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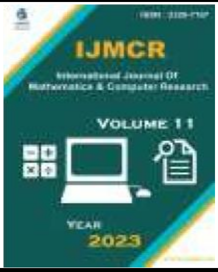

An Application of Maximal Numerical Range on Norm of an Elementary Operator of Length Two in Tensor Product

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ARTICLE INFO	ABSTRACT
Published Online: **03 November 2023** Corresponding Author: **Musundi Sammy Wabomba**	Many researchers in operator theory have attempted to determine the relationship between the norm of an elementary operator of length two and the norms of its coefficient operators. Various results have been obtained using varied approaches. In this paper, we attempt this problem by the use of the Stampfli's maximal numerical range in a tensor product.

KEYWORDS: Elementary Operator, Maximal Numerical Range, and Tensor Product.

1.0 INTRODUCTION

1.1 Tensor products of Hilbert spaces

Definition 1.1.1. Tensor product. (Muiruri *et al*, 2019)

If $Z = \{u_{1,}u_2 \ldots\}$ and $L = \{v_1, v_2 \ldots\}$ are complex Hilbert spaces. Define their inner products $< u_1 u_2 >$ and $< v_2 v_2 >$ respectively. A tensor product of Z and L is a Hilbert space $Z \otimes L$ where $\otimes : Z \times L \to Z \otimes L, \otimes\ (u, v) \to u \otimes v$ is a bilinear mapping:

i). The vectors $u \otimes v$ form a total subset of $Z \otimes L$

ii). $< u_1 \otimes v_1,\ u_2 \otimes v_2 >=< u_1, v_1 >< u_2, v_2 >$ $, \forall\ u_1,\ u_2 \in Z, v_1, v_2 \in L$. This implies that $\| u \otimes v \| = \| u \| \| v \|\ \ \forall\ u \in Z, v \in L$.If $E \in B(Z), F \in B(L)$, then $B(Z \otimes L)$ is a Hilbert space and for $E \otimes F \in B(Z \otimes L)$ we have $E \otimes F(u \otimes v) = Eu \otimes Fv\ \forall\ u \in Z, v \in L$.

The following properties of members of $B(Z \otimes L)$ hold:

i). $(E \otimes F)(G \otimes H) = EG \otimes FH, \forall\ \ E \in B(Z), G \in B(Z)$ and $F \in B(L), H \in B(L)$.This property is both associative and commutative.

ii). $\| E \otimes F \| = \| E \| \| F \|\ \ \ \forall\ \ E \in B(Z)$ and $F \in B(L)$. This property indicates that norm is distributive under tensor product.

The linearity of the map, $\otimes\ (u, v) \to u \otimes v$ shows that $\otimes$ is linear with respect to the two coordinates, that is

(i) $(u_1 + u_2) \otimes v = (u_1\ \otimes v) + (u_2 \otimes v)$

(ii) $(\psi u) \otimes v = \psi(u\ \otimes v)$.

(iii) $u\ \otimes (v_1 + v_2) = u\ \otimes v_1 + u\ \otimes v_2$

(iv) $u\ \otimes (\psi v) = \psi(u\ \otimes v)$.

(v) The set of all vectors $\otimes\ (u, v), u \epsilon Z$ and $v \epsilon L$ form a total subset of $Z \otimes L$.

Definition 1.1.2. Elementary operator in a tensor product. (Muiruri *et al*, 2019)

Let Z be a complex Hilbert space and L be complex Hilbert space, $B(Z \otimes L)$ be the collection of all bounded operators that are linear on the complex Hilbert space $Z \otimes L$ and $E \otimes F$, $G \otimes H$ be fixed elements of, $B(Z \otimes L)$ where $E, G \epsilon B(Z)$ and $F, H \epsilon B(K)$, the collection of bounded operators which are linear on Z and L respectively. Define the elementary operator as;

$E_n(Z \otimes L) = \sum_{i=1}^{n} (E_i \otimes F_i)(U \otimes V)(G_i \otimes H_i)$
(1)

for every $U \otimes V \epsilon B(Z \otimes L)$, $E_i \otimes F_i, G_i \otimes H_i$ being fixed elements of $B(Z \otimes L)$.

Now substituting for $n = 1$ in (1) we obtain the basic elementary operator,

$E(Z \otimes L) = (E \otimes F)(U \otimes V)(G \otimes H)$
(2).

From equation (2) the basic elementary operator can be expressed as,

$E(Z \otimes L) = (E \otimes F)(U \otimes V)(G \otimes H) = (EUG) \otimes (FVH)$

Definition 1.1.3. Stampfli's maximal numerical range of an operator. (Stampfli, 1970)

The Stampfli's maximal numerical range of an operator $G \in \mathcal{B}(Z)$ is the set

$W_{\circ}(G) = \{\zeta \epsilon \mathbb{C} : < Gg_n, g_n > \to \zeta, \| g_n \| = 1, \| Gg_n \| \to \| G \|$
$\}$.

and Stampfli's maximal numerical range of an operator $G \in \mathcal{B}(L)$ is the set
$W_\circ(H) = \{\xi \epsilon \mathbb{C}: < Hh_n, h_n > \to \xi, \| h_n \| = 1, \| Hh_n \| \to \| G \|\}$.

2.0 NORM OF AN ELEMENTARY OPERATOR OF LENGTH n.

Mathieu (2001) determined the norm of elementary operator T on Calkin algebra using hangerup tensor norm $\|.\|_h$ and proved the Theorem below;

Theorem 2.1. (Mathieu, 2001)
Let T be an elementary operator on $B(H)$, then $||T|| = inf\{\|\sum_{i=1}^{n} A_i A_i^*\|^{1/2}\|\sum_{i=1}^{n} B_i^* B_i\|^{1/2}\}$
where the infimum is taken over all representations of T as $T = \sum_{i=1}^{n} M_{A_i B_i}$.

Moreover, Timoney (2007) determined the norm of elementary operator on a C*-algebra using the notion of matrix valued numerical ranges and a kind of tracial geometric mean for a positive matrices and prove theorem 2.2;

Theorem 2.2. (Timoney, 2007)
For $x = (x_1 \dots x_n) \in B(H)^n$(a row matrix of operators $x_i \in B(H)$), $Y = (y_1 \dots y_n) \in B(H)^n$(a row column matrix of operators $y_i \in B(H)$ and elementary operator, $T_{A,B}(x)$ we have,

$$\| T \| = \text{Sup}[tgm\{Q(A^*, \varepsilon), Q(B, \eta)\}: \varepsilon, \eta \epsilon H$$

where $\| \varepsilon \| = 1$ and $\| \eta \| = 1$.
Further, Nyamwala and Agure (2008) used the spectral resolution theorem to calculate the norm of elementary operator induced by normal operators in a finite dimensional Hilbert space and gave the following result.

Theorem 2.3. (Nyamwala and Agure, 2008)
Let $E_{A,B} : B(H) \to B(H)$ be an elementary operator defined by

$$E_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i, \forall X \in B(H)$$

where A_i, B_i are normal operators and H a finite dimensional Hilbert space then

$$||E|| = \left(\sum_{j=1}^{k} |\alpha_{i,j}|^2 |\beta_{i,j}|^2\right)^{\frac{1}{2}}$$

where $\alpha_{i,j}, \beta_{i,j}$ are distinct eigen values of A_i and $B_i$respectively.

King'ang'i *et al* (2014) employed the concept of finite rank operators to determine the norm of elementary operator of length two in an arbitrary C^*-algebra and proved the theorem 2.4;

Theorem 2.4. (King'ang'i *et al.*, 2014)
Let H be a complex Hilbert space, $B(H)$be the algebra of bounded linear operators on H. Let E_2 be the elementary operator on $\mathcal{B}(H)$ of length two. If for an operator $x \in \mathcal{B}(H)$ with $\| x \| = 1$ we have $X(x) = x$ for all $x \in H$ then

$$\| E_2 \| = \sum_{i=1}^{2} \| A_i \| \| B_i \|$$

$\forall A_i, B_i$ fixed in $\mathcal{B}(H)$ and $i = 1,2$.

King'ang'i (2017) employed the concept of the maximal numerical range of A^*B relative to B to determine the norm of an elementary operator of length two in an arbitrary C^*-algebra and proved theorem 2.5;

Theorem 2.5. (King'ang'i., 2017)
Let E_2 be an elementary operator of length two on $\mathcal{B}(H)$ then

$$\| E_2 \| \geq \sup_{\lambda \in W_{B_1}(B_2 * B_1)} \left\| B_1 A_1 + \frac{\bar{\lambda}}{\| B_1 \|} A_2 \right\|$$

where $\forall A_i, B_i$ fixed in $\mathcal{B}(H)$ and $i = 1,2$.
King'ang'i (2018) employed the concept of Stampfli's maximal numerical range to determine the norm of an elementary operator of length two in an arbitrary C^*-algebra and proved theorem 2.4.8;

Theorem 2.6. (King'ang'i., 2018)
Let E_2 be an elementary operator of length two on $\mathcal{B}(H)$ and $s_1, s_2 \in B(H)$, then if $\lambda_i \in W_{\circ(S_I)}$ for every $\lambda_i \epsilon \mathbb{C}$ $i = 1,2$, then $\| E_2 \| \geq \sup_{\lambda \in W_\circ(s_i)} \{\| \sum_{i=1}^{2} \lambda_i T_i \|\}: T_i \in B(H)$ $i = 1,2$.
Kawira *et al.*, (2018) extended the work of King'ang'i, *et al* 2014) to finite length and determined the norm of an elementary operator of an arbitrary length in a C^*-algebra using finite rank one operators and proved theorem 2.7;

Theorem 2.4.9. (Kawira *et al.*, 2018)
Let H be a complex Hilbert space and $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H. Let E_n be the elementary operator on $\mathcal{B}(H)$. If for an operator $X \in \mathcal{B}(H)$ with $\| x \| = 1$ we have $X(f) = f$ for all unit vectors $f \in H$ then
$\| E_n \| = \sum_{i=1}^{n} \| A_i \| \| B_i \|, n \in N$.

Muiruri *et al.*, (2018) determined the norm of an elementary operator in a tensor product by employing the techniques of tensor products and finite rank one operators and also expressed the norm of an elementary operator in terms of its coefficient's operators. Muiruri *et al* (2018) proved the theorem 2.8;

Theorem 2.8. (Muiruri *et al.*, 2019)
Let H and K be complex Hilbert spaces and $\mathcal{B}(H \otimes K)$be the set of bounded linear operators on $H \otimes K$ then $\forall X \otimes Y \in \mathcal{B}(H \otimes K)$ with $\| X \otimes Y \| = 1$ we have

$$\| M_{A \otimes B, C \otimes D} \| = \| A \| \| B \| \| C \| \| D \|$$

where A, B and C, D are fixed elements in $\mathcal{B}(H)$ and $\mathcal{B}(K)$ respectively.

Corollary 2.9. (Muiruri *et al.*, 2019)

Let H and K be complex Hilbert spaces and $\mathcal{B}(H \otimes K)$ be the set of bounded linear operators on $(H \otimes K)$. If for all $X \otimes Y \in \mathcal{B}(H \otimes K)$ with $\| X \otimes Y \| = 1$, then we have

$\| M_{A\otimes B, C\otimes D} \| = \| M_{A,C} \| \| M_{B,D} \|$

where $M_{A,C}$ and $M_{B,D}$ are basic elementary operators on $\mathcal{B}(H)$ and $\mathcal{B}(K)$ respectively.

Daniel *et al.*, (2022) determined the norm of the basic elementary operator in a tensor product by employing the techniques of tensor products and maximal numerical range and also expressed the norm of an elementary operator in terms of its coefficient's operators. Daniel *et al.*, (2018) proved the theorem 2.10;

Theorem 2.10.(Daniel *et al.*, 2022)

Let Z and L be complex Hilbert spaces and let, $O_{E\otimes F,G\otimes H}$ be the basic elementary operator on $\mathcal{B}(Z \otimes L)$ the set of bounded operators which are linear on a complex Hilbert space $Z \otimes L$. If $\forall\, U \otimes V \in \mathcal{B}(Z \otimes L)$ with $\|U \otimes V\| = 1, E, G \in B(Z), F, H \in B(L), \zeta \in W_\circ(G), \xi \in W_\circ(H)$ then we have,

$$\left\|O_{E\otimes F,G\otimes H} \backslash B(Z \otimes L)\right\| = \sup_{\zeta \in W_\circ(G)} \sup_{\xi \in W_\circ(H)} \{|\zeta||\xi|\|E\|\|F\|\}.$$

3.0 MAIN RESULTS

3.1 Norm of an elementary operator of length two in a tensor product.

In this section the lower and upper bound of an elementary operator of length two is determined using the Stampfli's Maximal numerical range.

Theorem 3.2

Let Z and L be complex Hilbert spaces and let, $\sum_{i=1}^{2} O_{E_i\otimes F_i, G_i\otimes H_i}$ be the elementary operator of length two on $\mathcal{B}(Z \otimes L)$ the set of bounded operators which are linear on a complex Hilbert space $Z \otimes L$. If $\forall\, U \otimes V \in \mathcal{B}(Z \otimes L)$ with $\|U \otimes V\| = 1, E_i, G_i \in B(Z), F_i, H_i \in B(L)$, $\zeta_i \in W_\circ(G_i), \xi_i \in W_\circ(H_i)\ \forall\, i = 1,2$ then we have,

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}} \backslash B(Z \otimes L)\right\| = \sup_{\zeta \epsilon W_\circ(G)} \|\textstyle\sum_{i=1}^{2} \zeta_i E_i\| \sup_{\xi \epsilon W_\circ(H)} \|\textstyle\sum_{i=1}^{2} \xi_i F_i\|$$

Proof.

By definition of the norm, $\forall\ U \otimes V \in \mathcal{B}(Z \otimes L), U \in B(Z), V \in B(L), \|U\| = 1, \|V\| = 1$then we have,

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}} \backslash B(Z \otimes L)\right\| = Sup \left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}} (U \otimes V)\right\|. \quad (1)$$

This implies that for every rank one operator $(m \otimes g_n)g_n = \langle g_n, g_n\rangle m \in B(H)$ and $(f \otimes h_n)h_n = \langle h_n, h_n\rangle f$ then,

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}} \backslash B(Z \otimes L)\right\| \geq \left\|\sum_{i=1}^{2} O_{E_i\otimes F_i, G_i\otimes H_i} (m \otimes g_n)(g_n) \otimes (f \otimes h_n)(h_n)\right\|$$

$$\geq \|E_1 \otimes F_1(m \otimes g_n)g_n \otimes (f \otimes h_n)h_n G_1 \otimes H_1 + E_2 \otimes F_2(m \otimes g_n)g_n \otimes (f \otimes h_n)h_n G_2 \otimes H_2\|$$

$$\geq \|\{E_1(m \otimes g_n)G_1 g_n\} \otimes \{F_1(f \otimes h_n)H_1 h_n\} + \{E_2(m \otimes g_n)G_2 g_n\} \otimes \{F_2(f \otimes h_n)H_2 h_n\}\|$$

$$\geq \|\langle G_1 g_n, g_n\rangle E_1 m \otimes \langle H_1 h_n, h_n\rangle F_1 f + \langle G_2 g_n, g_n\rangle E_2 m \otimes \langle H_2 h_n, h_n\rangle F_2 f\| \quad (2)$$

By the definition **1.0** , if $\zeta_i \in W_\circ(G_i) \forall\ i = 1,2$ we have,

(i) $\lim_{n\to\infty} \langle G_1 g_n, g_n\rangle = \zeta_1$ and
(ii) $\lim_{n\to\infty} \|G_1 g_n\| = \|G_1\|$
(iii) $\lim_{n\to\infty} \langle G_2 g_n, g_n\rangle = \zeta_2$ and
(iv) $\lim_{n\to\infty} \|G_2 g_n\| = \|G_2\|$

and $\xi_i \in W_\circ(H_i) \forall i = 1,2$ We have,

(i) $\lim_{n\to\infty} \langle H_1 h_n, h_n\rangle = \xi_1$ and
(ii) $\lim_{n\to\infty} \|H_1 h_n\| = \|H_1\|$
(iii) $\lim_{n\to\infty} \langle H_2 h_n, h_n\rangle = \xi_2$ and
(iv) $\lim_{n\to\infty} \|H_2 h_n\| = \|H_2\|$

Now taking limits as $n \to \infty$ on both sides of inequality (2)we have

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}} \backslash B(Z \otimes L)\right\| \geq \|\langle G_1 g_n, g_n\rangle E_1 m \otimes \langle H_1 h_n, h_n\rangle F_1 f + \langle G_2 g_n, g_n\rangle E_2 m \otimes \langle H_2 h_n, h_n\rangle F_2 f\| \geq \|\zeta \boldsymbol{E m} \otimes \xi \boldsymbol{F f}\|$$

$$\geq \|(\zeta_1 \otimes \xi_1)(E_1 \otimes F_1)(\boldsymbol{m} \otimes \mathbf{f}) + (\zeta_2 \otimes \xi_2)(E_2 \otimes F_2)(\boldsymbol{m} \otimes \mathbf{f})\|$$

So $\forall \in > 0$,

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}} \backslash B(Z \otimes L)\right\| - \epsilon < \|(\zeta_1 \otimes \xi_1)(E_1 \otimes F_1)(\boldsymbol{m} \otimes \mathbf{f}) + (\zeta_2 \otimes \xi_2)(E_2 \otimes F_2)(\boldsymbol{m} \otimes \mathbf{f})\|$$

$$\leq \|(\zeta_1 \otimes \xi_1)(E_1 \otimes F_1)(\boldsymbol{m} \otimes \mathbf{f}) + (\zeta_2 \otimes \xi_2)(E_2 \otimes F_2)(\boldsymbol{m} \otimes \mathbf{f})\|$$

$$\leq \|\textstyle\sum_{i=1}^{2} (\zeta_i \otimes \xi_i)(E_i \otimes F_i)(\boldsymbol{m} \otimes \mathbf{f})\| \quad (3)$$

where m and f are unit vectors in $B(Z)$ and $B(L)$ respectively.

Now since ϵ is arbitrary chosen and the unit vectors are chosen arbitrary then we get the supremum, this implies that

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}} \backslash B(Z \otimes L)\right\| \leq \sup_{\zeta \epsilon W_\circ(G)} \sup_{\xi \epsilon W_\circ(H)} \left[\sup_{\|\mathbf{m}\otimes\mathbf{f}\|=1} \{\|\textstyle\sum_{i=1}^{2} (\zeta_i \otimes \xi_i)(E_i \otimes F_i)(\boldsymbol{m} \otimes \mathbf{f})\|\}\right] = \sup_{\zeta \epsilon W_\circ(G)} \sup_{\xi \epsilon W_\circ(H)} [\|\textstyle\sum_{i=1}^{2} (\zeta_i \otimes \xi_i)(E_i \otimes F_i)\|] = \sup_{\zeta \epsilon W_\circ(G)} \sup_{\xi \epsilon W_\circ(H)} [\|\textstyle\sum_{i=1}^{2} (\zeta_i E_i \otimes \xi_i F_i)\|]$$

=

$$\sup_{\zeta\epsilon W_\circ(G)}\sup_{\xi\epsilon W_\circ(H)}[\|\sum_{i=1}^{2}\zeta_i E_i\|]\|\sum_{i=1}^{2}\xi_i F_i\|$$

$$=\sup_{\zeta\epsilon W_\circ(G)}\|\sum_{i=1}^{2}\zeta_i E_i\|\sup_{\xi\epsilon W_\circ(H)}\|\sum_{i=1}^{2}\xi_i F_i\|$$

thus

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i,G_i\otimes H_i}}\backslash B(Z\otimes L)\right\|\leq \sup_{\zeta\epsilon W_\circ(G)}\|\sum_{i=1}^{2}\zeta_i E_i\|\sup_{\xi\epsilon W_\circ(H)}\|\sum_{i=1}^{2}\xi_i F_i\| \quad (4)$$

Conversely, let $\{g_n\}_{n>1}$be a sequence of vectors of length one in a complex Hilbert space Z and let $\{h_n\}_{n>1}$ be a sequence of vectors of length one in a complex Hilbert space L.Define rank one operator,$(m\otimes g_n)\in B(Z)$ and $(f\otimes h_n)\in B(L)$ for a unit vector **m** $\in Z$ and **g** $\in L$ as $(\boldsymbol{m}\otimes g_n)x=\langle x,g_n\rangle\boldsymbol{m}$ and $(\boldsymbol{f},h_n)y=\langle y,h_n\rangle\boldsymbol{f}$ $\forall\ x\in Z$ and $\forall\ y\in L$.

Define the $W_\circ(G_i)$ and $W_\circ(H_i)$ of G_i and H_i $\forall\ i=1,2$is defined as

$$W_\circ(G_1)=\{\zeta_1\epsilon\mathbb{C}:\langle G_1g_n,g_n\rangle\to\zeta_1,\|g_n\|=1\ and\ \|G_1g_n\|=\|G_1\|\}$$
$$W_\circ(G_2)=\{\zeta_2\epsilon\mathbb{C}:\langle G_2g_n,g_n\rangle\to\zeta_2,\|g_n\|=1\ and\ \|G_2g_n\|=\|G_2\|\}$$

and

$$W_\circ(H_1)=\{\xi_1\epsilon\mathbb{C}:\langle H_1h_n,h_n\rangle\to\xi_1,\|h_n\|=1\ and\ \|H_1h_n\|=\|H_1\|\}$$

$$W_\circ(H_2)=\{\xi_2\epsilon\mathbb{C}:\langle H_2h_n,h_n\rangle\to\xi_2,\|h_n\|=1\ and\ \|H_2h_n\|=\|H_2\|\}$$

Now $\forall\,\zeta_i\epsilon Z$ and $\xi_i\epsilon L$, if $\zeta_i\epsilon W_\circ(G_i)$ $\forall\,G_i\in B(Z)$ $\forall\,i=1,2$ then a sequence $\{g_n\}_{n>1}$ of vectors of length one exists in Z such that;

(i) $\lim_{n\to\infty}\langle G_ig_{n,}g_n\rangle=\zeta_1$
(ii) $\lim_{n\to\infty}\|G_1g_n\|=\|G_1\|$
(iii) $\lim_{n\to\infty}\langle G_2g_{n,}g_n\rangle=\zeta_2$
(iv) $\lim_{n\to\infty}\|G_2g_n\|=\|G_2\|$

and if $\xi_i\epsilon W_\circ(H_i)$ $\forall\,H_i\in B(L)\forall i=1,2$ then a sequence $\{h_n\}_{n>1}$ exists of vectors of length one in L such that

(i) $\lim_{n\to\infty}\langle H_1h_{n,}h_n\rangle=\xi_1$
(ii) $\lim_{n\to\infty}\|H_1h_n\|=\|H_1\|$
(iii) $\lim_{n\to\infty}\langle H_2h_{n,}h_n\rangle=\xi_2$
(iv) $\lim_{n\to\infty}\|H_2h_n\|=\|H_2\|$

By finite rank one operator, the basic elementary operator norm of is given as

$$\|O_2\{(m\otimes g_n)g_n\otimes(f\otimes h_n)h_n\|=$$
$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i,G_i\otimes H_i}}(\boldsymbol{m}\otimes g_n)g_n\otimes(\boldsymbol{f}\otimes h_n)h_n\right\|$$

$$=\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i,G_i\otimes H_i}}(\boldsymbol{m}\otimes g_n)\otimes(\boldsymbol{f}\otimes h_n)(h_n\otimes g_n)\right\|$$

$$\leq\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i,G_i\otimes H_i}}(\boldsymbol{m}\otimes g_n)\otimes(\boldsymbol{f}\otimes h_n)\right\|\|(h_n\otimes g_n)\|$$

$$\leq\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i,G_i\otimes H_i}}(\boldsymbol{m}\otimes g_n)\right\|\|\boldsymbol{f}\|\|h_n\|\|h_n\|\|g_n\|$$

$$\leq\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i,G_i\otimes H_i}}\right\|\|\boldsymbol{m}\|\|g_n\|\|\boldsymbol{f}\|\|h_n\|\|h_n\|\|g_n\|$$

$$\leq\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i,G_i\otimes H_i}}\right\| \quad (5)$$

since the $\boldsymbol{m},\boldsymbol{f}$ are unit vectors and g_n h_n are unit sequences such that $\|\boldsymbol{f}\|=1,\|\boldsymbol{m}\|=1,\|h_n\|=1$ and $\|g_n\|=1$.

Therefore;

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i,G_i\otimes H_i}}\right\|\geq\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i,G_i\otimes H_i}}(\boldsymbol{m}\otimes g_n)g_n\otimes(\boldsymbol{f}\otimes h_n)h_n\right\|$$

$$\geq\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i,G_i\otimes H_i}}\{(\boldsymbol{m}\otimes g_n)g_n\otimes(\boldsymbol{f}\otimes h_n)h_n\}\right\|$$

$$\geq\|E_1\otimes F_1\{(\boldsymbol{m}\otimes g_n)g_n\otimes(\boldsymbol{f}\otimes h_n)h_n\},G_1\otimes H_1+E_2\otimes F_2\{(\boldsymbol{m}\otimes g_n)g_n\otimes(\boldsymbol{f}\otimes h_n)h_n\},G_2\otimes H_2\|$$

$$\geq\|E_1\otimes F_1\{(\boldsymbol{m}\otimes g_n)\otimes(\boldsymbol{f}\otimes h_n)\}g_n\otimes h_nG_1\otimes H_1+E_2\otimes F_2\{(\boldsymbol{m}\otimes g_n)\otimes(\boldsymbol{f}\otimes h_n)\}g_n\otimes h_nG_2\otimes H_2\|$$

$$\geq\|E_1\otimes F_1\{(\boldsymbol{m}\otimes g_n)\otimes(\boldsymbol{f}\otimes h_n)\}G_1\otimes H_1(g_n\otimes h_n)+E_2\otimes F_2\{(\boldsymbol{m}\otimes g_n)\otimes(\boldsymbol{f}\otimes h_n)\}G_2\otimes H_2(g_n\otimes h_n)\|$$

$$\geq\|\{E_1(\boldsymbol{m}\otimes g_n)G_1g_n\}\otimes\{F_1(\boldsymbol{f}\otimes h_n)H_1h_n\}+\{E_2(\boldsymbol{m}\otimes g_n)G_2g_n\}\otimes\{F_2(\boldsymbol{f}\otimes h_n)H_2h_n\}\|$$

$$\geq\left\|\{\langle G_1g_{n,}g_n\rangle E_1\boldsymbol{m}\}\otimes\{\langle H_1h_n,h_n\rangle F_1\boldsymbol{f}+\{\langle G_2g_{n,}g_n\rangle E_2\boldsymbol{m}\}\otimes\{\langle H_2h_n,h_n\rangle F_2\boldsymbol{f}\right\|$$ thus

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i,G_i\otimes H_i}}\right\|\geq\left\|\{\langle G_1g_{n,}g_n\rangle E_1\boldsymbol{m}\}\otimes\{\langle H_1h_n,h_n\rangle F_1\boldsymbol{f}+\{\langle G_2g_{n,}g_n\rangle E_2\boldsymbol{m}\}\otimes\{\langle H_2h_n,h_n\rangle F_2\boldsymbol{f}\right\| \quad (6)$$

by taking the limits both sides as $n\to\infty$ for the inequality (6) and $\forall\,\zeta_i\epsilon Z$ and $\xi_i\epsilon L$, if $\zeta_i\epsilon W_\circ(G_i)$ $\forall\,G_i\in B(Z)$ $\forall\,i=1,2$ then $\exists$ a sequence $\{g_n\}_{n>1}$ of vectors of length one in Z such that

(i) $\lim_{n\to\infty}\langle G_ig_{n,}g_n\rangle=\zeta_1$
(ii) $\lim_{n\to\infty}\|G_1g_n\|=\|G_1\|$
(iii) $\lim_{n\to\infty}\langle G_2g_{n,}g_n\rangle=\zeta_2$
(iv) $\lim_{n\to\infty}\|G_2g_n\|=\|G_2\|$

and similarly, for every $\xi_i\epsilon W_\circ(H_i)$ $\forall\,H_i\in B(K)\forall i=1,2$ then $\exists$ a sequence $\{h_n\}_{n>1}$ of vectors of length one in L such that

(i) $\lim_{n\to\infty}\langle H_1h_{n,}h_n\rangle=\xi_1$

(ii) $\lim_{n\to\infty}\|H_1 h_n\| = \|H_1\|$

(iii) $\lim_{n\to\infty}\langle H_2 h_n, h_n\rangle = \xi_2$

(iv) $\lim_{n\to\infty}\|H_2 h_n\| = \|H_2\|$

then we have;

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}}\right\| \geq \|\zeta_1 E_1 \boldsymbol{m}\otimes \xi_1 F_1 \boldsymbol{f} + \zeta_2 E_2 \boldsymbol{m}\otimes \xi_2 F_2 \boldsymbol{f}\|$$

$$\geq \|(\zeta_1\otimes\xi_1)(E_1\otimes F_1)(\boldsymbol{m}\otimes \mathbf{f}) + (\zeta_2\otimes \xi_2)(E_2\otimes F_2)(\boldsymbol{m}\otimes\mathbf{f})\|$$

$$\geq \|\textstyle\sum_{i=1}^{2}(\zeta_i\otimes\xi_i)(E_i\otimes F_i)(\boldsymbol{m}\otimes\mathbf{f})\|$$

This is true for any $\zeta_i \epsilon W_\circ(G_i)$, $\xi_i \epsilon W_\circ(F_i)\ \forall\ i = 1,2$ and for any unit vector $\boldsymbol{m}\epsilon Z, \boldsymbol{f}\epsilon L$.Since $\zeta_i, \xi_i \forall\ i = 1,2$ and the unit vectors are chosen arbitrarily, then we get the double supremum for the lower bound;

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}}\right\| \geq \sup_{\zeta\epsilon W_\circ(G)}\ \sup_{\xi\epsilon W_\circ(H)}\left[\sup_{\|\mathbf{m}\otimes\mathbf{f}\|=1}\{\|\textstyle\sum_{i=1}^{2}(\zeta_i\otimes\xi_i)(E_i\otimes F_i)(\boldsymbol{m}\otimes\mathbf{f})\|\}\right] = \sup_{\zeta\epsilon W_\circ(G)}\ \sup_{\xi\epsilon W_\circ(H)}[\|\textstyle\sum_{i=1}^{2}(\zeta_i\otimes\xi_i)(E_i\otimes F_i)\|] = \sup_{\zeta\epsilon W_\circ(G)}\ \sup_{\xi\epsilon W_\circ(H)}[\|\textstyle\sum_{i=1}^{2}(\zeta_i E_i\otimes \xi_i F_i)\|]$$

$$= \sup_{\zeta\epsilon W_\circ(G)}\ \sup_{\xi\epsilon W_\circ(H)}[\|\textstyle\sum_{i=1}^{2}\zeta_i E_i\|]\|\textstyle\sum_{i=1}^{2}\xi_i F_i\|$$

$$= \sup_{\zeta\epsilon W_\circ(G)}\|\textstyle\sum_{i=1}^{2}\zeta_i E_i\|\ \sup_{\xi\epsilon W_\circ(H)}\|\textstyle\sum_{i=1}^{2}\xi_i F_i\|$$

thus

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}}\right\| \geq \sup_{\zeta\epsilon W_\circ(G)}\|\textstyle\sum_{i=1}^{2}\zeta_i E_i\|\ \sup_{\xi\epsilon W_\circ(H)}\|\textstyle\sum_{i=1}^{2}\xi_i F_i\|$$

(7)

From inequality (4) and (7) we have,

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}}\backslash B(Z\otimes L)\right\| = \sup_{\zeta\epsilon W_\circ(G)}\|\textstyle\sum_{i=1}^{2}\zeta_i E_i\|\ \sup_{\xi\epsilon W_\circ(H)}\|\textstyle\sum_{i=1}^{2}\xi_i F_i\| \qquad []$$

Corollary 3.3

Let Z and L be a complex Hilbert space, $B(Z\otimes L)$ be the set of bounded linear operators on $Z\otimes l$.If for all $U\otimes V\in B(Z\otimes L)$ with $\|U\otimes V\| = 1$, then we have $\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}}\backslash B(Z\otimes L)\right\| = \|\sum_{i=1}^{2} O_{E_iG_i}\backslash B(Z)\|\|\sum_{i=1}^{2} O_{F_iH_i}\backslash B(L)\|$ where $\sum_{i=1}^{2} O_{E_iG_i}\backslash B(Z)$ and $\sum_{i=1}^{2} O_{E_iG_i}\backslash B(Z)$ are the elementary operators of length two on $B(Z)$ and $B(L)$ respectively.

Proof

Recall that from King'angi (2018) **Theorem 2.4.** $\|\sum_{i=1}^{2} O_{E,G}\| = \sup_{\zeta\epsilon W_\circ(G)}\|\sum_{i=1}^{2}\zeta_i E_i\|$ and $\|\sum_{i=1}^{2} O_{F,H}\| = \sup_{\xi\epsilon W_\circ(H)}\|\sum_{i=1}^{2}\xi_i F_i\|$.Now from theorem 4.3.2, we have

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}}\backslash B(Z\otimes L)\right\| = \sup_{\zeta\epsilon W_\circ(G)}\|\textstyle\sum_{i=1}^{2}\zeta_i E_i\|\ \sup_{\xi\epsilon W_\circ(H)}\|\textstyle\sum_{i=1}^{2}\xi_i F_i\|$$

We can rearrange this as

$$\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}}\backslash B(Z\otimes L)\right\| = \sup_{\zeta\epsilon W_\circ(G)}\|\textstyle\sum_{i=1}^{2}\zeta_i E_i\|\ \sup_{\xi\epsilon W_\circ(H)}\|\textstyle\sum_{i=1}^{2}\xi_i F_i\|$$

.Notice that $E, G\in B(Z)$ and $F, H\in B(L)$.Thus $\|\sum_{i=1}^{2} O_{E,G}\| = \sup_{\zeta\epsilon W_\circ(G)}\|\sum_{i=1}^{2}\zeta_i E_i\|$while $\|\sum_{i=1}^{2} O_{F,H}\| = \sup_{\xi\epsilon W_\circ(H)}\|\sum_{i=1}^{2}\xi_i F_i\|$

Then substituting, we obtain $\left\|\sum_{i=1}^{2} O_{2_{E_i\otimes F_i, G_i\otimes H_i}}\backslash B(Z\otimes L)\right\| = \|\sum_{i=1}^{2} O_{E,G}\|\|\sum_{i=1}^{2} O_{F,H}\|$

4.0. CONCLUSION

In this paper, we have determined the application of Stampfli's maximal numerical range on the norm of an elementary operator of length two in a tensor product.

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