*, Appl. Math. Model. 10(4) (1986) 266-270.

[7] M. Tatari, M. Dehghan, *Numerical solution of Laplace equation in a disk using the Adomian decomposition method*, Phys. Scripta 72(5) (2005) 345.

[8] M. Sohail, S.T.M. Din, *Reduced differential transform method for Laplace equations*, Int. J. Mod. Theor. Phys. 1(1) (2012) 6-12.

[9] A. Aminataei, M.M. Mazarei, *Numerical solution of Poisson's equation using radial basis function networks on the polar coordinate*, Comp. Math. Appl. 56(11) (2008) 2887-2895.

[10] H. Bennour, M.S. Said, *Numerical solution of Poisson equation with Dirichlet boundary conditions*, Int. J. Open Problems Compt. Math. 5(4) (2012) 171-195.

[11] R.S. Sumana, L.N. Achala, *A short report on different wavelets and their structures*, Int. J. Res. Engg. Sci. 4(2) (2016) 31-35.

[12] C.F. Chen, C.H. Hsiao, Haar wavelet method for solving lumped and distributed-parameter systems, IEE Proc.-Control Theory Appl. 144(1) (1997) 87-94.

[13] U. Lepik, Numerical solution of differential equations using Haar wavelets, Math. Comput. Simul. 68 (2005) 127-143.

[14] U. Lepik, Applications of the Haar wavelet transform to solving integral and differential equations, Proc. Estonian Acad. Sci. Phys. Math. 56(1) (2007) 28-46.

[15] U. Lepik, Numerical solution of evolution equations by Haar wavelet method, Appl. Math. Comput. 185 (2007) 695-704.

[16] U. Lepik, Haar wavelet method for solving stiff differential equations, Math. Model. Anal. 14 (2007) 467-481.

[17] N.M. Bujurke, C.S. Salimath, S.C. Shiralashetti, Computation of eigenvalues and solutions of regular

Sturm-Liouville problems using Haar wavelets, J. Comput. Appl. Math. 219 (2008) 90-101.

[18] G. Hariharan, Haar wavelet method for solving the Klein-Gordon and the Sine-Gordon equations, Int. J. Nonlin. Sci. 11(2) (2011) 180-189.