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On Metacompactness in Topological Spaces

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ABSTRACT
In this work, we intend to study metacompact spaces and locally metacompact spaces,and study their properties and their relations with order topological spaces.Several examples are discussed and many wellknown theorems are generalized concerning metacompact [7, 2, 8, 3, 4, 56].

KEYWORDS: metacompact, countably metacompact, metalindel¨of locally metacompact.

I. INTRODUCTION

The notion of a metacompact in topological space (X,τ) introduced by [Arthur, 1945].

Further intensive study of such spaces has been done since then see for examples ([Engleking, 1989], [Willard, 1970], [Matveev, 1994]).

In this paper, we study the notion of metacompact, and locally metacompact in topological spaces and derive some related results.

In section one, In this section, we study the concept of metacompactness in topological spaces, and introduce some of their properties, and relate it to other spaces. We study a well-known definitions which will be used in the sequel.

In section two, we study the concept of locally metacompactness in topological spaces, and prove several properties of these spaces.

the terms τ_u, τ_{dis}, τ_{cof} and τ_{coc} will denote the usual, discrete, cofinite and the co-countable topologies, respectively.

II. METACOMPACTNESS IN TOPOLOGICAL SPACES

In this section, we will study the concept of meatcompactness in topological spaces,and studt some of their properties.

Definition 1. [11] A topological spaces (X,τ) is called metacompact, if every open cover of the space (X,τ) has point a finite parallel refinement.

Theorem 1. [9] A countable compact metacompact space is compact.

Example 1. The topological space (R,τ_{cof}) is metacompact space, since (R,τ_{cof}) is compact.

Definition 2. [11] A topological space (X,τ) is called separable, if it has a dense countable subset D. It is clear that the topological spaces (R,τ_{cof}) and (R,τ_{coc}) are separable.

Theorem 2. [10] A separable metacompact space (X,τ) is Lindel¨of.

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of X. Assume that $\tilde{U}$ has no countable subcover of X. Let $\tilde{U} = \{V_\beta : \beta \in \Gamma\}$ be a point finite parallel refinement of $\tilde{U}$. Let D be a countable dense subset of X. Then $V_\beta \cap D$ 6= φ for each $\beta \in \Gamma$. Thus D is an uncountable set because $\tilde{U}$ is uncountable, which is a contradiction. Hence the claim.

Definition 3. [11] A topological space (X,τ) is called countably metacompact, if every countable-open cover of the space (X,τ) has a point finite parallel refinement.

Example 2. The topological space (N,τ_{dis}) is metacompact space. It is also countably metacompact.

Theorem 3. Every Lindel¨of countably metacompact space (X,τ) is metacompact space.

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of X. Since X is Lindel¨of, then $\tilde{U}$ has a countable subcover, say $\tilde{A} = \{A_{\alpha i}\}_{i=1}^{\infty}$. Since X is countably metacompact. Then $\tilde{A}$ has a point finite parallel refinement G of $\tilde{U}$. Hence (X,τ) is metacompact.

Theorem 4. [1] Every metalindel¨of countably metacompact space (X,τ) is metacompact space.

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of X. Since X is metalindelo¨f, then $\tilde{U}$ has a point countable parallel refinement $\tilde{A} = \{A_{\alpha i}\}_{i=1}^{\infty}$, which is also an open cover of (X,τ). Since X is countably metacompact, then $\tilde{A}$ has a point finite parallel refinement G of $\tilde{U}$. Hence (X,τ) is metacompact.

In [9] Engleking defined a cover $\{A_s : s \in S\}$ of a space X to be irreducible, if $\cup_{s \in S_\circ} A_s 6= X$ for every proper subset $S_\circ$ of the set S. The following theorem is easily proved.

Theorem 5. [11] Every point finite cover $\{A_s : s \in S\}$ of a space X has an irreducible subcover.

Example 3. The topological space (R,τ_{dis}) is metacompact, since τ_{dis} is open cover $V = \{\{x\} : x \in R\}$ of R, it is also countably metacompact. It is clear that (R,τ_{dis}) is metaLindel¨of space.

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Theorem 6. [9] Every countably compact metacompact topological space (X,τ) is compact.

Proof. Let $U^{˜} = \{U_\alpha : \alpha \in \Delta\}$ be any open cover of X, where $\{U_\alpha : \alpha \in \Delta\}$ is a set of open members of $U^{˜}$. Now, since f is perfect, then for every $y \in Y$, we have $f^{-1}(y)$ is compact subset of X. So there exist finite subsets of Δ, such that $f^{-1}(y) \subseteq \{\cup_{i=1}^{n} U_i : i \in \Delta\}$. Now, $O_y = Y - f(X -$ is an open subset of Y and $f^{-1}(O_y) \subseteq \{\cup_{i=1}^{n} U_i : i \in \Delta\}.y \in O_y$. So, $O^{˜} = \{O^{˜} = \{O_y : y \in Y\}\}$ is an open cover of Y . Since Y is metacompact, then $O^{˜}$ has a point finite parallel refinement $\tilde{O}^* = \tilde{O} = \{O_y^* : y \in Y\}$. Now, O_y^* is an open subset of X. Since f is perfect, then the set $\{f^{-1}(O_y^*) : y \in Y\}$ is a point finite parallel refinement of X. Then, X is metacompact.

Definition 4. [11] Let (X,τ) and (Y,σ) be topological spaces. Then the cartesian product of (X,τ) and (Y,σ) is the topological space $(X \times Y,\tau \times \sigma)$.

Lemma 1. [9] If A is a compact subset of a topological space (X,τ) and B is a compact subset of a topological space (Y,σ) and $A{\times}B \subseteq W$, where W is open subset of $X \times Y$, then there exist open sets U and V in X and Y respectively, such that $A \times B \subseteq U \times V \subseteq W$.

Theorem 7. [9] If X is a compact, then the projection function $f : X{\times}Y \to Y$ is closed, where (X,τ) and (Y,σ) are topological space.

Proof. To show that the projection function $f : X \times Y \to Y$ is closed, we show that the projection function $f : (X \times Y,\tau \times \sigma) \to (Y,\sigma)$ is closed.

Let $y \in Y$ and let U be an open set in $(X{\times}Y,\tau{\times}\sigma)$, such that $f^{-1}(\{y\}) \subseteq U$. Then, by (Wallace Lemma), there exists σ−open set in Y , say V_y such that $f^{-1}(\{y\}) = X{\times}\{y\} \subseteq X{\times}V_y \subseteq U$. Then, $y \in V_y$ and $f^{-1}(\{V_y\}) = X{\times}\{V_y\} \subseteq U$. So $f : (X \times Y,\tau \times \sigma) \to (Y,\sigma)$ is closed function.

Theorem 8. [9] The product of a compact space X and a metacompact space Y is metacompact, where (X,τ) and (Y,σ) are topological spaces.

Proof. Let $f : X \times Y \to Y$ be the projection function, such that (x,y). Then $f : X \times Y \to Y$ is perfect function. Since Y is metacompact, then $X \times Y$ is metacompact.

Theorem 9. [9] Let $f : (X,\tau) \to (Y,\sigma)$ be a continuous, closed and onto function. Then Y is metacompact if X is so.

Proof. Let $A^{˜} = \{U_\alpha : \alpha \in \Delta) \cup \{V_\beta : \beta \in \Gamma\}$ be any open cover of Y , where $\{U_\alpha : \alpha \in \Delta\}$ is σ−open members of $A^{˜}$. Since f is continuous and onto function. Then, the set $U^{˜} = \{f^{-1}(U\alpha) : \alpha \in \Delta\} \cup \{f^{-1}(V_\beta) : \beta \in \Gamma\}$ is an open cover of X. Since X is metacompact space, then there exists a point finite open parallel refinement of $U^{˜}$, say $\tilde{U}^* = \{f^{-1}(U_\alpha^*) : \alpha \in \Delta\} \cup \{f^{-1}(V_\beta^*) : \beta \in \Gamma\}$. Thus, $\tilde{A}^* = \{U_\alpha^* : \alpha \in \Delta\} \cup \{V_\beta^* : \beta \in \Gamma\}$ is a point finite open parallel refinement of $A^{˜}$. Then Y is metacompact.

Lemma 2. [9] Let $f : (X,\tau) \to (Y,\sigma)$ be a continuous and onto function. If $A^{˜} = \{A_\alpha : \alpha \in \Delta\}$ is a point finite family subset of X, then $\{f(A_\alpha) : \alpha\Delta\}$ is a point finite family subset

Definition 5. [9] A space (X,τ) is said to be metacompact, if every open cover of x has an open locally finite refinement.

Definition 6. [11] A subset z of a space (X,τ) is said to be paracompact relative to X, if every open cover of z by members of τ has a locally finite parallel refinement in X by members of τ.

Corollary 1. [11] Every paracompact spaces is metacompact.

Theorem 10. [10] Every closed subspace of a metacompact space (X,τ) is metacompact.

Proof. This result follows directly from the fact that every metacompact space (X,τ) is ortho-compact, and every closed subspace of an ortho-compact space
(X,τ) is ortho-compact.

Theorem 11. [9] Every metacompact subset of a Hausdorff locally indiscrete space (X,τ) is closed.

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ be a bijection and continuous map, if (Y,σ) is Hausdorff and locally indiscrete space and (X,τ) is metacompact, then f is homeomorghism. It is sufficient to show that f is closed. Let A be a closed proper subset of X. Then A is a metacompact subset of X, since $f : (X,\tau) \to (Y,\sigma)$ is continuous, then we have f(A) is metacompact subset of Y . So f(A) is a closed subset of Y . Then $f : (X,\tau) \to (Y,\sigma)$ is closed function. Hence the resul

Corollary 2. [9] Let $f : (X,\tau) \to (Y,\sigma)$ be a bijection continuous map, if (Y,σ) is Hausdorff and locally indiscrete space and (X,τ) is paracompact, then f is homeomorphism.

Corollary 3. [9] Let $f : (X,\tau) \to (Y,\sigma)$ be a bijection continuous map, if (Y,σ) is Hausdorff and (X,τ) is compact then f is homomorphism.

II. LOCALLY METACOMPACT SPACE

In this section, we study the concept of locally metacompactness in topological space, and prove several properties of these spaces.

A. **Definition 7.** [11] If (X,τ) is a topological space, then τ is said to be locally compact, if each point of X has an open neighborhood whose closure is compact. Note: every compact space is locally compact.

Definition 8. [11] If (X,τ) is a topological space, then τ is said to be locally metacompact, if each point of X, has an open neighborhood whose closure is metacompact.

Example 4. The topological space (R,τ_{dis}) is locally metacompact.

Theorem 12. [9] If a topological space (X,τ) is metacompact and A is a subset of X which is closed, then it is metacompact. If moreover A is a proper subset of X, then A is also metacompact.

Proof. Let $\tilde{U}$ be any open cover of the subspace (A,τ^*), where $\tau^* = \{U \cap A : U \in \tau\}$. Then $\tilde{U} \cup \{X - A\}$ is a open cover of the metacompact space (X,τ) which has a point finite open parallel refinement for X and hence $\tilde{U}$ for A.

Corollary 4. [9] Every metacompact space is locally metacompact.

Proof. We show that τ is locally metacompact. Let $x \in X$ and U be any open neighborhood of x. Then CLU is a closed proper subset of a metacompact space X. So CLU is metacompact. Hence the result. Thus (X,τ) is locally metacompact. The following example shows that the converse of the above theorem needs not be true.

Example 5. The topological space $(R,\tau_{r.r})$ is locally metacompact but not metacompact.

Theorem 13. [9] A topological space (X,τ) is regular, if for each point $x \in X$ and open set U containing x, there exists an open set V containing x, such that $x \in V \subset CLV \subset U$.

Theorem 14. [9] Let $f : (X,\tau) \to (Y,\sigma)$ be an onto, continuous and open function. If (X,τ) is locally metacompact, then (Y,σ) is so.

Proof. First we show that σ is locally metacompact. Let $y \in Y$. Then $f^{-1}(y) \in X$, since (X,τ) is locally metacompact, then there is an open set U containing $f^{-1}(y)$, such that CLU is metacompact. Now, let $f : (X,\tau) \to (Y,\sigma)$ is open, then f(U) is an open subset of Y and $y \in f(U)$. Since $f : (X,\tau) \to (Y,\sigma)$ is onto continuous, then f(CLU) is metacompact. Thus $y \in f(U) \subset CLf(U) \subset f(CLU)$ and f(CLU) is metacompact. So (Y,σ) is locally metacompact.

Theorem 15. [9] Let $f : (X,\tau) \to (Y,\sigma)$ be a perfect function. Then (X,τ) is locally metacompact if (Y,σ) is so.

CONCLUSIONS

In this paper,the metacompact spaces and the locally metacompact spaces as well as study their properties along withe their relations with some other topological spaces have been studied and discussed to outline several theoretical results.These results are generalization of several well known theorems concerning with metacompact spaces.

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