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Some Summability Techniques in Infinite Series & Sequence

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ABSTRACT

The study of summability techniques plays a vital role in the fields of Functional Analysis, Fourier series, and Engineering. This paper is mainly focused on the comparison between Cesaro summability and Able summability. Besides providing a theorem, and giving examples related to Able method and Cesaro method, we give a theorem to prove the completeness of a sequence space.

KEYWORDS: Grandi's series, Cesaro summability, Able summability, Regular.

1. INTRODUCTION AND MOTIVATION

The study of the convergence of infinite series is a classical technique. In early times, people were more anxious with traditional examinations of convergence of infinite series. Series that did not converge were of no interest until the arrival of L. Euler (1707–1783)[9]. He took up a serious analysis of the series that did not converge. Euler was followed by a galaxy of great mathematicians, such as C.F. Gauss (1777–1855), A.L. Cauchy (1789–1857), and N.H. Abel (1802–1829).In the second half of the 19th century, the interest in the study of divergent series almost declined [10]. later, in 1890 E. Cesaro reanimated the study of divergent series and introduced the idea of (C,1) convergence. [8]Since then, many other mathematicians have been contributing to the study of divergent series.

Summability theory has an important role in real analysis as well as applied mathematics[10]. Mathematicians, researchers, engineers, or physicists who deal with Fourier transforms or Fourier series and analytic continuation may find summability theory a very useful for their research. Newton and Leibniz were the first to apply the concept of infinite series, with a brief mention of divergent series. Later it became a crucial component of mathematical analysis. Cesaro sum was first used explicitly by Leibniz in 1713, and then implicitly by Frobenius and Holder in 1882. Holder developed a new summability approach called Holder summability, which was an extension of Cesaro summability. Cauchy and Frobenius addressed the problem of convergence of infinite series in 1880, introducing the arithmetic mean technique of summability, which Cesaro expanded in 1890 as the (C,K) method of summability[8].In 2016, Mishra, Tripathi, and Gupta studied some of the properties of Cesaro and Holder's mean-of-product summability methods[10]. In 2019, Sudhakar, Mallik, and Misra studied the summability techniques and their applications in different fields of science and engineering [9].

Summability theory is a part of the mathematical analysis that generalizes the concept of convergence to all sequences even non-convergent ones. It attempts to create an algorithm that assigns a limit to non-convergent sequences; The theory makes a non-convergent series converge, in a general sense, whenever a sequence of positive linear operators doesn't ordinarily converge. Several workers like Ghimire and Pahari ([5] ,[6]) Pahari, Pahari ([11],[12]), Paudel, Pahari,and Kumar ([13], [14]), etc. have made their contribution and enriched the theory in sequences.

2. PRELIMINARIES

Before proceeding with the work, we recall some of the basic notations and definitions that are used in this paper.

Definition 2.1 [3]: Let (a_n) be a given real or complex-valued sequence. Then an expression of the form $\sum_{n=1}^{\infty} x_n$ or $\sum a_n = a_1 + a_2 + a_3 + a_4 + \ldots$ is called an "infinite series" . If all of the terms of the sequence (a_n) after a certain terms are zero, then $\sum a_n$is called a "finite series" and is written simply as $\sum_{n=1}^{m} a_m$.

Definition 2.2[1]: A sequence (a_n) in a normed space $(X, \|.\|)$ is said to be Cauchy if $\forall\ \epsilon > 0, \exists N \in \mathbb{N}$ such that $\|a_n - a_m\| < \epsilon$, $\forall m, n > N$. It is said to be convergent if $\forall\ \epsilon > 0, \exists N \in \mathbb{N}$ such that $\|a_n - a\| < \epsilon$, $\forall n \geq N$. Otherwise it is called divergent.

Definition2.3[2]: For any natural number n, if the sequence (s_n) of n-th partial sum $s_n = \sum_{i=1}^{n} a_I$ of the series $\sum_{n=1}^{\infty} a_n$ assign a finite value S, then the series is said to have sum S. In this case, the series is said to converge to the sum S.

Definition2.4[3]: A summation method is called regular if it sums every convergent series (in the normal sense) to its ordinary sum.

Definition2.5[4]: A summation method S is said to be absolutely regular if it is regular and, moreover, if for all sequences (a_n) with partial sums (s_n) for which $\lim_{n ® ¥} S_n = \infty$ we have $\lim_{n ® ¥} S_n = \infty(S)$.

Definition2.6[7]: An infinite series whose terms are alternately positive and negative is called an alternating series. Thus, it may be written in the form $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \ldots$ $(a_n > 0)$.

3. SOME BASIC TECHNIQUES IN THE INFINITE SERIES

In this section, we are introducing some summability techniques, which are as follows.

Cesaro Summability[7]: The Cesaro summation method is an averaging method based on the arithmetic mean of the sequence of partial sum. Let (a_n) be a sequence, and let

$$s_k = a_1 + a_2 + a_3 + \ldots + a_k = \sum_{n=1}^{k} a_n$$

beits k^{th} partial sum. Then $\sum a_n$ is called Cesaro sum A if the sequence of arithmetic mean of arithmetic mean of its first n partial sum tends to A as n tends to infinity.

That is $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} s_k = A$.

Here the limit A is called Cesaro sum of the series $\sum_{n=1}^{\infty} a_n$ and we write $\sum_{n=1}^{\infty} a_n = A(C)$.

Example:(Cesaro summable series): Consider the Grandi's series

$G = \sum_{n=0}^{\infty}(-1)^n = 1-1+1-1+\ldots$ has partial sum s_n given by

$s_n = \sum_{i=0}^{n} g_n = \frac{(-1)^i+1}{2}$ where $g_n = (-1)^i$.

Now by using Cesaro summable method we can observe that,

$$c_n = \frac{1}{n}\sum_{i=0}^{n} s_i = \frac{1}{n}\sum_{i=0}^{n} \frac{(-1)^i+1}{2}$$
$$= \frac{1}{2n}\left(n + \sum_{i=0}^{n}(-1)^i\right)$$
$$= \frac{1}{2} + \frac{1}{2n}\sum_{i=0}^{n}(-1)^i$$
$$= \frac{1}{2} + \frac{1}{2n}s_n$$

As we know that $|s_n| \leq 1$, we can observe that c_n tends to $\frac{1}{2}$ as $n \to \infty$.

Therefore, we conclude that $\sum_{n=0}^{\infty} g_n = \frac{1}{2}(C, 1)$.

In this way, a new branch of mathematical analysis came into the existence with the aim to assign a limit(in some sense) to divergent series. The same idea can be developed for infinite series. This research had grown very fast in the last century and many mathematicians played pioneer role in the development of many summability methods for divergent sequences and series.

The following example is a milestone in which the sequence is $(C, 1)$-summable, although it is divergent.

Example (Cesaro summable Sequence)**:**

If $a_k = (-1)^k (k \in \mathbb{N})$, then consider a sequence (b_n) defined by $b_n = 0$, for $n = 2k$ and $b_n = -\frac{1}{n}$ if for $n = 2k + 1$.

We see $\lim_{n\to\infty} b_n = 0$, which gives that the sequence (a_k) is Cessaro summable i.e., $(C, 1)-$ summable to 0 and is written as $\lim_{k\to\infty} a_k = 0\ (C, 1)$.

In particular if $a_k \to S$ (C, 1) as $k\to\infty$, it is written as $\lim_{k\to\infty} a_k = S\ (C, 1)$.

It is also noted that $(C, 1)$-summability preserve the usual convergence i.e. every convergent sequence is always $(C, 1)$-summable to their own limit. Thus, with the help of arithmetic means, we see a divergent sequence may be treated as a convergent sequence. Moreover, $(C, 1)$-summability can be extended to $(C, 2)$-summability and further up to (C, k)-summability.

Example: (Series which is not Cesaro summable): Consider the series

$$F \equiv \sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots \infty$$

then the sequence of partial sums (s_k) is obtained (1,3,6,10,…).

Since the sequence of partial sums grows without bounds, the series F diverges to infinity. Also, the sequence (t_n) of arithmetic mean of partial sums of the series F is $\{\frac{1}{1}, \frac{4}{2}, \frac{10}{3}, \frac{20}{4}, \ldots\}$. This sequence diverges to infinity as well. So F is not Cesaro summable.

Theorem 3.1: [7]If $K^1 > K > -1$, and $\sum a_n = A(C,K)$ then $\sum a_n = A(C,K^1)$.

Theorem 3.2:[2] If $K>0$, then (C,K) method is regular.

Now we will define another popular method of summability which is known as Able summability method.

Definition:[7] Consider a series,

$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \ldots$; we form a new series $a_0 + a_1 x + a_2 x^2 + \ldots$; by the help of the sequence $(1, x^2, x^3, \ldots)$.

If the new series converges for $0<x<1$ to a value A with a limit x tends to 1, then the limit is called the Able sum of the original series, $a_0 + a_1 + a_2 + \ldots$.

In other words a sequence a_n is said to be Able summable, written as (A) sumable to L if $\lim_{x\to 1^-} (1-x)\sum_{k=0}^{\infty} a_k x^k$ exists finitely and equals to L.

Example:[4]From the Grandi'sseries

$$F \equiv \sum_{n=1}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots,$$

we form a new series $1-x+x^2-x^3+\ldots$; by the help of the sequence $(1, x^2, x^3, \ldots;)$.

Now for $0 < x < 1$, the new series converges to the limit $A = \frac{1}{2}$, as $1 - x + x^2 - x^3 + x^4 - \ldots = \lim_{x\to 1} \frac{1}{1+x} = \frac{1}{2} = A$.

Theorem 3.3 :[7] [15] Able's method is regular.

But, if the series is Able summable may not be convergent. This is illustrated by the following example.

Let $a_n = 1+(-1)^n$, for all $n \in \mathbb{N}$, that is clear that $\sum_{n=0}^{\infty} a_n$ is divergent. However,

$$\lim_{x\to 1^-} \sum_{k=0}^{\infty} a_k x^k = \lim_{x\to 1^-} \left(2\sum_{k=0}^{\infty} (-1)^k x^k\right)$$

$$= 2\lim_{x\to 1^-} \left(\frac{1}{1+x}\right) = 1.$$

Which shows that $\sum_{n=0}^{\infty} a_n$ is (A) summable to 1.

Now we are at the position of mentioning our main result.

4. MAIN RESULT

In this section, we shall investigate some results that characterize the Able method, which is more stronger than Cesaro method. [4][7].

Theorem4.1: If $\sum_{n=0}^{\infty} a_n$is(C,1) summable to L, then it is Able Summable.

Proof: Let us consider $s_n = \frac{1}{(n+1)}\sum_{k=0}^{n} a_k$ then $(n+1)s_n = \sum_{k=0}^{n} a_k$.

Calculating we will get $a_k = (k+1).s_k - k.s_{k-1}$.

Also, suppose that

$$f(x) = (1-x)\sum_{k=0}^{\infty} a_k x^k.$$

We observe that

$$f(x) = (1-x)\sum_{k=0}^{\infty} a_k x^k$$

$$= (1-x)\left[a_0 + \sum_{k=1}^{\infty}\{(k+1)s_k - k s_{k-1}\}x^k\right]$$

Since, $\sum_k x^{k+1}$has radius of convergence 1, the series $\sum_k (k+1)x^{k+1}$ has radius of convergence 1 and so, $\sum_k (k+1)s_k x^{k+1}$ and $\sum_k k s_{k-1} x^k$ each has radius of convergence at least 1as s_k converges to a finite limit.

Thus the power series $\left[a_0 + \sum_{k=1}^{\infty}\{(k+1)s_k - k s_{k-1}\}x^k\right]$, has a radius of convergence at least 1 and so, the power series $(1-x)\sum_k a_k x^k$ converges for $|x|<1$.

Now, we assert that $\lim_{x\to 1^-} (1-x)\sum_k a_k x^k = L$.

Now, $\frac{1}{(1-x)^2} f(x) = \frac{1}{(1-x)^2}(1-x)\sum_k a_k x^k$

$$= \frac{1}{(1-x)}\sum_k a_k x^k$$

$$= \sum_k x^k \sum_k a_k x^k$$

$$= \sum (a_0 + a_1 + a_2 + \cdots + a_k) x^k$$

$$= \sum_k (k+1) s_k x^k$$

We know that $\frac{1}{(1-x)^2} = \sum_k (k+1)x^k$

Hence for $0 < x < 1$, we get

$$|f(x) - L| = \left|(1-x)^2\sum_k (k+1)s_k x^k - L(a-x)^2\sum_k (k+1)x^k\right|$$

$$= \left|(1-x)^2\sum_{K+1}(s_k - L)x^k\right|$$

$$\le (1-x)^2 \sum_k (k+1)\,|s_k - L| x^k$$

Since $\lim_{n®¥} s_n = L$. Given that $\varepsilon > 0$, $\exists$ a positive integer N such that

$|s_n - L| < \frac{\varepsilon}{2}$, $n \ge N$.

Also as (s_n) is bounded , therefore there exists a $M > 0$ such that $|s_n| \le M$, $\forall\, n \in \mathbb{N}$.

Now let us choose

$$\delta = \min\left[\frac{1}{2}\left\{\frac{\varepsilon}{4(M+1)(N+1)}\right\}^{\frac{1}{2}}\right].$$

If $1-\delta < x < 1$, then

$$|f(x)-L| \leq (1-x)^2\left[\sum_0^{N-1}(k+1)|s_k - L|x^k + \sum_{k=N}^{\infty}(k+1)|s_k - L|x^k\right]$$

$$<\delta^2 N(2M)\sum_{k=0}^{N-1} x^k + (1-x)^2\frac{\varepsilon}{2}\frac{1}{(1-x)^2}$$

$$\leq \frac{\varepsilon}{4(M+1)(N+1)^2} 2N^2M + \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, we observed that

$\lim_{x\to 1^-} f(x) = L$ or $\lim_{x\to 1^-}(1-x)\sum_k a_k x^k = L$.

Which enable us to write that (a_n) is Able summable too. This completes the proof of the theorem.

The converse of the theorem may not hold. That is, there are some series which are Able summable but not Cesaro summable.

This result is shown by the following example:

$$a_n = \begin{cases} (k+1), if\ \ n = 2k \\ -(k+1), if\ \ n = 2k+1 \end{cases}$$

Let $s_n = \frac{1}{(n+1)}\sum_{k=0}^{n} a_k$ .Thenwe have

$$s_n = \begin{cases} \frac{k+1}{2k+1}, if\ \ n = 2k \\ 0,\ \ if\ \ n = 2k+1 \end{cases}$$

Since (s_{2k}) converges to $\frac{1}{2}$ and $(s_{2k+1})$is convergent to 0, so (s_n) does not converge.

Thus (a_n) is not $(C,1)$ summable. Now for $|x|<1$, we have

$$\sum_k a_k x^k = 1 - x + 2x^2 - 2x^3 + 3x^4 - 3x^5 + \ldots..$$

$$=(1-x)+2x^2(1-x)+3x^4(1-x)+\ldots..$$

$$=(1-x)\frac{1}{(1-x^2)^2}$$

Thus we have $(1-x)\sum_k a_k x^k = (1-x)^2\frac{1}{(1-x^2)^2} = \frac{1}{(1+x)^2}$

Consequently, $\lim_{x\to 1^-}(1-x)\sum_k a_k x^k = \lim_{x\to 1^-}\frac{1}{(1+x)^2} = \frac{1}{4}$

That is the sequence (a_n) is Able-summable to $\frac{1}{4}$. This completes the proof.

In the following, for a fixed positive number n, we define a new class of complex sequences

$a = (a_k)$ as follows:

$$(S, \|.\|) = \left\{ a = (a_k) \in \mathbb{C} : \sup_{\substack{k\geq 1 \\ n\geq 0}} \left| \frac{1}{k}\sum_{i=1}^{k} a_{n+i} \right| < \infty \right\}$$

Moreover, we define

$$\|a\| = \sup_{\substack{k\geq 1 \\ n\geq 0}} \left| \frac{1}{k}\sum_{i=1}^{k} a_{n+i} \right|, a \in (S, \|.\|) \qquad \ldots(4.1)$$

Then one can easily show that the class $(S, \|.\|)$forms a linear space with respect to the norm $\|a\|$ on $(S, \|.\|)$.

Theorem 4.2 : The linear space$(S, \|.\|)$ is complete space with respect to the norm (4.1).

Proof: Let $(a^r)_{r=1}^{\infty}$,be a Cauchy sequence in $(S, \|.\|)$,

where $a^r = (a_i^r)_{i=1}^{\infty}$, $r = 1, 2, 3, \ldots\ldots$

Given $0 < \varepsilon < 1$, there exists a positive integer N_0 such that

$$\|a^r - a^s\| < \varepsilon, \forall r, s \geq N_0$$

$$\Rightarrow \sup_{\substack{k\geq 1 \\ n\geq 0}} \left| \frac{1}{K}\sum_{i=1}^{k} a_{n+i}^r - a_{n+i}^s \right| < \varepsilon, \forall r, s\geq N_0$$

$$\Rightarrow \left| a_{n+i}^r - a_{n+i}^s \right| < \varepsilon, \forall n\geq 0 \text{ and } \forall r, s\geq N_0 \qquad \ldots(4.2)$$

This shows that for a fixed $i(1\leq i < \infty)$, the sequence $(a_i^r)_{i=1}^{\infty}$ is a Cauchy sequence of complex numbers.

Since the space of complex numbers is complete, therefore converges in it.

Let $a_i^r = a_i$ as $r\to\infty$. Define $a = (a_i)_{i=1}^{\infty}$ and taking limit $s\to\infty$ in (4.2), we get $\left|a_{n+i}^r - a_{n+i}\right| \leq \varepsilon$, $\forall n \geq 0$ and $\forall r \geq N_0$ therefore, we have

$$\sup_{\substack{k\geq 1 \\ n\geq 0}} \left| \frac{1}{K}\sum_{i=1}^{k} a_{n+i}^r - a_{n+i} \right| \leq \varepsilon, \forall r\geq N_0$$

or, $\|a^r - a\| \leq \varepsilon, \forall r \geq N_0$

or, $a^r \to a$ as $r\to\infty$

Now in view of (4.1) and(4.2), we have

$$\|a\| = \sup_{\substack{k\geq 1 \\ n\geq 0}} \left| \frac{1}{K}\sum_{i=1}^{k} a_{n+i} \right| \leq \sup_{\substack{k\geq 1 \\ n\geq 0}} \left| \frac{1}{K}\sum_{i=1}^{k} a_{n+i} - a_{n+i}^r \right| + \sup_{\substack{k\geq 1 \\ n\geq 0}} \left| \frac{1}{K}\sum_{i=1}^{k} a_{n+i}^r \right| < \infty.$$

This shows that $a \in (S, \|.\|)$ and hence $(S, \|.\|)$ is a complete normed space.

This completes the proof.

5. CONCLUSION

In this paper, we have studied some methods of summability techniques of infinite series and sequences. In fact, these results can be used in the fields of Functional Analysis, Fourier series, and Engineering to investigate other properties of the infinite series and sequences.

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