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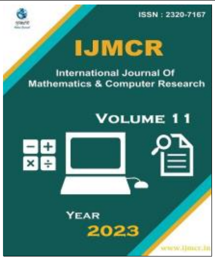

A Wavelet-Based Numerical Algorithm for Some Fractional Models

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ABSTRACT
This article presents an efficient numerical algorithm based on Legendre wavelets operational matrix for solving some fractional models. Fractional integration operational matrix of Legendre wavelets is derived and it is employed to reduce fractional differential equations into a system of algebraic equations. Mixing problems, Newton law of cooling problems and sugar inversion problems are included to elucidate the applicability and the simplicity of the Legendre wavelet-based numerical algorithm.

KEYWORDS: Caputo fractional derivative, Legendre wavelets, Operational matrix. AMS Subject Classification (2010): 26A33, 34A08, 65T60

1. INTRODUCTION

Fractional calculus which originated at the time of development of the classical calculus is the theory of differentiation and integration of arbitrary order. In the last few decades, fractional models have a great attention in various disciplines, such as Medicine, Economics, Dynamical problems, Mathematical physics, Traffic model, Fluid flow, Bio Sciences, Bio Engineering, Electro chemistry, Electromagnetism, Viscoelasticy and so on.

Moreover, most fractional models have no exact solutions. Owing this fact, many researchers have engaged in developing the numerical techniques for fractional models. Among these numerical techniques are variational iteration method, Adomian decomposition method, Finite element method, Finite difference method, Homotopy analysis method, Homotopy perturbation method, Spectral tau method and Spline collocation method. Some polynomials namely, Shifted Chebyshev polynomials[1], Laguerre polynomials[4], Bernstein Polynomials[5], fractional-order Lagrange polynomials[16], Bernoulli polynomials[17] and Chelyshkov polynomials [18] are also employed to solve fractional models numerically.

Recently, orthogonal wavelets have become more popular numerical techniques for solving differential and integral equations due to their excellent properties. Many researchers have employed the operational matrices of fractional integrations of Chebyshev wavelets[3], Haar wavelets[9], Müntz-Legendre wavelets[13], Bernoulli wavelets[12], Legendre wavelets[14,15,20], Taylor wavelets[19], Second kind Chebyshev wavelets[21] and Euler wavelets[22] to find the approximate solutions of arbitrary order differential and integral equations.

The primary goal of this study is to develop numerical solutions of mixing problems, cooling problems and problems involving sugar inversion using the Legendre wavelet operational matrix.

This article is classified as follows. Section 2 provides a quick overview of fundamental concepts definitions and characteristics of fractional calculus. Section 3 provides a brief description of function approximation using Legendre wavelets and the operational matrix for fractional integration of Legendre wavelets. Section 4 explains the simplicity and applicability of the suggested technique with reference to a few fractional models. In section 5, the conclusion is reached.

2. PRELIMINARIES

Here, we cover fractional order integral and differential operators along with some fundamental fractional calculus concepts.

Definitions 2.1. [2,8,10,11] *The Riemann derivative of fractional order $\mu \geq 0$, of the function $h(t) \in L^1([0,\infty))$, is given by*

$$J^{\mu}h(t) = \begin{cases} \dfrac{1}{\Gamma(\mu)}\displaystyle\int_0^t (t-v)^{\mu-1}h(v)\,dv, & \mu > 0. \\ h(t), & \mu = 0. \end{cases}$$

Let $h(t), g(t) \in L^1([0,\infty)),\ \gamma, \lambda, v > -1, \mu \geq 0 \in \mathbb{R}$.

Then, the following properties are attained.

i. $J^{\mu}(\gamma h(t) + \lambda g(t)) = \gamma J^{\mu}h(t) + \lambda J^{\mu}g(t)$.

ii. $J^{\mu}t^{v} = \frac{\Gamma(v+1)}{\Gamma(\mu+v+1)}t^{\mu+v}$.

Definitions 2.2. [2,8,10,11] *The caputo derivative of fractional order $\mu \geq 0$, of the function $h(t) \in L^1([0,\infty))$, is given by*

$$D^{\mu}h(t) = J^{m-\mu}(D^m h(t)) = \begin{cases} \dfrac{1}{\Gamma(m-\mu)}\displaystyle\int_0^t \frac{h^{(m)}(v)}{(t-v)^{\mu-m+1}}dv, & m-1 < \mu < m,\ m \in \mathbb{N}. \\ \dfrac{d^m}{dt^m}h(t), & \mu = m \in \mathbb{N}. \end{cases}$$

Let $g(t), h(t) \in L^1([0,\infty)),\ \gamma, \lambda \in \mathbb{R}, v >,\ \mu \geq 0$.

Then, the following properties are attained.

i. $D^{\mu}k = 0,$ *where k is constant.*

ii. $D^{\mu}(\gamma h(t) + \lambda g(t)) = \gamma(D^{\mu}h(t)) + \lambda(D^{\mu}g(t))$.

iii. $D^{\mu}\left((J^{\mu}h(t)) = h(t)\right)$.

iv. $D^{\mu}\left((J^{\mu}h(t)) = h(t)\right) - \sum_{\ell=0}^{m-1} h^{(\ell)}(0^+)\frac{t^{\ell}}{\ell!}, \quad \mu \in (m-1, m],$

where t > 0, $m \in \mathbb{N}$ and $h^{(\ell)}(0^+) := \lim_{t\to 0^+} D^{(\ell)}h(t)$, $\ell = 0$ *to* $m-1$.

3. LEGENDRE WAVELETS APPROXIMATIONS OF FUNCTIONS

The Legendre wavelet and its fractional integral operational matrix are now covered.

3.1 wavelets

A family of functions made up of dilations and translations of a mother wavelet function $\psi(t)$ over $\mathbb{R}$ is known as wavelets. The following family of continuous wavelets are obtained by continually varying the translation parameter c and dilation parameter b.

$$\psi_{bc}(t) = |b|^{-\frac{1}{2}}\psi\left(\frac{t-c}{b}\right), b \neq 0, c \in \mathbb{R}.$$

Taking $b = b_0^{-u}$, $c = vc_0b_0^{-u}$, $b_0 > 1$, $c_0 > 0$, $u, v \in \mathbb{N}$, the following family of discrete wavelets are arrived.

$$\psi_{uv}(t) = |b_0|^{-\frac{u}{2}}\psi(b_0^u t - vc_0),$$

Where $\{\psi_{uv}(t)\}$ forms a basis for $L^2(\mathbb{R})$. Particularly, when $b_0 = 2$ and $c_0 = 1$, $\{\psi_{uv}(t)\}$ forms an orthonormal basis for $L^2(\mathbb{R})$.

The Legendre wavelets over the interval $[0,T)$, $T \in \mathbb{R}^+$ are defined as

$$\psi_{pq}(t) = \begin{cases} 2^{\frac{k}{2}}\left(q+\frac{1}{2}\right)^{\frac{1}{2}} 2^{\frac{k}{2}}L_q(2^k t - \hat{n}), & t \in \left[\frac{(\hat{n}-1)T}{2^k}, \frac{(\hat{n}+1)T}{2^k}\right). \\ 0, & otherwise, \end{cases} \quad (1)$$

where $p = 1$ *to* 2^{k-1}, $q = 0$ *to* $M-1$, $M, k \in \mathbb{N}$ and $\hat{n} = 2p-1$. The coefficient $\left(q+\frac{1}{2}\right)^{\frac{1}{2}}$ in (1) is utilized for the condition of orthonormality. Additionally, $L_q(t)$ are mutually perpendicular Legendre polynomials of order q with respect to the weight function $w(t) = 1$ over $[-1,1]$ and fulfil the following recurrence formulae.

For $q = 0,1,2,3,\ldots, L_{q+1}(t) = \left(\frac{2q+1}{q+1}\right)tL_q(t) - \left(\frac{q}{q+1}\right)L_{q-1}(t)$, where $L_0(t) = 1, L_1(t) = t$.

3.2 Approximation of square integrable functions

Legendre wavelets can be applied to represent every function $h(t)$ from $L^2([0,T))$ as follows

$$h(t) = \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} c_{pq}\,\psi_{pq}(t), \quad (2)$$

where the following inner product resolves the coefficients c_{pq}.

$$\langle h(t), \psi_{pq}(t)\rangle = \int_0^T h(t)\psi_{pq}(t)dt.$$

If series in (2) is truncated, $h(t)$ can be roughly calculated as

$$h(t) \approx \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} c_{pq}\, \psi_{pq}(t). \tag{3}$$

To make things easier,(3) be expressed as

$$h(t) \approx \sum_{i=1}^{N} c_i\, \psi_i(t) = C\chi(t), \tag{4}$$

where$N = 2^{k-1}M$, $\psi_i(t) = \psi_{pq}(t)$, $\chi(t) = [\psi_1(t), \psi_2(t), \dots, \psi_N(t)]'$, $c_i = c_{pq}$, $C = [c_1, c_2, .., c_N]_{1\times N}$, is the coefficient vector and the index i is determined by the relation $i = (p-1)M + q + 1$. By discretising (4)at the collocation points $t_i = \frac{2i-1}{2N}$, $i = 1\ to\ N$, we attain

$$H \approx C\Phi_{N\times N}, \tag{5}$$

where $H = [h(t_1), h(t_2), \dots, h(t_N)]_{1\times N}$ and $\Phi_{N\times N} = [\chi(t_1), \chi(t_2), \dots, \chi(t_N)]$ is a Legendre wavelet coefficient matrix of order N. Especifically, for $k = 2$, $M = 3$ and $T = 1$ the Legendre wavelet coefficient matrix becomes

$$\Phi_{6\times 6} = \begin{pmatrix} 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 \\ -1.6330 & 0 & 1.6330 & 0 & 0 & 0 \\ 0.5270 & -1.5811 & 0.5270 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & -1.6330 & 0 & 1.6330 \\ 0 & 0 & 0 & 0.5270 & -1.5811 & 0.5270 \end{pmatrix}$$

3.3 Legendre wavelets operational matrix of fractional integration

According to [6], the 'N' Block pulse functions (BPFs) on $[0, T)$ are given by

$$b_\ell(t) = \begin{cases} 1, & \frac{(\ell-1)T}{N} \le \mathrm{t} < \frac{\ell T}{N}, \\ 0, & otherwise, \end{cases}$$

where $\ell = 1\ to\ N, N \in \mathbb{N}$.
For $t \in [0, T)$,

$$b_\ell(t)b_m(t) = \begin{cases} b_\ell(t), & \ell = m. \\ 0, & \ell \ne m. \end{cases}$$

and

$$\int_0^T b_\ell(t)b_m(t)dt = \begin{cases} \frac{T}{N}, & \ell = m. \\ 0, & \ell \ne m. \end{cases}$$

Every function $h(t) \in \mathrm{L}^2([0,\mathrm{T}))$ can be written in terms of BPFs as

$$h(t) \approx \sum_{\ell=1}^{N} h_\ell\, b_\ell(t) = h' B_N(t),$$

where $B_N(t) = [b_1(t), b_2(t), \dots, b_N(t)]'$, $h = [h_1, h_2, \dots, h_N]'$and $h_\ell = \frac{N}{T}\int_{\frac{(\ell-1)T}{N}}^{\frac{\ell T}{N}} h(t)b_\ell(t)dt$.

The relationship between the Legendre wavelets and the BPFs is given by

$$\chi(t) = \Phi_{N\times N}B_N(t). \tag{6}$$

According to [6] , the integration J^μ with fractional order $\mu \ge 0$ of $B_N(t)$ is approximated

$$J^\mu(B_N(t)) \approx F^\mu_{N\times N}B_N(t), \tag{7}$$

where $F^\mu_{N\times N}$ is the Block pulse operational matrix of $J^\mu(B_N(t))$,

$$F^\mu_{N\times N} = \left(\frac{T}{N}\right)^\mu \frac{1}{\Gamma(\mu+2)} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \dots & \xi_{N-1} \\ 0 & 1 & \xi_1 & \xi_2 & \dots & \xi_{N-2} \\ 0 & 0 & 1 & \xi_1 & \dots & \xi_{N-3} \\ 0 & 0 & 0 & 1 & \dots & \xi_{N-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \xi_1 \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

with $\xi_s = (s-1)^{\mu+1} - 2s^{\mu+1} + (s+1)^{\mu+1}$.

The integration J^{μ} with fractional order $\mu \geq 0$ of $\chi(t)$ is approximated as,

$$J^{\mu}\left(\chi(t)\right) \approx P^{\mu}_{N\times N}\chi(t), \tag{8}$$

where $P^{\mu}_{N\times N}$ is referred to as the Legendre wavelets operational matrix of order N for fractional integration with order $\mu \geq 0$. Using (6) and (7), we arrive

$$J^{\mu}\big(\chi(t)\big) = J^{\mu}\big(\Phi_{N\times N}B_N(t)\big) = \Phi_{N\times N}J^{\mu}\big(B_N(t)\big) \approx \Phi_{N\times N}F^{\mu}_{N\times N}B_N(t). \tag{9}$$

Thus, combining (8) and (9) we obtain

$$P^{\mu}_{N\times N}\chi(t) \approx J^{\mu}\big(\chi(t)\big) \approx \Phi_{N\times N}F^{\mu}_{N\times N}B_N(t) = \Phi_{N\times N}F^{\mu}_{N\times N}(\Phi_{N\times N})^{-1}\chi(t)$$

and so

$$P^{\mu}_{N\times N} \approx \Phi_{N\times N}F^{\mu}_{N\times N}(\Phi_{N\times N})^{-1}. \tag{10}$$

Especially, for $M = 3$, $k = 2$, $T = 1$ and $\mu = 0.5$, the Legendre wavelet operational matrix for fractional integration becomes,

$$P^{0.5}_{6\times 6} = \begin{pmatrix} 0.5282 & 0.1819 & -0.0298 & 0.4438 & -0.0871 & 0.0256 \\ -0.1452 & 0.2243 & 0.1329 & 0.0799 & -0.0449 & 0.0198 \\ -0.0598 & -0.0964 & 0.1688 & -0.0417 & -0.0002 & 0.0029 \\ 0 & 0 & 0 & 0.5282 & 0.1819 & -0.0298 \\ 0 & 0 & 0 & -0.1452 & 0.2243 & 0.1329 \\ 0 & 0 & 0 & -0.0598 & -0.0964 & 0.1688 \end{pmatrix}$$

4. ILLUSTRATIVE EXAMPLES

The applicability and the simplicity of the Legendre wavelet-based numerical algorithm are elucidated by the following fractional models.

Example 4.1. [7] ***Mixing Problem:*** 200 litres of water with 40 kilograms of dissolved salt are contained in a tank. Five litres of brine, each containing 2 kilograms of dissolved salt run into the tank per minute and the mixture which is kept uniform by stirring flows out at the same rate. Find the amount of salt $y(t)$ in the tank at any time t.

Solution:

First step.(Modeling). The fractional order time rate of change $\frac{d^{\mu}y(t)}{dt^{\mu}}$ of the amount of salt $y(t)$ in the tank at any time t equals the difference between salt inflow and salt outflow. The salt inflow rate is 10 kilograms per minute. Since the tank always contains 200 litres of brine, the salt out flow rate is $0.025y(t)$.

Thus,

$$\frac{d^{\mu}y(t)}{dt^{\mu}} = 10 - 0.025y(t), \qquad 0 < \mu \leq 1, \quad 0 \leq t < T, \tag{11}$$

with the initial condition $y(0) = 40$. When $\mu = 1$, the exact solution of (11) is $y(t) = 400 - 360e^{-0.025t}$.

Suppose

$$\frac{d^{\mu}y(t)}{dt^{\mu}} \approx C\chi(t). \tag{12}$$

Then

$$y(t) = J^{\mu}\left(\frac{d^{\mu}y(t)}{dt^{\mu}}\right) \approx CP^{\mu}\chi(t) + y(0). \tag{13}$$

Using(12), (13) in (11), we have

$$C\chi(t) + 0.025(CP^{\mu}\chi(t) + 40) - 10 = 0. \tag{14}$$

We can attain the coefficient vector C by solving the equation (14) at $t_i = \frac{(2i-1)T}{2N}$, $i = 1$ to N. Using the coefficient vector C in (13), we get the numerical solutions of (11) for any time t in $[0, T)$.

Table 1: The achieved absolute errors and numerical solutions of Example 4.1 using the proposed strategy for $M = 4, k = 2$ and $T = 1$.

t	$\mu = 0.25$	$\mu = 0.5$	$\mu = 0.75$	$\mu = 0.95$	$\mu = 1$	Exact	Absolute error $\mu = 1$
0.0625	44.6621	42.3778	41.1725	40.6521	40.5616	40.5621	4.3831e-04
0.1875	46.4034	44.3347	42.7662	41.8652	41.6831	41.6836	4.3603e-04
0.3125	47.2756	45.5979	44.0579	43.0272	42.8011	42.8015	4.3376e-04
0.4375	47.9058	46.6142	45.2154	44.1613	43.9156	43.9160	4.3150e-04
0.5625	48.4094	47.4878	46.2869	45.2752	45.0266	45.0271	4.2926e-04
0.6875	48.8333	48.2652	47.2958	46.3731	46.1342	46.1346	4.2702e-04
0.8125	49.2017	48.9719	48.2560	47.4574	47.2383	47.2387	4.2479e-04
0.9375	49.5291	49.6240	49.1766	48.5299	48.3390	48.3394	4.2257e-04

Example 4.2.[7] **Mixing Problem:** 400 litres of brine containing 100 kilograms of dissolved salt are kept in a tank. Two litres of fresh water per minute flow into the tank and the mixture which is essentially kept uniform by stirring also flows out at the same rate. Determine the salt content $y(t)$ in the tank at the time t.

Solution:

First step.(modeling).The fractional order time rate of change $\frac{d^\mu y(t)}{dt^\mu}$ of the amount of salt $y(t)$ in the tank at any time t equals the difference between salt inflow and salt outflow. The salt inflow rate is 2 litres per minute. Since the tank always contains 400litres of brine, the salt out flow rate is $0.005y(t)$

Thus,

$$\frac{d^\mu y(t)}{dt^\mu} = 0 - 0.005y(t), \quad 0 < \mu \leq 1, \quad 0 \leq t < T, \tag{15}$$

with the initial condition $y(0) = 100$. When $\mu = 1$, the exact solution of (15) is $y(t) = 100e^{-\frac{1}{200}t}$.

Second step.(Numerical solution)

Suppose

$$\frac{d^\mu y(t)}{dt^\mu} \approx C\chi(t). \tag{16}$$

Then

$$y(t) = J^\mu\left(\frac{d^\mu y(t)}{dt^\mu}\right) \approx CP^\mu\chi(t) + y(0). \tag{17}$$

Using(16), (17) in (15), we have

$$C\chi(t) + 0.005\big(CP^\mu\chi(t) + y(0)\big) = 0. \tag{18}$$

We can attain the coefficient vector C by solving the equation (18) at $t_i = \frac{(2i-1)T}{2N}$, $i = 1\ to\ N$. Using the coefficient vector C in (17), we get the numerical solutions of (15) for any time t in $[0, T)$.

Table 2: The achieved absolute errors and numerical solutions of Example 4.2 using the proposed strategy for $M = 3, k = 2$ and $T = 1$.

t	$\mu = 0.25$	$\mu = 0.5$	$\mu = 0.75$	$\mu = 0.95$	$\mu = 1$	Exact	Absolute error $\mu = 1$
0.0833	99.7188	99.8467	99.9190	99.9523	99.9584	99.9584	8.6745e-06
0.2500	99.6127	99.7199	99.8086	99.8635	99.8751	99.8751	8.6625e-06
0.4167	99.5592	99.6375	99.7187	99.7782	99.7919	99.7919	8.6505e-06
0.5833	99.5204	99.5709	99.6379	99.6948	99.7088	99.7088	8.6384e-06
0.7500	99.4893	99.5135	99.5629	99.6126	99.6257	99.6257	8.6264e-06
0.9167	99.4631	99.4623	99.4921	99.5314	99.5427	99.5427	8.6145e-06

Example 4.3. [7] **Newton’s law of cooling:** Assume that you turn off the heat in your home at night 2 hours before you go to bed; this time is designated as $t = 0$.what temperature can you anticipate in the morning say 8 hours later $(t = 0)$ if the temperature at time $t = 0$ is $66^\circ F$ and at the time $t = 2$ has decreased to $63^\circ F$? Naturally the outside temperature H_A, which we assume to remain constant at $32^\circ F$ will affect this cooling process.

Physical details:

According to experiments the fractional order time rate of change $\frac{d^\mu H}{dt^\mu}$ of the temperature H of a body is directly related to the difference between the temperature H and the surrounding medium temperature H_A.

Thus,

$$\frac{d^\mu H}{dt^\mu} = \rho(H - H_A) = k(H - 32), \quad 0 < \mu \leq 1, \quad 0 \leq t < T, \tag{19}$$

Where ρ is the constant of proportionality, which is assumed to be -0.046187 when $0 < \mu \leq 1$, with the initial condition $H(0) = 66$. When $\mu = 1$, the exact solution of (19) is $H(t) = 32 + 34e^{-0.046187t}$.

Suppose

$$\frac{d^\mu H}{dt^\mu} \approx C\chi(t). \tag{20}$$

Then

$$H(t) = J^\mu\left(\frac{d^\mu H}{dt^\mu}\right) \approx CP^\mu\chi(t) + H(0). \tag{21}$$

Using(20), (21) in (19), we have

$$C\chi(t) - \rho(CP^{\mu}\chi(t) + 66 - 32) = 0. \quad (22)$$

We can attain the coefficient vector C by solving the equation (22) at $t_i = \frac{(2i-1)T}{2N}$, $i = 1 \; to \; N$. Using the coefficient vector C in (21), we get the numerical solutions of (19) for any time t in $[0, T)$. From (21), the expected temperature at $t = 10$ is $H(10) = 53.4383$ for $M = 3, k = 2$ and $\mu = 1$.

Table 3: The achieved absolute errors and numerical solutions of Example 4.3 using the proposed strategy for $M = 3, k = 2$ and $T = 1$

t	$\mu = 0.25$	$\mu = 0.5$	$\mu = 0.75$	$\mu = 0.95$	$\mu = 1$	Exact	Absolute error $\mu = 1$
0.0833	65.1369	65.5245	65.7472	65.8508	65.8696	65.8694	2.5023e-04
0.2500	64.8186	65.1361	65.4051	65.5741	65.6099	65.6097	2.4704e-04
0.4167	64.6606	64.8877	65.1297	65.3103	65.3522	65.3519	2.4387e-04
0.5833	64.5469	64.6890	64.8843	65.0541	65.0964	65.0962	2.4075e-04
0.7500	64.4563	64.5190	64.6584	64.8037	64.8426	64.8424	2.3765e-04
0.9167	64.3802	64.3684	64.4467	64.5580	64.5908	64.5906	2.3459e-04

Example 4.4. According to experiments, the concentration $y(t)$ of unmodified sugar at any given time t is directly related to the rate of inversion of cane sugar in dilute solution. Find $y(t)$ if the concentration is $\frac{1}{100}$ at $t = 0$ and $\frac{1}{300}$ at $t = 4$.

Modeling:

Thus,

$$\frac{d^{\mu}y(t)}{dt^{\mu}} = \rho y(t), \qquad 0 < \mu \leq 1, \quad 0 \leq t < T\,, \quad (23)$$

Where ρ is the constant of proportionality, which is assumed to be -0.02747 when $0 < \mu \leq 1$, with the initial condition $T(0) = \frac{1}{100}$. When$\mu = 1$, the exact solution of (23) is $y(t) = \frac{1}{100}e^{-0.2747t}$.

Suppose

$$\frac{d^{\mu}y(t)}{dt^{\mu}} \approx C\chi(t). \quad (24)$$

Thus

$$y(t) = J^{\mu}\left(\frac{d^{\mu}y(t)}{dt^{\mu}}\right) \approx CP^{\mu}\chi(t) + y(0). \quad (25)$$

Using (24), (25) in (23), we have

$$C\chi(t) - \rho\left(CP^{\mu}\chi(t) + \frac{1}{100}\right) = 0. \quad (26)$$

We can attain the coefficient vector C by solving the equation (26) at $t_i = \frac{(2i-1)T}{2N}$, $i = 1 \; to \; N$. Using the coefficient vector C in (25), we get the numerical solutions of (23) for any time t in $[0, T)$.

Table 4: The achieved absolute errors and numerical solutions of Example 4.4 using the proposed strategy for $M = 3, k = 2$ and $T = 1$.

t	$\mu = 0.25$	$\mu = 0.5$	$\mu = 0.75$	$\mu = 0.95$	$\mu = 1$	Exact	Absolute error $\mu = 1$
0.0833	0.0087	0.0092	0.0096	0.0097	0.0098	0.0098	2.0024e-03
0.2500	0.0082	0.0086	0.0090	0.0093	0.0093	0.0093	1.5648e-03
0.4167	0.0080	0.0083	0.0086	0.0089	0.0089	0.0089	1.1468e-03
0.5833	0.0079	0.0080	0.0082	0.0085	0.0085	0.0085	7.4751e-04
0.7500	0.0078	0.0078	0.0079	0.0081	0.0081	0.0081	3.6610e-04
0.9167	0.0077	0.0076	0.0076	0.0071	0.0078	0.0078	1.7667e-06

5. CONCLUSION

In this article, an efficient numerical algorithm based on Legendre wavelet operational matrix was derived and successfully employed to solve some fractional models. The Legendre wavelet operational matrix is structurally a sparse matrix, which reduces the computational complexity in solving the system of algebraic equations. Moreover, the Legendre wavelet-based numerical algorithm gives solutions for fractional models with high precision of accuracy. Error tables of numerical examples reveal that the Legendre wavelet-based numerical algorithm is efficient for fractional models.

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