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The Connected Neighbourhood Polynomial of Graphs.

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Abstract

Graph polynomial is one of the algebraic representations for graph. In this paper, we introduce a new type of graph polynomial called connected neighbourhood polynomial and we defined the connected neighbourhood roots of a graph. The connected neighbourhood polynomial of some standard graphs is obtained and some properties of this polynomial are established.

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1. Introduction

Throughout this paper we will consider only a simple connected graph, finite and undirected, without loops and multiple edges. In general, we use ⟨X⟩ to denote the subgraph induced by the set of vertices X. $N(v)$ *and* $N[v]$ denote the open and closed neighbourhood of a vertex v, respectively. For terminology and notations not specifically defined here, we refer reader to [3] and [5]. For more details about neighbourhood number and its related parameters, we refer to [4] and [7].

A set of vertices in a graph G is a neighbourhood set if $G = \cup_{v\in s}\langle N[v]\rangle$ where $\langle N[v]\rangle$ is the subgraph of G induced by v and all vertices adjacent to v. The neighbourhood number $\eta(G)$ of G is the minimum cardinality of a neighbourhood set. The neighbourhood set s is called an independent neighbourhood set of G if $\langle s\rangle$ is totally disconnected graph and the independent neighbourhood number $n_i(G)$ is the minimum cardinality of an independentneighbourhood set. A neighbourhood set s of G is called connected neighbourhood set if $\langle s\rangle$ is connected. The connected neighbourhood number $n_i(G)$ is the minimum cardinality of a connected neighbourhood set.

The Independent neighbourhood number is not defined for any graph, for example c_5 has no Independent neighbourhood set. Recently graph polynomials are studied by many authors, domination polynomial is introduced in [1] as:

Let $G = (V, E)$ be any graph with p vertices, Then the domination polynomial of G, $D(G,x) = \sum_{i=\gamma(G)}^{p} d\,(G,i)x^{ji}$, where $\gamma(G)$ is the domination number of G and $d(G,i)$ is the number of dominating sets in G of size i.Similarly the neighbourhood polynomial in graph is introduced in [2].

In this paper we introduce the independent neighbourhood polynomial in graphs. Some graphs classification and the independent neighbourhood polynomial of some standard graphs are obtained. The Independent neighbourhood number is not defined for any graph, for example c_5 has no Independent neighbourhood set.

A graph G is called an IN-graph if G has an Independent neighbourhood set, i.e., n_e exist. The neighbourhood polynomial of a graph G has introduced by Anwar Alwardi et al, in [6]. In this paper we study special cases of the neighbourhood polynomial in graph.

Definition 1.1. *A set of vertices D in a graph G is a neighbourhood set if $G = \bigcup_{v \in D} < N[v] >$, where $N[v]$ is the closed neighbourhood of the vertex v in G. The neighbourhood number $\eta(G)$ is the minimum cardinality of a neighbourhood set.*
Definition 1.2. *Let $G = (V, E)$ be a connected graph. A connected neighbourhood set s is a neighbourhood set of G such that $< s >$ is connected graph .The connected neighbourhood number $\eta_c(G)$ is the minimum cardinality of a connected neighbourhood set of G.*
Example 1.3. Let G be a graph as in Figure 1.

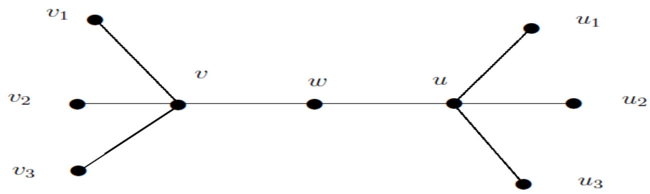

Figure. 1: $\eta_c(G) = 3$

From the figure 1, the set $\{v, u\}$ is the minimum neighbourhood set of G. i.e, $\eta(G) = 2$, the minimum connected neighbourhood set is $\{v, w, u\}$ and $\eta_c(G) = 3$.
As we know that if G is a connected graph then the connected dominating set of G is a dominating set D such that the induced subgraph $< D >$ is connected and the minimum cardinality of a connected dominating set is called the connected domination number and denoted by $\gamma_c(G)$.
Clearly, from the definition of connected dominating set and the connected neighbourhood set, any connected neighbourhood set is connected dominating set that means $\gamma_c(G) \leq \eta_c(G)$.
For example $G \cong C_4$, obviously, $\gamma_c(G) = 2$, but $\gamma_c(G) = 3$.
Example 1.4. *Let G be a graph of order n and let $\eta(G, i)$ denoted to the number of neighbourhood sets with cardinality i . Then the neighbourhood polynomial $N(G, x)$ of G is defined as $N(G, x) = \sum_{i=\eta(G)}^{n} n(G, i) x^i$, where $\eta(G)$ is the neighbourhood number of G.*

Definition 1.5. *For any connected graph $G = (V, E)$ the connected neighbourhood polynomial denoted by $N_c(G, x)$ and defined as : $N_c(G, x) = \sum_{i=\eta_n(G)}^{n} n_c(G, i) x^i$, $n_c(G, i)$ is the number of connected neighbourhood sets of size i in the graph G and $\eta_c(G)$ is the neighbourhood number of G. The roots of the polynomial $N_c(G, x)$ are called connected neighbourhood roots of the graph G and denoted by $(N_c(G, x))$.*

Example 1.6. Let $G = (V, E)$ be a graph as in figure 2.

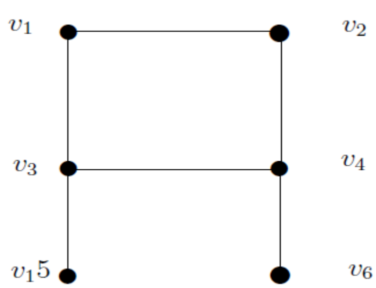

Figure 2: $\gamma_c(G) \neq \eta_c(G)$

From the figure 2, clearly there is only two minimum connected neighbourhood sets $\{v_1, v_3, v_4\}$ and $\{v_2, v_3, v_4\}$, that means $\eta_c(G) = 3$. But there is only one minimum connected dominating set of size two which is $\{v_3, v_4\}$ that implies to $\gamma_c(G) = 2$.
In figure 2, $\gamma_c(G) \neq \eta_c(G)$.
To get the connected neighbourhood polynomials, first we have, $\eta_c(G, 3) = 2$.
Now the connected neighbourhood sets of G of size four are $\{v_1, v_3, v_4, v_2\}$, $\{v_1, v_3, v_4, v_5\}$, $\{v_1, v_3, v_4, v_6\}$, $\{v_2, v_3, v_4, v_5\}$, $\{v_2, v_3, v_4, v_6\}$, $\{v_1, v_2, v_4, v_5\}$, $\{v_1, v_2, v_3, v_6\}$ and $\{v_2, v_3, v_5, v_6\}$ and {1,4,5,6}.
Therefore, $\eta_c(G, 4) = 9$.
Also there are only connected neighbourhood sets of size five that means $\eta_c(G, 5) = 6$.
The connected neighbourhood sets are $\{v_1, v_2, v_3, v_4, v_5\}$,$\{v_1, v_2, v_3, v_4, v_6\}$ $\{v_1, v_3, v_4, v_5, v_6\}$,$\{v_2, v_3, v_4, v_5, v_6\}$,$\{v_1, v_2, v_3, v_5, v_6\}$ and $\{v_1, v_2, v_4, v_5, v_6\}$ and there is only one connected neighbourhood set of size six $\{v_1, v_2, v_3, v_4 v_5, v_6\}$.
Hence the connected neighbourhood polynomial of the figure 6.4 is $N_c(G) = 2x^3 + 9x^4 + 6x^5 + x^6$ and to get the connected neighbourhood roots of G , we will solve the equation $2x^3 + 9x^4 + 6x^5 + x^6 = x^3(x^3 + 6x^2 + 9x + 2) = 0$, that means $x^3 + 6x^2 + 9x + 2 = (x + 2)(x^2 + 4x + 1) = 0$. Hence, the connected neighbourhood roots of the graph figure 6.4 are $0, 0, 0, -2, -2 + \sqrt{3}$ and $-2 - \sqrt{3}$.

Theorem 1.7. *For any star graph $k_{1,n}$ where $n \geq 2$ the connected neighbourhood roots of $k_{1,n}$ are zero and -1 of multiplicity n .*
Proof: Let $G \cong k_{1,n}$, where $n \geq 2$ be a star then clearly, $\eta_c(G) = 1$ and there are n connected neighbourhood sets of size two by taking the center vertex and another vertex from the vertices of the star.
The number of connected neighbourhood sets in G of size three can be selected in $\binom{n}{2}$ ways, that means in general for any connected neighbourhood set of size i , we have to select the center and $i - 1$ vertices from the other n vertices.
That means $d_c(G, i) = \binom{n}{i-1}$ for all $i \geq 2$.
Therefore,
$$\begin{aligned} N_c(G, x) &= x + \binom{n}{1} x^2 + \binom{n}{2} x^3 + \ldots\ldots\ldots + \binom{n}{n} x^{n+1} \\ &= x\,[1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots\ldots\ldots + \binom{n}{n} x^n\,] \\ &= x \sum_{k=0}^{n} \binom{n}{k} x^k \\ &= x(1 + x)^n. \end{aligned}$$
Hence, the connected neighbourhood roots of $k_{1,n}$,$n \geq 2$ is zero and -1 with *multiplicity* n.

Proposition 1.8. *For any complete graph k_n , $N_c(k_n, x) = (x + n)^n - 1$.*
Proof: Obviously any neighbourhood set in k_n is connected and $\eta_c(k_n) = 1$,
then $n_c(k_n, i) = \binom{n}{i}$.
Hence ,
$$\begin{aligned} N_c(k_n, x) &= \sum_{i=1}^{n} \binom{n}{i} x^i \\ &= \left(\sum_{i=1}^{n} \binom{n}{i} x^i \right) - 1 \\ &= (x + 1)^n - 1 . \end{aligned}$$
In other words we can generalize Proposition 1.8 to the following.
Proposition 1.9. *Let $G = (V, E)$ be a connected graph of n vertices. Then $N_c(G, x) = (x + 1)^n - 1$ if and only if $G \cong k_n$.*
Proof: If $G \cong k_n$, then by proposition 1.8, $N_c(G, x) = (x + 1)^n - 1$.
Conversely, if $N_c(G, x) = (1 + x^n) - 1$,

Then $N_c(G,x) = \sum_{i=1}^{n} \binom{n}{i} x^i = \binom{n}{1} x + \binom{n}{2} x^2 + \cdots \ldots + \binom{n}{n} x^n$, that means any subset of the vertices of G is connected neighbourhood set.
Hence, $G \cong k_n$.

Proposition 1.10. *For any cycle graph C_n with $n \geq 4$ vertices the connected neighbourhood roots of C_N are zero with multiplicity $n-1$ and $-n$*
Proof: Obviously, for any cycle C_n , $n \geq 4$ $\eta_c(C_n) = n-1$, then there are n ways to construct connected neighbourhood set of size $n-1$ every time we will miss one vertex in C_n.
Therefore, $d_c(C_n, n-1) = n$ and there is only one connected neighbourhood set of size n.
Therefore , $N_c(G,x) = x^n + nx^{n-1}$.
To get the connected neighbourhood roots of C_n , we will solve the equation
$x^n + nx^{n-1} = 0$
$x^{n-1}(x+n) = 0$.
Hence, the connected neighbourhood roots of C_n, where $n \geq 4$ are zero with multiplicity $n-1$ and $-n$.

Theorem 1.11. *For any tree $G \cong T_n$ with n vertices and t pendant vertices the connected neighbourhood polynomial is $N_c(G,x) = x^{n-t}(1+x)^t$.*
Proof: The connected neighbourhood number $G \cong T_n$ is $n-t$, $\eta_c(T_n) = n-t$, where t is the pendant vertices in T_n.
Now, to construct connected neighbourhood set of size $n+1-t$, we have $\binom{t}{1} = t$ way and to construct connected neighbourhood set of size $n-t+2$ there are $\binom{t}{2}$ ways . In general we have $\binom{t}{j}$ ways to select any connected neighbourhood set of size $n-t+j$.

$$\begin{aligned}
\text{Therefore,} \quad N_c(T_n,x) &= x^{n-t} + \binom{t}{1} x^{n-t+1} + \binom{t}{2} x^{n-t+2} + \cdots \ldots + \binom{t}{t} x^n \\
&= x^{n-t}\left[1 + \binom{t}{1} x + \binom{t}{2} x^2 + \cdots \ldots . + \binom{t}{t} x^t \right] \\
&= x^{n-t} \sum_{i=0}^{t} \binom{t}{i} x^i \\
&= x^{n-t}(1+x)^t.
\end{aligned}$$

Corollary 1.12. *For any path P_n with $n \geq 3$ vertices then*
$N_c(G,x) = x^{n-2} + 2x^{n-1} + x^n$.
Corollary 1.13. *For any tree T_n with n vertices and t pendant vertex, the connected neighbourhood roots are zero with $n-t$ multiplicities and -1 with multiplicity t .*
Corollary 1.14. *For any bi-star $B(m,n)$, $m,n \geq 2$ $N_c(B(m,n),x) = x^2(1+x)^{m+n}$.*
Proof: The number of vertices in $B(m,n)$ is $m+n+2$ and the number of pendant vertices is $m+n$.
Therefore, by applied theorem 1.11, we have, $N_c(B(m,n),x) = x^2(1+x)^{m+n}$.
Corollary 1.15. *Let $G \cong s(B(m,n))$, where $s(B(m,n))$ is the subdivision of the bi-star. Then $N_c(G,x) = x^{m+n+3}(1+x)^{m+n}$.*
Proof: By applying the theorem as the number of vertices in $s(B(m,n))$ is $2m+2n+3$ and the number of pendant vertices is $m+n$.

$$N_c(G,x) = x^{m+n+3}(1+x)^{m+n}.$$

We can generalize the corollary 1.15 to the following proposition.

Proposition 1.16. *For any tree T_n with n vertices the connected neighbourhood roots of the subdivision $s(T_n)$ are zero with $2n-t-1$ multiplicity and -1 with t multiplicity, where t is the number of pendant vertices in T_n.*

Proof: As it is known that the subdivision of T_n is $s(T_n)$ is also tree of same number of pendant vertices t and $m+n-1$ vertices.

Then by the theorem, $N_c(s(T_n),x) = x^{2n-1-t}(1+x)^t$

So if $x^{2n-t-1}\,(1+\mathrm{x})^t = 0.$

We get that the connected neighbourhood roots are zero with $2n-t-1$ multiplicities and -1 with t multiplicity.

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