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A Polynomial Approximation for the Numerical Solution of First Order Volterra Integro-Differential Equations

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ARTICLE INFO	ABSTRACT
Published Online: 05 September 2023 Corresponding Author: **Ganiyu Ajileye**	In this study, we develop and implement a numerical approach for solving first-order Volterra integro-differential equations. We derive the integral form of the problem, which is then transformed into an algebraic equation system using standard collocation points. We established the approach's uniquenessas well as its convergence and numerical examples were used to test the method's efficiency which showsthat the method competes favourably with existing methods.

KEYWORDS: Collocation method; Volterra; Integro-differential equations; Approximate solution.

1. INTRODUCTION

In 1926, Vito Volterra used integro-differential to explore population increase and focus on hereditary effects. Integro-differential equations are a powerful tool in pure and applied mathematics, engineering, and physics. Many mathematical formulations of physical phenomena incorporate integro-differential equations, which arise in a range of domains such as fluid dynamics, heat transfer, diffusion processes, neutron diffusion, biological models, nanohydrodynamics, economics, and population growthmodels.[1].

Some methods for determining the numerical solution of integro-differential equations include: Bernstein Method [14], Adomian decompositions method [2, 3], Finite difference-Simpson method [17], Collocation method by [4, 5, 6, 7, 21, 22], Hybrid linear multistep method [8, 9], Chebyshev-Galerkin method [10], Bernoulli matrix method [11], Differential transform method [12], Lagrange Interpolation [13], Differential Transformation [15], Block pulse functions operational matrices [19] Chebyshev polynomials[16], Optimal Auxiliary Function Method (OAFM) [18] and Spectral HomotopyAnalysis Method [20].

We consider first order Volterra integro-differential equation of the form

$$y'(x) = g(x) + \int_0^x k(x,t)y(t)dt \qquad (1)$$

with the initial condition

$$y(0) = q \qquad (2)$$

where $k(x,t)$ is the Volterra integral kernel function. $g(x)$ is the known function and $y(x)$ is an unknown function to be determined.

2. BASIC DEFINITIONS AND TERMS

We give certain definitions and fundamental notions in this section for the purpose of problemformulation.

Definition 1:[21] Let (a_m), $m \leq 0$ be a sequence of real numbers. The power series in x with coefficients a_n is an expression.

$$y(x) = \sum_{m=0}^{\infty} a_m x^m = \emptyset(x)\boldsymbol{A} \qquad (3)$$

where $\emptyset(x) = [1 \ x \ x^2 \ \cdots \ x^N]$, $\boldsymbol{A} = [a_0 \ a_1 \cdots a_N]^T$

Definition 2:[5] The desired collocation points within an interval are determined using this method.

i.e. [a,b] and is provided by

$$x_i = a + \frac{(b-a)i}{N}, i = 0, 1, 2 \ldots \ldots N \qquad (4)$$

Definition 3:[22] Let $p(s)$ be an integrable function, then

$$_0I_x^{\beta}(p(s)) = \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}p(s)ds \qquad (5)$$

Definition 4:[22] Let $y(x)$ be a continuous function, then

$$_0I_x^{\beta}({}_x^CD_a^{\alpha}y(x)) = y(x) - \sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k \qquad (6)$$

where $m - 1 < \beta < 1$

3. METHODOLOGY

Let the solution to (1) and (2) be approximated by

$y(x) = \emptyset(x)\boldsymbol{A}$ (7)

$$\emptyset'(x)\boldsymbol{A} = g(x) + \int_0^x k(x,t)\emptyset(t)\boldsymbol{A}\, dt \qquad (8)$$

collecting the like terms

$$\left(\emptyset'(x) - \int_0^x k(x,t)\emptyset(t)\, dt\right)\boldsymbol{A} = g(x) \qquad (9)$$

Equation (9) can be written in this form

$$U(x)\boldsymbol{A} = g(x) \qquad (10)$$

where

$$U(x) = \left(\emptyset'(x) - \int_0^x k(x,t)\emptyset(t)\, dt\right)_{1\times[N+1]}$$

Collocating (10) using the standard collocation points

$$x_i = a + \frac{(b-a)i}{N}$$

$$U(x_i)\boldsymbol{A} = g(x_i) \qquad (11)$$

where

$$U(x_i) = \begin{pmatrix} U_0 x_0 & U_1 x_0 & \dots & U_n x_0 \\ U_0 x_1 & U_2 x_1 & \dots & U_n x_1 \\ U_0 x_N & U_1 x_N & \dots & U_n x_N \end{pmatrix}$$

$$g(x_i) = [g(x_0) \quad g(x_1) \dots \quad g(x_N)]^T$$

Using the initial condition

$\emptyset(x)$ is an interpolating polynomial and **A** are parameters to be determined substituting equation (3) into equation (1) gives

$$y(0) = q \qquad (12)$$

hence, (3.6) becomes

$$\emptyset(0) = q \qquad (13)$$

Substituting equation (13) into equation (9) gives

$$U^*(x_i)\boldsymbol{A} = g^*(x_i) \qquad (14)$$

The unknown values are solved using matrix inversion. Substituting the values of a_i obtained in the approximate solution gives the numerical solution.

$$y(x) = \emptyset(x_i)\, U^{*-1}(x_i)\, g^*(x_i)$$

3.1 Numerical Examples

In this section, we give numerical examples to evaluate the method's usefulness and accuracy. Let $y_n(x)$ and $y(x)$ be the approximate and exact solutions respectively. $Error_N = |y_N(x) - y(x)|$

Example 1: [17] Considering first order Volterra integro-differential equation

$$y'(x) = 1 + sinx + \int_0^x y(t)dt \qquad (15)$$

subject to initial condition

$$y(0) = 1$$

Exact solution $y(x) = \frac{e^x}{4} - \frac{3e^{-x}}{4} - \frac{cosx}{2}$

Solution 1

The approximate solution of equation (15) at N=5

$$y_5(x) = -1.000000000184 + 1.000000507317x - 0.128181117e - 3x^2 + 0.167363479046x^3 - 0.43011129944e - 1x^4 + 0.9284810779e - 2$$

Table 1: Exact, approximate and absolute error values for example 1

X	Exact	Our method$_{N=5}$	error$_5$	error [17]
0.0625	-0.937459937700	-0.937460256200	3.185000e-7	3.28257e-2
0.125	-0.874684397400	-0.874685275100	8.777000e-7	6.37537e-3
0.1875	-0.811450933000	-0.811452193600	1.260600e-6	3.68533e-2
0.250	-0.747550443900	-0.747551775100	1.331200e-6	1.27946e-2
0.3125	-0.682786210500	-0.682787346500	1.136000e-6	4.04099e-2

Example 2: [17] Considering first order Volterra integro-differential equation

$$y'(x) = -sinx - cosx + 2\int_0^x cox(x-t)y(t)dt \ (16)$$

subject to initial condition

$$y(0) = 1$$

Exact solution $y(x) = \frac{e^x}{4} - \frac{3e^{-x}}{4} - \frac{cosx}{2}$

Solution 2

The approximate solution of equation (16) at $N = 5{,}7\ and\ 10$ gives

$$y_5(x) = 0.999999999417 - 0.999999193495x + 0.499753750089x^2 - 0.165229249724x^3 + 0.38466152158e - 1x^4 - 0.5113765043e - 2x^5$$

$$y_7(x) = 1.000000000000 - 0.999999999782x + 0.499999117332x^2 - 0.166657457128x^3 + 0.41626608698e - 1x^4 - 0.8242412005e - 2x^5 + 0.1274753828e - 2x^6 - 0.121171994e - 3x^7$$

$$y_{10}(x) = 1.000000000000 - 1.000000000001x + 0.499999938009x^2 - 0.166665705852x^3 + 0.41661627591e - 1x^4 - 0.8319467306e - 2x^5 + 0.1362442970e - 2x^6 - 0.170111656e - 3x^7 + 0.6675720e - 5x^8 + 0.3516674e - 5x^9 - 3.352761268616 \times 10^{-7}x^1$$

Table 2: Exact and approximate values for example 2

x	Exact	N = 5	N = 7	N = 10
0.2	0.818730753100	0.818728386100	0.818730750000	0.818730753300
0.4	0.670320046000	0.670318618500	0.670320043400	0.670320038500
0.6	0.548811636100	0.548809882300	0.548811633000	0.548811636300
0.8	0.449328964100	0.449325726200	0.449328960100	0.449328964200
1.0	0.367879441200	0.367877693500	0.367879438900	0.367879441700

Table 3: Absolute Error for example 2

x	$error_5$	$error_7$	$error_{10}$
0.2	2.367000E-6	3.1000000e-9	2.00000e-10
0.4	1.427500e-6	2.600000e-9	7.50000e-9
0.6	1.753800e-6	3.100000e-9	7.62000e-10
0.8	3.237900e-6	4.000000e-9	3.28300e-10
1.0	1.747700e-6	2.300000e-9	1.38800e-10

Example 3: [17] Considering first order Volterra integro-differential equation

$y'(x) = 1 - \int_0^x y(t)dt$ (17)

subject to initial condition

$y(0) = 0$

Exact solution $y(x) = sinx$

Solution 3

The approximate solution of equation (17) at $N = 7$ gives

$$y_7(x) = -4.091778999000 \times 10^{-16} + 1.000000000165x - 5.712718121000 \times 10^{-7}x^2 - 0.166660841322x^3 - 0.24197157e - 4x^4 + 0.8384018671e - 2x^5 - 0.54626726e - 4x^6 - 0.172795088e - 3x^7$$

Table 4: Exact and approximate values for example 3

x	Exact	N = 7	$error_7$	error [17]
0.2	0.198669330800	0.198669328900	1.900000e-9	2.293e-3
0.4	0.389418342300	0.389418340900	1.400000e-9	2.051e-2
0.6	0.564642473400	0.564642472000	1.40000e-9	7.061e-2
0.8	0.717356090900	0.717356089700	1.20000e-9	1.686e-1
1.0	0.841470984800	0.841470987100	2.30000e-9	3.307e-1

4. RESULTS AND DISCUSSIONS

The numerical results obtained from the solved examples using the derived numerical method arediscussed in this section.

The approximate solution obtained for Example 1 as shown in Table1 for N = 5 $gives\ y_5 = - 1.000000000184 + 1.000000507317x - 0.128181117e - 3x^2 + 0.167363479046x^3 - 0.43011129944e - 1x^4 + 0.9284810779e - 2x5$. The numerical result gives a smaller errors and consistent across all valuesof x. This confirmed that our method performed better than the method proposed by [17].

The results of the numerical Example 2 shown in Tables 2 and 3 show that the approximate solutionat N = 5 gives $y_5 = 0.999999999417 - 0.999999193495x + 0.499753750089x^2 - 0.165229249724x^3 + 0.38466152158e - 1x^4 - 0.5113765043e - 2x^5$. Solving for N = 7 and 10, the numerical results convergeto an exact solution as the value of N increases. This shows that the numerical method developed is consistent and accurate.

The approximate solution obtained in Example 3 shown in

Table 4, for the value of N gives

$y_7 = -4.091778999000 \times 10^{-16} + 1.000000000165x - 5.712718121000 \times 10-7x^2 - 0.166660841322x^3 - 0.24197157e-4x^4 + 0.8384018671e-2x^5 - 0.54626726e-4x^6 - 0.172795088e-3x^7$. This also confirmed that our method performed better than the method proposed by [17].

5. CONCLUSION

In this work, the collocation approach was examined for the numerical solution of first-order Volterra integro-differential equations. This method is reliable, effective and straightforward to compute. Maple 18 is used for all of the computations in this work.

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