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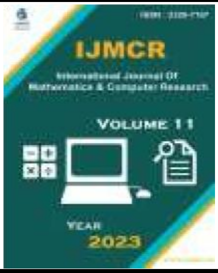

Dynamics of Evolution of Chialvo System: Study of Chaos and Complexity

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ABSTRACT
Chialvo map modelling a neuron has been investigated numerically as it exhbit generically some of the intriguing/ complex features of several excitable biological systems. Bifurcation diagram obtained for this map shows pattern of repeated period-doubling route to chaos. Appearance of complicated periodic windows within bifurcation suggests presence of complexity within the map. Regular and chaotic attractors are drawn here in different parameter space. Lyapunov exponents describing the stability of the systems have been computed for different cases, clearly revealing complexity involved dynamics. Numerical simulation have been extended to calculate topological entropy as a measure of complexity and presented through graphics. Topological entropy graphs showing significant increase within certain parameter range. Correlation dimension of certain chaotic attractors also obtained.

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KEYWORDS: Chaos, Lyapunov Exponents, Bifurcation, Topological Entropy

1. INTRODUCTION

Numerous nonlinear systems studied recently showing chaos in some parameter space as well as properties like bi-stability, intermittency, cascading effects, coexistence of multiple attractors etc. due to their internally multicomponent structure. Such nonlinear systems are termed as complex system, [1–7]. Thus, complexity is a property of a nonlinear system. Chaos in a system is measured by Lyapunov exponents (LCEs)- a evolving system display chaos if LCE > 0 and the evolution be regular if LCE< 0, [8–13]. Presence of complexity in a system may be measured by increment in topological entropy: more increase or fluctuations of topological entropy signifies the system is more complex [14–18].

Neurons are granular cells and are fundamental units of the nervous system. A neuron is an electrically excitable cell which fires electric signals called action potentials across a neural network [19]. Neurons are main components of nervous tissue in all living creatures. The *Chialvo map* which is a two-dimensional map, proposed by Dante R. Chialvo in 1995, describes the dynamics of excitable systems and represent a model for evolution of neurons, [20–23]. Most of the models describing evolution of neuron in cells are internally multicomponent of nature and comes within domain of complex systems. Such models display chaos as well as the complexity property.

The objective of this article to perform dynamical study on Chialvo map. Starting from bifurcation analysis numerical investigations have been carried out to obtain sets of regular and chaotic attractors. For chaotic attractors, Lyapunov exponents (LCEs) are obtained and presented through graphics. Numerical calculations have been further be extended to calculate topological entropies as measures of complexity. Finally, correlation dimensions of chaotic attractors computed clearly the fractal characteristic of the evolving dynamics.

2. DESCRIPTION OF CHIALVO MAP: STEADY STATES AND BIFURCATIONS

Chialvo map consists of two-dimensional discrete mathematical equations represented by

$$\begin{aligned} x_{n+1} &= x_n^2 \exp(y_n - x_n) + k\,, \\ y_{n+1} &= a\, y_n - b\, x_n + c \quad . \end{aligned} \tag{2.1}$$

Here, x is activation or action potential variable, and y is the recovery variable. Then parameters, k stands for steady or time-dependent additive perturbation, a stands for time constant of recovery $(a < 1)$, b stands for activation-dependence of the recovery $(b < 1)$ process and c stands for an offset constant. The Chialvo model presents a very rich dynamics, from an oscillatory to chaotic behavior in different parameter space, [19–21].

As first equation in (2.1) contain exponential terms, fixed points are infinite in number for any set of values of parameters. However, in case $k = 0$ and $b \ll a$, (i. e. when b is very small compared to a and be neglected), the model mimics the lack of ‘voltage-dependence inactivation’ for real neurons and the evolution of the recovery variable is fixed at

$$y^* = \frac{c}{1-a} = r, \text{ say.}$$

Then, system (2.1) is described by a single equation

$$x_{n+1} = f(x_n, r) = x_n^2 \exp(r - x_n). \quad (2.2)$$

which evolve into a period-doubling bifurcation, Figure 1.

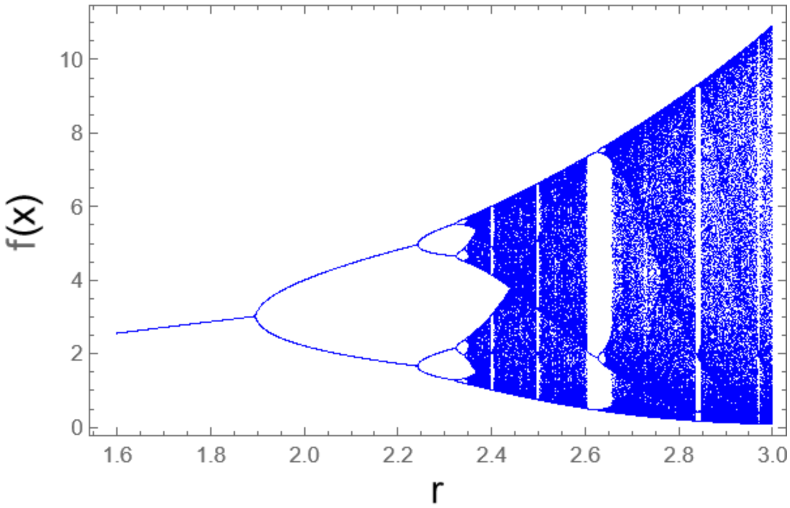
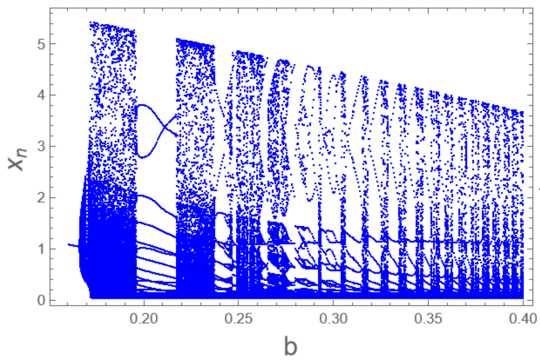


Figure 1: Period-doubling bifurcations of map (2,2) for $1.6 \leq r \leq 3.0$.

Bifurcation of Chialvo system (2.1) for fixed parameter values $a = 0.89, c = 0.28, k = 0.026$ and varying parameter b, $0.16 \leq b \leq 0.4$, along both x – and y – axes shown in Figure 2. One finds here a very complex pattern of bifurcation. We find repeated period-doubling routs to chaos when b is increased from 0.16 to 0.4. Within chaos, also, we observe complicated types of muti-periodic windows. To have better display of such periodic windows, bifurcations are shown in close ranges of b; $0.18 \leq b \leq 0.25$ and $0.26 \leq b \leq 0.32$. The characteristic patterns appearing within periodic windows display mixed phenomena like that of bi-stability, intermittency, cascading effects, exhibit of hysteresis properties etc. These effects are in confirmation of presence of complexity in the system.



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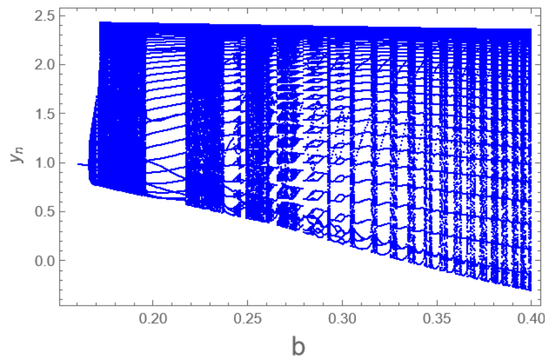

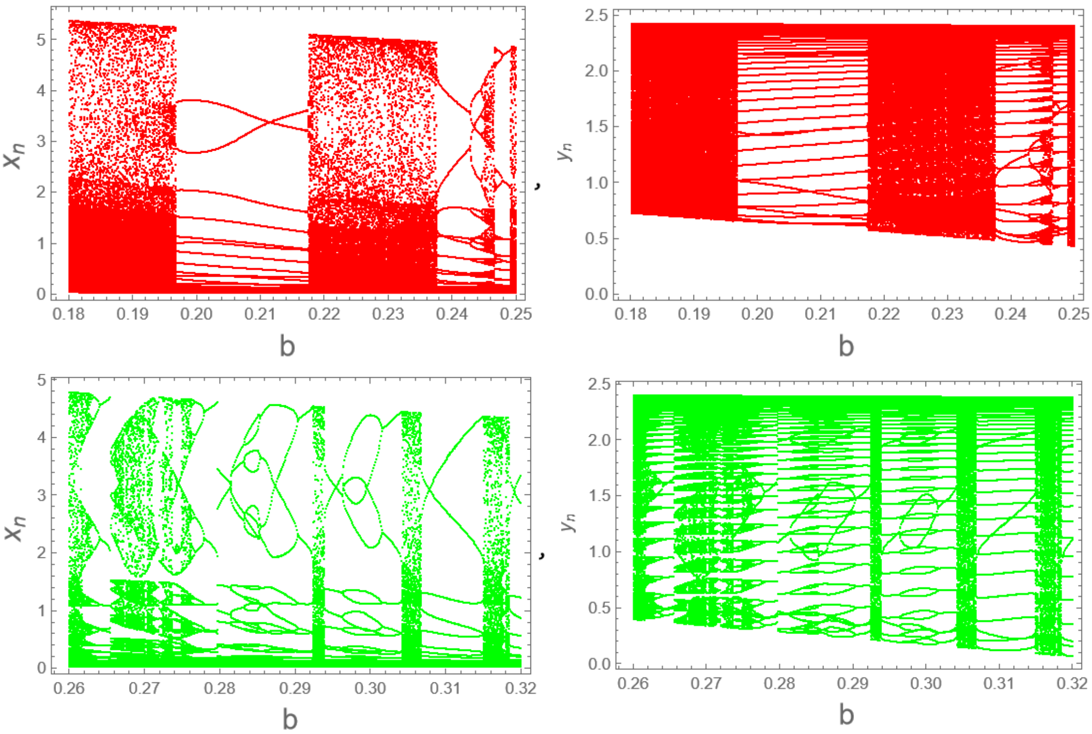

Figure 2: Bifurcation diagrams of Chialvo map for $a = 0.89, c = 0.28, k = 0.026$ and varying b : (a) for upper figures $0.16 \le b \le 0.4$, (b) for middle figures $0.18 \le b \le 0.25$, (c) for lower figures $0.26 \le b \le 0.32$.

3. NUMERICAL SIMULATIONS

(a) **Attractors:** Regular and chaotic attractors of system (2.1) obtained by fixing parameters $a = 0.89, c = 0.28, k = 0.026$ and different values of parameter b are shown in Figure 3. Here we observe, attractor formations from stable point attractor to stable focus, to chaotic attractors which later changing into stable cycles as we increase the value of parameter b from 0.1 to 0.46. Such formation of attractors would repeat as it appears from the bifurcation scenario, Figure 2.

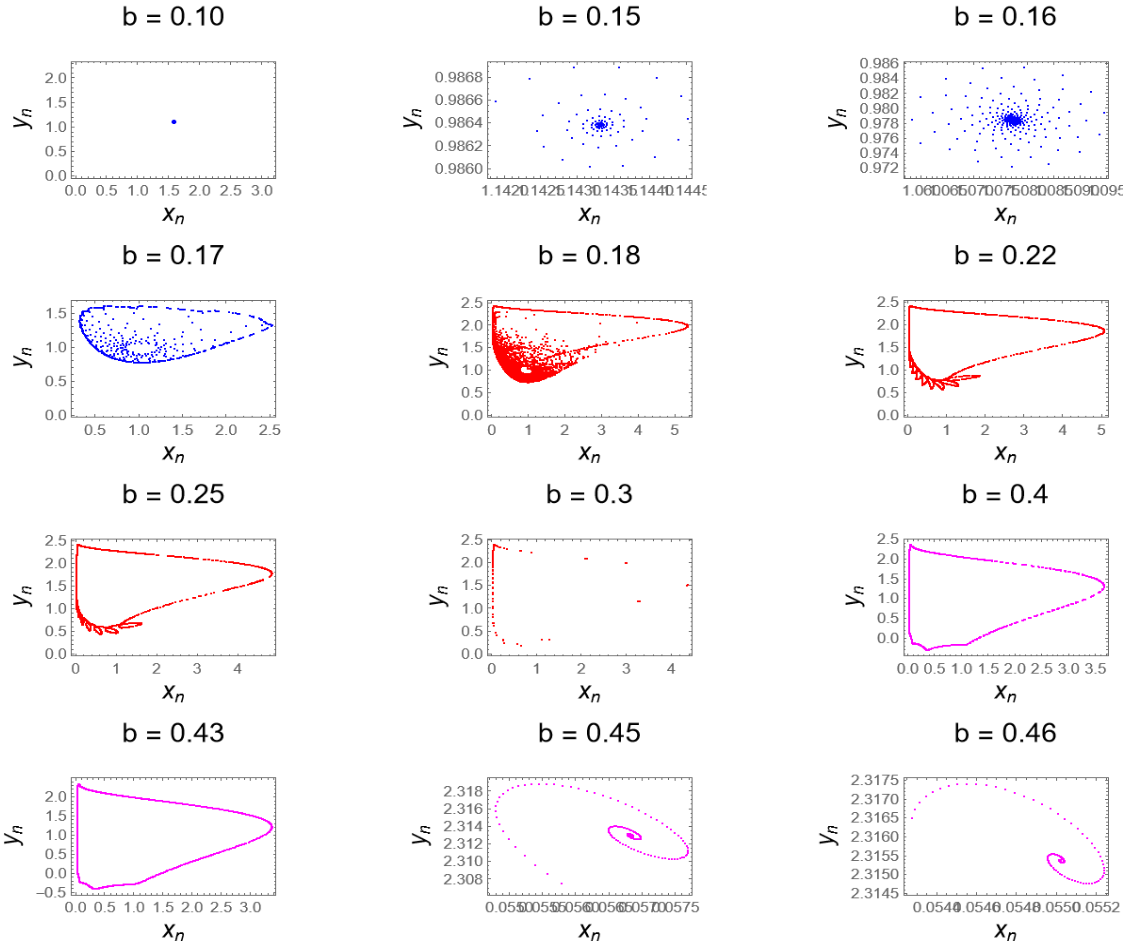

Figure 3: Attractors of Chialvo map for different values of b when other fixed parameters are $a = 0.89, c = 0.28, k = 0.026$.

(b) **Lyapunov Exponents (LCEs):** In case of chaos, system shows *sensitivity to initial conditions*, i.e. that the two trajectories initiated extremely close to each other show divergence behavior during long term evolution. Lyapunov exponents provide a measure of divergence of trajectories initiated closely and so if the system evolves chaotically then LCEs > 0. If the system evolves regularly then LCEs < 0. In Figure 4, plots of Lyapunov exponents are shown against cases of regular and chaotic attractors. To calculate Lyapunov exponents and plotting those we have used Mathematica codes of Martelli [24].

Then, in Figure 5, plots of time series curves shown for two values of b along both x- and y- axes for chaotic cases when $a = 0.89$, $c = 0.28, k = 0.028$. These time series curves showing certain characteristic complex patterns.

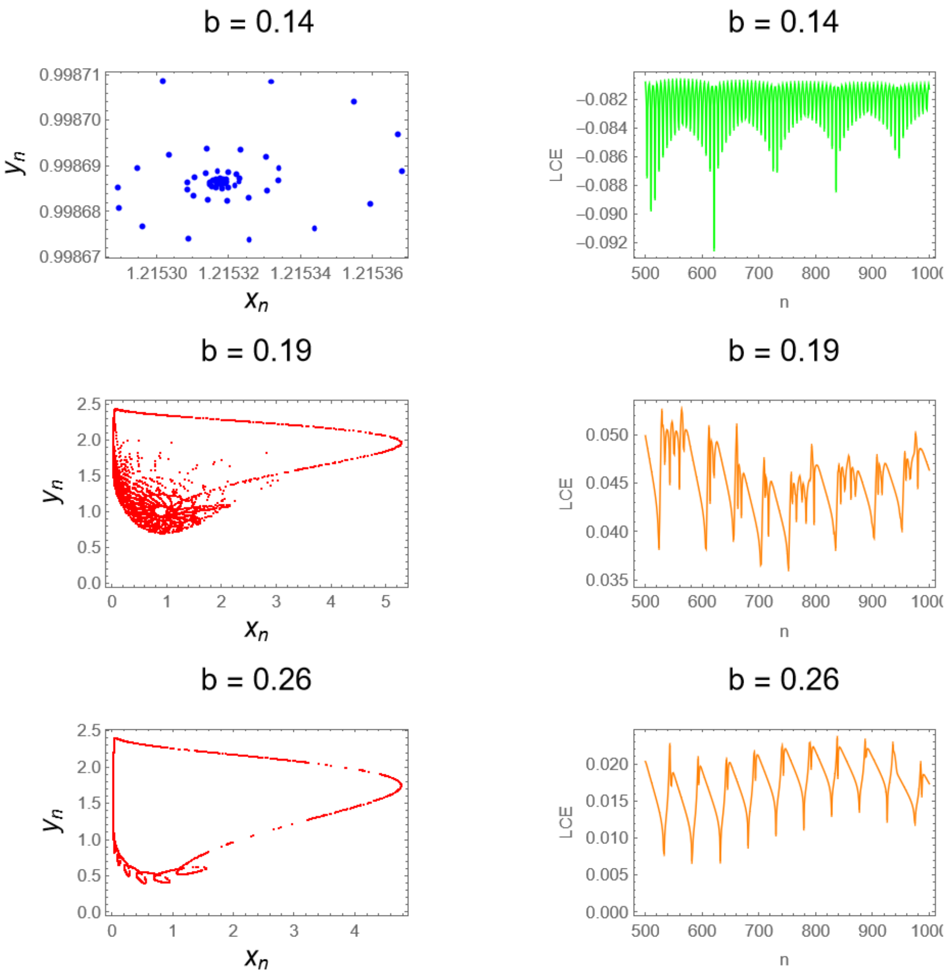

Figure 4: Plots of chaotic attractors and corresponding plots of LCEs for fixed parameters $a = 0.89, c = 0.28, k = 0.028$ and three different values of b.

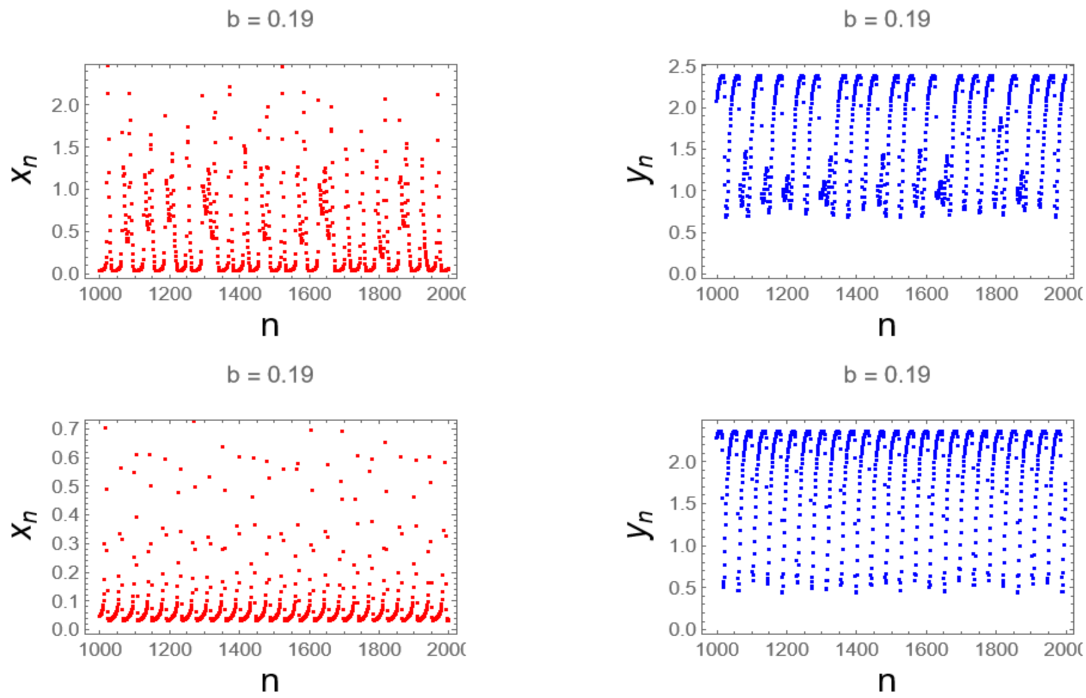

Figure 5: Time series curves for $a = 0.89,\ b = 0.25, c = 0.28, k = 0.028$.

(c) **Topological Entropies:** Complex systems internally composed structurally multi-components and during evolution their individual elements evolve in their own independent way displaying mixed nonlinearity properties, chaos as well as complexity. So, measure of such presence of complexity in the system should be done with some new perspective and the law of probability be applied to describe the rate of mixing of evolutions. The notion of topological entropy was introduced and suggested that the topological entropy describes measure of the *rate of mixing* of a dynamical system, [4–7, 14–18]. Real systems are mostly nonlinear and many are complex because they are internally composed with multicomponent structure, (e. g. neural networks). During evolution their individual elements evolve in their own independent way and display mixed properties of nonlinearity, chaos as well as complexity. Topological entropy measures the exponential growth rate of the number of distinguishable orbits as time advances. Topological entropy is also called Kolmogorov – Sinai entropy [25]).

In order to explain method to measure topological entropy, consider a finite partition of a state space X denoted by $P = \{A_1, A_2, A_3, \ldots, A_N\}$. Then a measure μ on X with total measure $\mu(X) = 1$ defines the probability of a given reading as

$$p_i = \mu(A_i), i = 1, 2, .., N. \quad (3.1)$$

Then the entropy of the partition be given by

$$H(P) = -\sum_{i=0}^{N} p_i \operatorname{Log} p_i \quad (3.2)$$

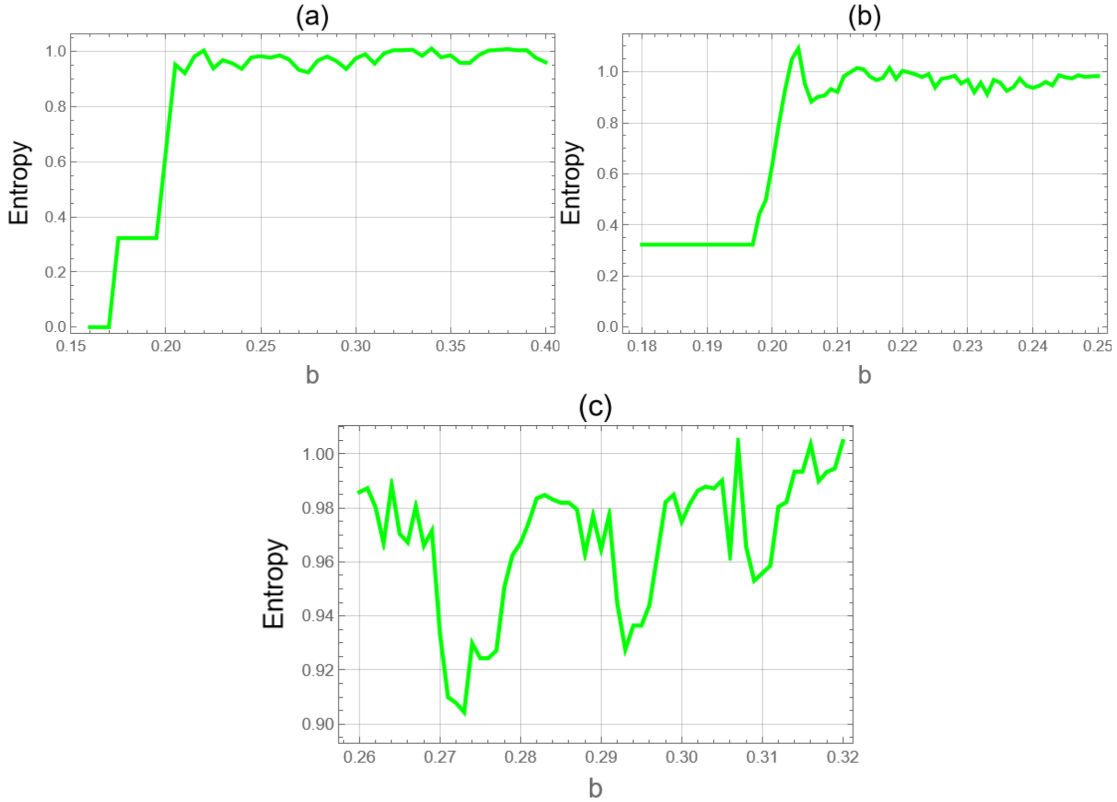

Figure 6: Plots of topological entropy, figure (a) for $0.16 \le b \le 0.4$ and for other parameters values $a = 0.89, c = 0.28, k = 0.028$. Figures (b) and (c) , respectively, corresponds to subintervals $0.18 \le b \le 0.25$ and $0.26 \le b \le 0.32$.

Figure 6 illustrates the variation of topological entropy of the system as the parameter *b* varies for fixed values of the parameters *a, c* and *k*.

(d) **Correlation Dimensions:** Correlation dimension provides the measure of dimensionality of the chaotic attractor. This is calculated statistically with the application of Heaviside function, [24]. To obtain this, first we have to calculate correlation integral data C(r), for a certain $r \ll 1$.

Then, to plot the curve $\frac{\log C(r)}{\log r}$ against r, shown in Figure 10. After this, to apply a linear fit criterion to the correlation data and obtained the equation of the straight line fitting the data points. The y-intercept of this line provides the correlation dimension.

Calculation of correlation dimension of a attractor Chialvo map for $a = 0.89$, $b = 0.25$, $c = 0.28, k = 0.028$ carried out here. A plot of correlation data for this case shown in Figure 7.

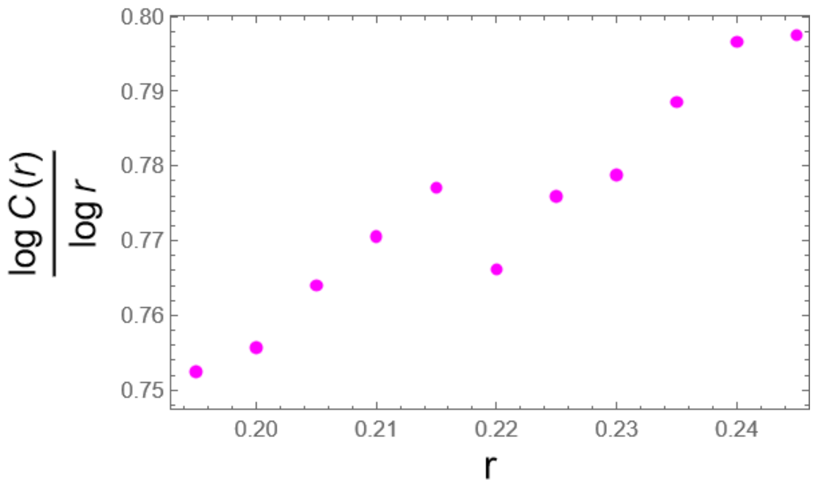

Figure 7: Plot of correlation data points of Chialvo map for $a = 0.89$, $b = 0.25$, $c = 0.28, k = 0.028$.

Equation of the straight line approximately fitting data points of Figure 6 given by

$$y = 0.868436\, x + 0.583815 \tag{3.3}$$

The y −intercept of this straight line is

$$0.583815 \cong 0.584.$$

Therefore, the correlation dimension of the chaotic attractor of Figure 3 obtained for $a = 0.89,\ b = 0.25,\ c = 0.28, k = 0.028$ is

$$D_C \cong 0.584.$$

Proceeding in similar way one can find the correlation dimension of the chaotic attractor of figure 4 obtained for parameters $a = 0.89, b = 0.19,\ c = 0.28, k = 0.028$ as

$$D_C \cong 1.488.$$

Therefore the observed dynamics in phase space in case of the forgoing parameter values exhibit fractal behaviour.

4. CONCLUDING REMARKS

Dynamics of evolution of Chialvo system, which represents model for dynamic evolution of neurons of nervous system, has been investigated in this article. Plots of bifurcations shown in Figure 2 show pattern of repeated period-doubling route to chaos. Numerous periodic windows within bifurcation diagrams indicating mixed phenomena like bi-stability, intermittency, cascading effects, which exhibit hysteresis characteristics etc. Then, plots of topological entropies Figure 6 show significant increase of topological entropies for $0.16 \le b \le 0.4$ and for other parameters values $a = 0.89, c = 0.28, k = 0.028$. In addition to increase of topological entropies, we also observe its fluctuations in the subinterval $0.26 \le b \le 0.32$. Thus, Chialvo map is of highly complex type. The map also evolves into regular and chaotic attractors at different parameter spaces shown through Figures 3 to 5. Calculation of Correlation dimension of certain chaotic attractors has been discussed at the end of section 3, and Mathematica codes described in [24] has been further used here to find it.

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