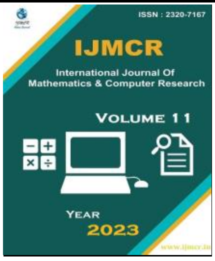
International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 11 Issue 08 August 2023, Page no. 3627-3631

Index Copernicus ICV: 57.55, Impact Factor: 7.362

DOI: 10.47191/ijmcr/v11i8.01



Generalized Beta Functions Involving New Generalized Hypergeometric Functions and Its Applications

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ARTICLE INFO

Published online: 01 August 2023

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ABSTRACT

The main of this research is to provide a systematic review of a new type of generalized beta function beta function involving a new generalized hypergeometric function. We also obtain a certain integral representation and summation formulas. Furthermore, we also study beta distributions and their properties.

KEYWORDS: Beta function, Integral representations.

1. INTRODUCTION

The Gaussian hypergeometric function ${}_2F_1$ is among most important special functions. In the recent years so many different author's [1-7] have been investigated numerous generalizations of the Gaussian hypergeometric function ${}_2F_1$. The Gaussian hypergeometric function can be used to express the probability distribution of the Beta distribution, which is a continuous probability distribution that is often used to model the distribution of proportions.

In 1994, Chaudhary and Zubair [2] introduced the following extension of the gamma function

$$\Gamma_\rho(\lambda_1) = \int_0^\infty t^{\lambda_1 - 1} exp(-t - \rho t^{-1}) dt, \quad (Re(\rho) > 0, Re(\lambda_1) > 0). \tag{1}$$

In 1997, Chaudhary et al. [8] presented the following extension of Euler's beta function

$$B_\rho(\lambda_1, \lambda_2) = \int_0^1 t^{\lambda_1 - 1} (1-t)^{\lambda_2 - 1} exp\left(-\frac{\rho}{t(1-t)}\right) dt, \tag{2}$$
$$(\mathrm{Re}(\rho) > 0, Re(\lambda_1) > 0, Re(\lambda_2) > 0),$$

and they demonstrated that this extension has connections to the Macdonald, Error and Whittakers functions. Apparently it seems that $\Gamma_0(\lambda_1) = \Gamma(\lambda)$ *and* $B_0(\lambda_1, \lambda_2)$.

In 2011, Ozergin [9] (see also Ozergin et al. [10] introduced and studied a further potentially useful extension of the gamma and beta functions as follows:

$$\Gamma_\rho^{k_1,k_2}(\lambda_1) = \int_0^\infty t^{\lambda_1 - 1} \, {}_1F_1(k_1, k_2; -t - \rho t^{-1}) \, dt, \tag{3}$$
$$(Re(k_1) > 0, Re(k_2) > 0, Re(\lambda_1) > 0, Re(\lambda_2) > 0 \text{ and } Re(\rho) > 0),$$

and

$$B_\rho^{k_1,k_2}(\lambda_1, \lambda_2) = \int_0^1 t^{\lambda_1 - 1} (1-t)^{\lambda_2 - 1} \, {}_1F_1\left(k_1; k_2; -\frac{\rho}{t(1-t)}\right) dt, \tag{4}$$
$$(Re(k_1) > 0, Re(k_2) > 0, Re(\lambda_1) > 0, Re(\lambda_2) > 0 \text{ and } Re(\rho) > 0),$$

correspondingly, where ${}_1F_1$represents the confluent hypergeometric function given in [11].

We clearly, have

$$\Gamma_\rho^{k_1,k_1}(\lambda_1) = \Gamma_\rho(\lambda_1), \tag{5}$$
$$\Gamma_0^{k_1,k_1}(\lambda_1) = \Gamma(\lambda_1), \tag{6}$$

and

$$B_\rho^{k_1,k_1}(\lambda_1, \lambda_2) = B_\rho(\lambda_1, \lambda_2), \quad B_0^{k_1,k_2}(\lambda_1, \lambda_2) = B(\lambda_1, \lambda_2). \tag{7}$$

2.Extension of Beta Function: In this section, we first define the new generalized beta function and show that it can be expressed in terms of the new generalized hypergeometric functions. We then derive a integral representation and properties of these function.

In the present paper the authors introduced a new extension of classical beta function associated with new generalized hypergeometric function in the form

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)=\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}3R2\big(\lambda,\alpha,\beta;\gamma,\tau;\omega;mt(1-t)\big)dt \qquad (8)$$

$where\ \{Re(\lambda)>0, Re(\alpha)>0, Re(\beta)>0, m\in\mathbb{C}, |m|<M;\ M\in R^+\ and$ less than 2.0335}.

If $m=0\ \omega=1, \beta=\tau, \lambda=a\ and\ a=c$ then equation (8) reduces to classical beta function.

2.1 Integral representation of Extended Beta Function

Theorem 1. If $Re(\lambda_1)>0, Re(\lambda_2)>0,\ Re(\lambda)>0, Re(\alpha)>0, Re(\beta)>0, Re(\lambda)>0, Re(\alpha)>0, Re(\beta)>0, m\in\mathbb{C}, |m|<M;\ M\in R^+\ and$ less than 2.0335, then we have the following relation,then

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)=2\int_0^{\frac{\pi}{2}} cos^{2\lambda_1-1}\theta\, sin^{2\lambda_2-1}\theta\ 3R2(\lambda,\alpha,\beta;\gamma,\tau;\omega;m\,cos^2\theta\, sin^2\theta)d\theta\ , \qquad (9)$$

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)=\int_0^\infty \frac{u^{\lambda_1-1}}{(1+u)^{\lambda_1+\lambda_2}}3R2\left(\lambda,\alpha,\beta;\gamma,\tau;\omega;\frac{m\,u}{(1+u)^2}\right)du \qquad (10)$$

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)=\frac{1}{2}\int_0^\infty \frac{u^{\lambda_1-1}+u^{\lambda_2-1}}{(1+u)^{\lambda_1+\lambda_2}}3R2\left(\lambda,\alpha,\beta;\gamma,\tau\omega;\frac{m\,u}{(1+u)^2}\right)du \qquad (11)$$

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)=$$

$$(l_1-l_2)^{1-\lambda_1-\lambda_2}\int_{l_1}^{l_2}(u-l_2)^{\lambda_1-1}(l_1-u)^{\lambda_2-1}\,3R2\left(\lambda,\alpha,\beta;\gamma,\tau;\omega;\frac{m\,(u-l_2)(l_1-u)}{(l_1-l_2)^2}\right)du \qquad (12)$$

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)=2^{1-\lambda_1-\lambda_2}\int_{-1}^{1}(1+u)^{\lambda_1-1}(1-u)^{\lambda_2-1}\,2\left(\lambda,\alpha,\beta;\gamma,\delta;\omega;\frac{m\,(1-u^2)}{4}\right)du \qquad (13)$$

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)=2^{1-\lambda_1-\lambda_2}\int_{-\infty}^{\infty}e^{(\lambda_1-\lambda_2)x}\,3R2\left(\lambda,\alpha,\beta;\gamma,\tau;\omega;\frac{m}{4cosh^2x}\right)\frac{dx}{(coshx)^{\lambda_1+\lambda_2}} \qquad (14)$$

Proof: Substituting $t=cos^2\theta$ in equation (8) we get equation (9); to obtained (10) choose $t=\frac{u}{u+1}$; Applying symmetry property in (10) then adding new one we obtained equation (11). By putting $t=\frac{u-l_2}{l_1-l_2}$ in equation (8) we achieved equation (12). Setting $l_2=1, l_1=-1$ in (12) gives the result (13). Result (14) can be obtained by setting $u=tanhx$ and using $tanhx=\left(\frac{e^{\lambda_1}-e^{\lambda_2}}{e^{\lambda_1}+e^{\lambda_2}}\right)$ in equation (13).

2.2 Properties of the New Extension of the Classical Beta Function

Theorem 2 If $Re(\lambda_1)>0, Re(\lambda_2)>0, m\in\mathbb{C}, |m|<M;\ M\in R^+\ and$ less than 2.0335, then we have the following relation

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2+1)+B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1+1,\lambda_2)=B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2) \qquad (15)$$

Proof: By LHS of equation (15) and using definition ,we have

$$=\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2}3R2\big(\lambda,\alpha,\beta;\gamma,\tau;\omega;mt(1-t)\big)dt$$
$$+\int_0^1 t^{\lambda_1}(1-t)^{\lambda_2-1}3R2\big(\lambda,\alpha,\beta;\gamma,\tau;\omega;mt(1-t)\big)dt \qquad (16)$$

$$=\int_0^1\left[t^{\lambda_1-1}(1-t)^{\lambda_2}+t^{\lambda_1}(1-t)^{\lambda_2-1}\right]3R2\big(\lambda,\alpha,\beta;\gamma,\tau;\omega;mt(1-t)\big)dt \qquad (17)$$

$$\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}3R2\big(\lambda,\alpha,\beta;\gamma,\tau;\omega;mt(1-t)\big)dt=B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m} \qquad (18)$$

Remark

1. If we set $\omega=1,\lambda=\alpha,\beta=\tau$, (15) reduces to the result given earlier by [1].

2. For $m=0,\omega=1,\lambda=\alpha,\beta=\tau$, (15) reduces to the classical beta function.

Theorem 3 If $Re(\lambda_1)>0, Re(1-\delta)>0, m\in\mathbb{C}, |m|<M;\ M\in R^+$, then we have the following summation relation

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,1-\delta)=\sum_{n=0}^{\infty}\frac{(\delta)_n}{n!}B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1+n,1) \tag{19}$$

Proof. By the help of equation (8) LHS of (19) gives

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,1-\delta)=\int_0^1 t^{\lambda_1-1}(1-t)^{1-\delta-1}\,3R2\big(\lambda,\alpha,\beta;\gamma,\tau;\omega;mt(1-t)\big)dt \tag{20}$$

$$=\int_0^1 t^{\lambda_1-1}(1-t)^{-\delta}\,3R2\big(\lambda,\alpha,\beta;\gamma,\tau;\omega;mt(1-t)\big)dt \tag{21}$$

Using the binomial expansion $(1-t)^{-\delta}=\sum_{n=0}^{\infty}\left(\frac{(\delta)_n}{n!}\right)t^n$ in (21) and then interchanging the order of summation and integration, result (21) reduced in the following form

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,1-\delta)=\sum_{n=0}^{\infty}\left(\frac{(\delta)_n}{n!}\right)t^n\int_0^1 t^{\lambda_1+n-1}\ 3R2\big(\lambda,\alpha,\beta;\gamma,\tau;\omega;mt(1-t)\big)dt \tag{22}$$

$$=\sum_{n=0}^{\infty}\frac{(\delta)_n}{n!}B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1+n,1) \tag{23}$$

Theorem 4 If $Re(\lambda_1)>0, Re(\lambda_2)>0, m\in\mathbb{C}, |m|<M;\ M\in R^+$, then we have the following second summation relation

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)=\sum_{n=0}^{\infty}B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1+n,1+\lambda_2) \tag{24}$$

Proof. The LHS of equation (24) can be written as

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)=\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}3R2\big(\lambda,\alpha,\beta;\gamma,\tau;\omega;mt(1-t)\big)dt \tag{25}$$

Considering the binomial series expansion,
$(1-t)^{\lambda_2-1}=(1-t)^{\lambda_1}\sum_{n=0}^{\infty}t^n$, $|t|<1$, and interchanging the order of summation and integration, (25) reduces to

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)=\int_0^1 t^{\lambda_1+n-1}(1-t)^{\lambda_2+1-1}3R2\big(\lambda,\alpha,\beta;\gamma,\tau;\omega;mt(1-t)\big)dt \tag{26}$$

$$=\sum_{n=0}^{\infty}B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1+n,1+\lambda_2) \tag{27}$$

3. The Beta distribution of generalized beta function $B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)$.

The generalized beta distribution is with parameters $\lambda_1$and λ_2 such that $-\infty<\lambda_1<\infty, -\infty<\lambda_2<\infty$
and $Re(\lambda)>0, Re(\alpha)>0, Re(\beta)>0, Re(\gamma)>0, Re(\tau)>0, m\in\mathbb{C}, |m|<M;\ M\in R^+$
will be applied to a random variable X with a probability density function (pdf) $f(t)$ is given by

$$f(t)=\begin{Bmatrix}\frac{t^{\lambda_1-1}(1-t)^{\lambda_2-1}3R2(\lambda,\alpha,\beta;\gamma,\tau;\omega;mt(1-t))}{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)}, & 0<t>1\\ 0, & otherwise\end{Bmatrix} \tag{28}$$

If r is any real number then

$$E(X^r)=\frac{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1+r,\lambda_2)}{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)} \tag{29}$$

In particular, for $r=1$, the expected value of the random variable X is obtained as

$$\mu=E(X)=\frac{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1+1,\lambda_2)}{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)} \tag{30}$$

and the variance of the random variable X, σ^2 is obtained as

$$\sigma^2=E(X^2)-[E(X)]^2$$

$$=\frac{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1+2,\lambda_2)}{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)}-\left\{\frac{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1+1,\lambda_2)}{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)}\right\}^2$$

$$=\frac{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1+2,\lambda_2)-\left\{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1+1,\lambda_2)\right\}^2}{\left\{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)\right\}^2} \quad (31)$$

The moment generating function is defined by

$$M(t)=\int_0^\infty e^{tx}f(x)dx=\sum_{n=0}^{\infty}\frac{t^nE(X^n)}{n!}$$

$$=\frac{1}{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)}\sum_{n=0}^{\infty}B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1+n,\lambda_2)\frac{t^n}{n!} \quad (32)$$

The commutative distributive function of (28) is given by

$$F(x)=\frac{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m,x}(\lambda_1,\lambda_2)}{B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m}(\lambda_1,\lambda_2)} \quad (33)$$

where

$$B^{MC,\omega}_{\lambda,\alpha,\beta;\gamma,\tau,m,x}(\lambda_1,\lambda_2)=\int_0^x t^{\lambda_1-1}(1-t)^{\lambda_2-1}{}_3R_2\big(\lambda,\alpha,\beta;\gamma,\tau;\omega;mt(1-t)\big)dt, \quad (34)$$

$$\{-\infty<\lambda_1<\infty,-\infty<\lambda_2<\infty,|m|<M;\ M\in R^+\}$$

is the new extended incomplete beta function, for $m=0,\omega=1,\lambda_1,\lambda_2>0$, equation (34) convergence, and $B^{MC,1}_{\lambda,\alpha,\beta;\gamma,\tau,0,x}(\lambda_1,\lambda_2)=B_x(\lambda_1,\lambda_2)$, where $B_x(\lambda_1,\lambda_2)$ is incomplete beta function defined by [5].

4. CONCLUSIONS

We believe that generalized beta functions and new generalized hypergeometric functions are a powerful tool for solving a wide range of problems in mathematics, physics, and engineering. We hope that this paper will stimulate further research on these functions and their applications.We conclude the paper by discussing the future directions of research on generalized beta functions and new generalized hypergeometric functions.

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